Crouzeix-Velte Decompositions and the Stokes Problem

PhD Thesis

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1 Abstract

In our dissertation the staggered grid approximation of the Stokes problem on several special domains and with different boundary conditions is investigated, using finite difference and box methods. (for finite element method see: Girault-Raviart, 1986) We deal with methods where the discrete Crouzeix-Velte decomposition (see: Stoyan-Strauber-Baran, 2004; or Subsection 1.2.) exists: we prove the existence of this decomposition and show numerical results to prove the effectiveness of these discretizations.

1.1 The Stokes problem

In our dissertation we consider at first the following first-kind Stokes problem in a cartesian coordinate system:

$$-\Delta\vec{u} + \operatorname{grad} p = \vec{f}, \text{ in } \Omega, \tag{1.1}$$

$$\operatorname{div}\vec{u} = 0, \text{ in } \Omega, \tag{1.2}$$

where $\Omega \subset \mathbb{R}^n$, $n = 2, 3$ is a bounded, simply-connected open domain, with Lipschitz – continuous boundary $\partial\Omega$, and

$$\vec{u}(x) = (u_1(x), \ldots, u_n(x))^T, \ \vec{f}(x) = (f_1(x), \ldots, f_n(x))^T,$$

defined for $x = (x_1, \ldots, x_n) \in \Omega$. On the boundary, homogeneous Dirichlet boundary conditions are imposed:

$$\vec{u} = 0, \text{ on } \partial\Omega. \tag{1.3}$$

The problem consists in finding a vector-function $\vec{u}(x)$ and a scalar function $p(x)$ that satisfy the system of partial differential equations above. The function $\vec{u}$ is the velocity of fluid and p is the (kinematic) pressure. For a constant also appearing in the first equation, the kinematic viscosity, we have chosen the value 1. Finally, f describes accelerations caused by an external force field, and the boundary conditions mean that the walls are impermeable and at rest.

A unique weak solution $\vec{u} \in V$ and $p \in P$ exists when, for example, $\vec{f} \in (L_2(\Omega))^n$, (see, e.g.,Varnhorn,1994), where $P := L_{2,0}(\Omega)$ is the subspace of $L_2(\Omega)$ of square integrable functions with zero integral over Ω , with the Hilbert space

$$L_2(\Omega) = \{\phi | (\phi, \phi) < \infty\}, (\phi, \psi) = \int_\Omega \phi\psi d\Omega$$

and where $V := (H_0^1(\Omega))^n$ is the well-known Sobolev space, with generalized derivatives in $(L_2(\Omega))^n$ and with zero boundary values in the sense of traces on the boundary $\partial\Omega$.

1.2 The Crouzeix–Velte decomposition

For an $n > 2$, let $(\cdot,\cdot)$ denote the Euclidean scalar product in $\mathbb{R}^n$, moreover, let $A, B, C \in \mathbb{R}^{n\times n}$ be matrices satisfying

$$\begin{aligned} A &= B + C, && (1.4)\\ A &= A^T > 0, && (1.5)\\ B = B^T \geq 0 \quad &, \quad C = C^T \geq 0, && (1.6)\\ \delta := \dim\ker B \geq 1 \quad &, \quad \rho := \dim\ker C \geq 1. && (1.7) \end{aligned}$$

Then, with a suitable subspace $W \subset \mathbb{R}^n$ (which may turn out to be empty) and with orthogonality to be understood in the sense of the scalar product $(A\cdot,\cdot)$, the following orthogonal decomposition of $\mathbb{R}^n$ can be derived:

$$\mathbb{R}^n = \ker B \oplus \ker C \oplus W. \tag{1.8}$$

This decomposition is called the *algebraic* Crouzeix–Velte decomposition of $\mathbb{R}^n$. (The definition of the *analytical* Crouzeix–Velte decomposition is described in the dissertation.)

For the eigenvalues λ_i of the generalized eigenvalue problem

$$\lambda A x = B x \tag{1.9}$$

we have

$$\lambda_i \in [0,1] \text{ for all } i \tag{1.10}$$

and also using the eigenvectors $x^{(i)}$, the subspaces in (1.8) can be characterized as follows:

$$\begin{aligned} \ker B &= span(x^{(i)}, \lambda_i = 0),\\ \ker C &= span(x^{(i)}, \lambda_i = 1),\\ W &= span(x^{(i)}, \lambda_i \in (0,1)). \end{aligned}$$

We can use the following correspondences connected with the matrices in (1.4):

$$A \sim -\Delta, \;\; B \sim -\operatorname{grad}\operatorname{div}, \;\; C \sim \operatorname{curl}\operatorname{rot},$$

where Δ is the (vector) Laplace operator and where the sign $\sim$ expresses only an analogy between a differential operator and a matrix, and is not necessarily a (good) approximation. In this sense (1.4) corresponds to the well known identity

$$-\Delta = -\operatorname{grad}\operatorname{div} + \operatorname{curl}\operatorname{rot} \tag{1.11}$$

of vector analysis.

The algebraic Crouzeix–Velte decomposition is *proper* in case $\dim W = n - \delta - \rho > 0$.

In the *discrete* case, the velocity space (which approximates $(H^1(\Omega))^n$ or a subspace of the latter) will be denoted by $\vec{V}_h$, the pressure space will be denoted by P_h, and div_h and rot_h will be written for the discrete equivalents of the divergence and rotation operator, Δ_h will denote the discrete Laplace operator. The matrix corresponding to the mapping $-\mathrm{div}_h$ from the velocity space into the pressure space is denoted by $\tilde{B}_h$ and we introduce the following notations: $\tilde{C}_h$ for the matrix of the operator rot_h and A_h for the matrix of the operator $-\Delta_h$. If A_h, $\tilde{B}_h$, $\tilde{C}_h$ matrices satisfy the following:

$$\begin{aligned} A_h &= B_h + C_h, \\ A_h &= A_h^T > 0, \end{aligned} \tag{1.12}$$

where $B_h = \tilde{B}_h^T \tilde{B}_h$ and $C_h = \tilde{C}_h^T \tilde{C}_h$ and $\ker \tilde{B}_h \neq \emptyset$ and $\ker \tilde{C}_h \neq \emptyset$, then a discrete Crouzeix-Velte decomposition exists and (1.8) takes the form

$$V_h = \ker \mathrm{div}_h \oplus \ker \mathrm{rot}_h \oplus W = V_{h,0} \oplus V_{h,1} \oplus V_{h,\beta}.$$

P_h is decomposed similarly into three orthogonal subspaces:

$$P_h = \ker \mathrm{grad}_h \oplus \mathrm{div}_h \ker \mathrm{rot}_h \oplus \mathrm{div}_h V_{h,\beta} = P_{h,0} \oplus P_{h,1} \oplus P_{h,\beta}.$$

After discretization by the finite element or finite difference methods, the Stokes problem takes the following form:

$$\begin{pmatrix} A_h & \tilde{B}_h^T \\ \tilde{B}_h & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}. \tag{1.13}$$

With these matrices (1.9) corresponds to

$$\lambda_h A_h x_h = B_h x_h,$$

which is transformed by $p_h := \tilde{B}_h x_h$ into

$$\lambda_h p_h = S_h p_h,$$

where S_h is the discrete Schur complement operator of the Stokes problem, $S_h := \tilde{B}_h A_h^{-1} \tilde{B}_h^T$. If the discrete Crouzeix-Velte decomposition exists then $\lambda_h \in [0, 1]$.

In our dissertation we deal with approximations of the Stokes problem on several domains, where the discrete Crouzeix-Velte decomposition exists. In this case in the spectrum of the discrete Schur complement operator, there is an eigenvalue 1 of high multiplicity (of the order of inner grid points). Then, the error components lying in the eigensubspace corresponding to this eigenvalue can be removed by one step of a simple damped Jacobi iteration or Uzawa-algorithm. The remaining error lies in an eigensubspace of much smaller dimension connected only with boundary effects. Moreover, the conjugate gradient iteration automatically takes advantage of such a spectrum and converges faster than for discrete spaces without a decomposition: as it is proved by the numerical results.

In our dissertation the following methods are used: finite difference and box methods for approximation of Stokes-problem, and Uzawa-algorithm (as outer iteration) in the numerical experiments, moreover the conjugate gradient method (as inner iteration) with an effective preconditioning matrix and FFT algorithm and alternatively multigrid method solving the discrete Poisson equations.

2 Results

2.1 First order staggered grid approximation on non-equidistant rectangular grids

First we consider the well-known staggered-grid approximation where Ω is a rectangle subdivided by a non-equidistant grid into $(n-1)(m-1)$ rectangular cells. The boundary conditions are homogeneous Dirichlet boundary conditions.

For pressure vectors p_h and velocity vectors $\vec{u}_h = (u_h, v_h)^T$ suitable discrete scalar products and the corresponding norms are introduced. Using these scalar products and norms the following Theorem is proved, with the notations $A_h, \tilde{B}_h, \tilde{C}_h$ for the matrices of the operators $-\Delta_h$, $-\operatorname{div}_h$ ill rot_h:

Theorem 1

$$(A_h\vec{u}_h, \vec{u}_h)_{0,h} = \|\tilde{B}_h\vec{u}_h\|^2_{0,h} + \|\tilde{C}_h\vec{u}_h\|^2_{0,\tilde{h}} \tag{2.1}$$

holds for all vectors $\vec{u}_h := (u_h, v_h)^T \in \vec{V}_h$ if and only if

$$h_{1,i+1/2} = \frac{h_{1,i-1/2} + h_{1,i+3/2}}{2} \quad \text{and} \quad h_{2,j+1/2} = \frac{h_{2,j-1/2} + h_{2,j+3/2}}{2}. \quad \bullet$$

Remark 1.

It is proved that $\dim(V_{h0}) = (n-2)^2$, $\dim(V_{h1}) = (n-3)^2$ and $\dim(V_{h,\beta}) = 4n-9$, where n denotes the number of grid points (including corner points) along a side of the square. $\bullet$

Remark 2.

We prove that A_h is symmetric in the sense of the scalar product $(\vec{u}_h, \vec{v}_h)_{0,h}$ above, and the following equation holds:

$$D_A A_h = \tilde{B}_h^T D_B \tilde{B}_h + \tilde{C}_h^T D_C \tilde{C}_h, \tag{2.2}$$

where D_A, D_B, D_C are diagonal matrices corresponding to the adequate norms. That is

$$D_A A_h =: \hat{A}_h = B_h + C_h, \tag{2.3}$$

where $B_h = \hat{B}^T \hat{B}$ és $C_h = \hat{C}^T \hat{C}$ with the notations: $\hat{B} = D_B^{1/2} \tilde{B}_h, \hat{C} = D_C^{1/2} \tilde{C}_h$. $\bullet$

Remark 3.

We prove that $D_A A_h = \hat{A}_h$ matrix is positive definite, that is $\hat{A}_h > 0$. Together with remark 1 and 2, it means that a proper Crouzeix–Velte decomposition of the velocity and the pressure space into three nontrivial parts exists, if $n > 3$ $\bullet$

2.2 First order staggered grid approximation based on the finite volume (box) method on non-equidistant rectangular grids

Then we apply finite volume method on the staggered grid, where Ω is also a rectangle subdivided by a non-equidistant grid into $(n-1)(m-1)$ rectangular cells. The boundary conditions are homogeneous Dirichlet boundary conditions.

We prove that using this approximation, the discrete Crouzeix–Velte decomposition exists, if $n > 3$, without any restriction of grid spacing.

Remark

The results above (using either finite difference or finite volume methods) hold if Ω is a union of rectangles such that all the boundary lines of the different rectangles fit on the same global grid with grid spacing h_1 and h_2. $\bullet$

2.3 Numericul results

We show with numerical experiments - using the staggered grid approximation based on the finite volume method - that both the Uzawa-type and the conjugate gradient-type methods are faster on such grids which are condensing in the center and coarser near the boundary of the domain. In the case of Uzawa-type methods we determine the optimal non-equidistant grid.

2.4 Second order staggered grid approximation based on the finite difference method on equidistant rectangular grids

Then we apply second order finite difference method on the staggered grid, where Ω is a unit square subdivided by an equidistant grid. The boundary conditions are homogeneous Dirichlet boundary conditions. After defining the suitable approximations, discrete scalar products and the corresponding norms, the following Theorem is proved:

Theorem For the second order staggered grid approximation,

$$\begin{aligned}(A_{\overline{h}}\vec{u}_h, \vec{u}_h)_{0,h} &- \|\tilde{B}_{\overline{h}}\vec{u}_h\|^2_{0,h} - \|\tilde{C}_{\overline{h}}\vec{u}_h\|^2_{0,h} = \\ &= -\sum_{i=2}^{n-1} 2(u^2_{i,1} + u^2_{i,n-1}) - \sum_{j=2}^{n-1} 2(v^2_{1,j} + v^2_{n-1,j}), \quad (2.4)\end{aligned}$$

where $A_{\overline{h}} := -\Delta_{\overline{h}}$, $\tilde{B}_{\overline{h}} := -div_{\overline{h}}$ and $\tilde{C}_{\overline{h}} := rot_{\overline{h}}$. •

Remark We introduce the notation $\tilde{A}_{\overline{h}}$, where

$$(\tilde{A}_{\overline{h}}\vec{u}_h, \vec{u}_h)_{0,h} = (A_{\overline{h}}\vec{u}_h, \vec{u}_h)_{0,h} + \sum_{i=2}^{n-1} 2(u^2_{i,1} + u^2_{i,n-1}) + \sum_{j=2}^{n-1} 2(v^2_{1,j} + v^2_{n-1,j}).$$

$\tilde{A}_{\overline{h}}$ is a symmetric positive definite matrix in the sense of the corresponding scalar product, together with $A_{\overline{h}}$. Since $\dim(V_{h0}) = (n-2)^2$, $\dim(V_{h1}) = (n-3)^2$ and $\dim(V_{h,\beta}) = 4n-9$, a proper Crouzeix–Velte decomposition exists in this case as well, for $n > 3$. In this case the algebraic decomposition exists not for the matrix $A_{\overline{h}}$,but for $\tilde{A}_{\overline{h}}$, hence $\tilde{A}_{\overline{h}}$ can be advantageous as a preconditional matrix solving $A_{\overline{h}}\vec{u}_h = b_h$. •

2.5 Finite difference approximation in the case of non-standard boundary conditions

We investigate the finite difference approximation of the Stokes problem on the staggered grid with the following non-standard boundary conditions:

$$u_{1,j} = u_{n,j} = v_{i,1} = v_{i,n} = 0, \tag{2.5}$$

$1 \le i \le n-1,\;\; 1 \le j \le n-1$ esetén, és

$$(\mathrm{rot}_h\,\vec{u}_h)_{0,j} = (\mathrm{rot}_h\,\vec{u}_h)_{n-1,j} = (\mathrm{rot}_h\,\vec{u}_h)_{i,1} = (\mathrm{rot}_h\,\vec{u}_h)_{i,n} = 0, \tag{2.6}$$

$0 \le i \le n-1,\;\; 2 \le j \le n-1$, where rot_h is defined as usual. (This boundary condition satisfies the Lopatinski-condition.)

We show that a proper discrete Crouzeix–Velte decomposition exists in the case of this boundary condition as well, using either first or second order approximation. •

2.6 Finite difference approximation in the case of periodical boundary conditions

As we show in this point, the results on the existence of an analytical Crouzeix-Velte decomposition for the Stokes problem along with Dirichlet and periodical boundary conditions carry over to the discrete case for the staggered grid approximation.

We apply first order finite difference approximation on the staggered grid, where Ω is a unit square subdivided by an equidistant grid. Periodical boundary conditions are assumed on the left and right sides of the unit square:

$$\begin{aligned} u_{1,j} = u_{n-1,j}, u_{2,j} = u_{n,j}, &\qquad 0 \le j \le n, \\ v_{1,j} = v_{n-1,j}, v_{2,j} = v_{n,j}, &\qquad 1 \le j \le n. \end{aligned} \tag{2.7}$$

On the upper and lower sides of the unit square we prescribe homogeneous Dirichlet conditions:

$$\begin{aligned} u_{i,0} = u_{i,n} = 0, &\qquad 1 \le i \le n, \\ v_{i,1} = v_{i,n} = 0, &\qquad 1 \le i \le n-1, \end{aligned}$$

where $u_{i,0}, u_{i,n}$ are values on two additional grid lines, which the grid has been supplemented with. We prove that a proper discrete Crouzeix–Velte decomposition exists in this case as well. •

2.7 Approximation on a nonequidistant grid in 3D with homogeneous Dirichlet boundary conditions

Then the well-known difference approximation on a staggered grid is considered in 3D case. In our case Ω is a rectangular parallelepipedon subdivided by a rectangular grid into $(n-1)(m-1)(l-1)$ cells of volume $h_1h_2h_3$ each, $h_1 := 1/(n-1), h_2 := 1/(m-1), h_3 := 1/(l-1)$. We assume $n, m, l \geq 3$.

First we assume homogeneous Dirichlet boundary conditions:

$$u_{1,j,k} = u_{n,j,k} = v_{i,1,k} = v_{i,m,k} = w_{i,j,1} = w_{i,j,l} = 0,$$

where $\vec{u}_h := (u_h, v_h, w_h)^T$ is the velocity vector and $1 \leq i \leq n-1,\ 1 \leq j \leq m-1,\ 1 \leq k \leq l-1$.

We prove the existence of the discrete Crouzeix–Velte decomposition. •

2.8 Approximation on an equidistant grid in 3D with periodical boundary conditions

Then we consider the first order staggered grid approximation on an equidistant, cubic grid. On the front-back sides and on the north-south sides of the cube we prescribe homogeneous Dirichlet conditions, and periodical boundary conditions are assumed on the east-west sides. We prove the existence of the discrete Crouzeix–Velte decomposition in this case as well. •

2.9 The Stokes problem in polar coordinates for the disk domain

Finally let Ω be the unit disk

$$\Omega = \{(r,\varphi) | 0 \leq r < 1, 0 \leq \varphi < 2\pi\},$$

and consider the following Stokes problem:

$$\Delta_{r\varphi} u - \frac{u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial \varphi} - \frac{\partial p}{\partial r} = f_1, \quad (2.8)$$

$$\Delta_{r\varphi} v - \frac{v}{r^2} + \frac{2}{r^2}\frac{\partial u}{\partial \varphi} - \frac{1}{r}\frac{\partial p}{\partial \varphi} = f_2, \quad (2.9)$$

$$\operatorname{div} \vec{u} = \frac{1}{r}\left(\frac{\partial}{\partial r}(ru) + \frac{\partial v}{\partial \varphi}\right) = 0, \quad (2.10)$$

where $(u, v) = \vec{u}$ and $(f_1, f_2) = \vec{f}$ and

$$\Delta_{r\varphi} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}.$$

On the boundary, homogeneous Dirichlet boundary conditions are imposed:

$$\vec{u} = 0, \text{ on } \partial\Omega. \tag{2.11}$$

The problem consists in finding a vector-function $\vec{u}(x)$ and a scalar function $p(x)$ that satisfy the system of partial differential equations above. In our dissertation we consider a suitable second order finite difference approximation and introduce the corresponding discrete scalar products and norms. The following Theorem - similarly to (2.1) - is proved:

Theorem

$$(\tilde{A}_h\vec{u}_h, \vec{u}_h)_{0,h} = \|\tilde{B}_h\vec{u}_h\|^2_{0,\bar{r},h} + \|\tilde{C}_h\vec{u}_h\|^2_{0,r,h} \tag{2.12}$$

holds for all vectors $\vec{u}_h := (u_h, v_h)^T \in \vec{V}_h$, where $\tilde{B}_h$ corresponds to the negative divergence operator, $\tilde{C}_h$ corresponds to the rotation operator and $\tilde{A}_h$ corresponds to the negative Laplace operator, which is in polar coordinates:

$$\Delta\vec{u} = \begin{pmatrix} \Delta_{r\varphi}u - \frac{u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial\varphi} \\ \Delta_{r\varphi}v - \frac{v}{r^2} + \frac{2}{r^2}\frac{\partial u}{\partial\varphi} \end{pmatrix}. \quad \bullet \tag{2.13}$$

Remark From the Theorem above we obtain that $\tilde{A}_h$ can be written in the following form:

$$D_{\tilde{A}}\tilde{A}_h =: A_h = B_h + C_h, \tag{2.14}$$

where $B_h = \hat{B}_h^T\hat{B}_h$, $C_h = \hat{C}_h^T\hat{C}_h$ and $\hat{B}_h = D_{\tilde{B}}^{1/2}\tilde{B}_h, \hat{C}_h = D_{\tilde{C}}^{1/2}\tilde{C}_h$ and where $D_{\tilde{A}}$, $D_{\tilde{B}}$, $D_{\tilde{C}}$ are diagonal matrices corresponding to the adequate scalar products. $\bullet$

2.10 Numerical results for approximation on the unit disk

2.10.1 Uzawa-algorithm

Using the notations (2.14) our problem consists in finding the solution of the following algebraic system:

$$A_h\vec{u}_h + \hat{B}_h^T p_h = \vec{f}_h, \tag{2.15}$$
$$\hat{B}_h\vec{u}_h = g_h. \tag{2.16}$$

We use the Uzawa-algorithm to solve (2.15), (2.16):

$$\begin{aligned} p_h^{(0)} &:= 0, \\ p_h^{(i+1)} &:= p_h^{(i)} + \omega(\hat{B}_h \vec{u}_h^{(i)} - g_h) \\ \vec{u}_h^{(i)} &:= A_h^{-1}(\vec{f}_h - \hat{B}_h^T p_h^{(i)}) \end{aligned} \tag{2.17}$$

$i = 0, 1, 2, \dots .$

Since the discrete Crouzeix-Velte decomposition exists, using the Uzawa-algorithm we can reach the third Crouzeix-Velte subspace after 1 step (with $\omega = 1$). In this subspace the spectrum of the Schur complement is closer, and the algorithm shows effective convergence. The optimal iteration parameter is also calculated using the smallest and the largest of the eigenvalues different from 0 and 1 of the discrete Schur complement. The numerical results show that the discretization obeying a discrete Crouzeix–Velte decomposition leads to effectively solvable systems of algebraic equations. The average number of the iterations using the Uzawa algorithm is 3-5, and the speed of convergence is growing together with the refinement of the grid.

2.10.2 Fourier transformation and conjugate gradient method

Instead of the calculation of A_h^{-1} in (2.17) in the first case the fast Fourier transformation is used in combination with the preconditioned conjugate gradient method. In the numerical experiments several preconditioning matrices are investigated: the number of inner iterations needed to reach the stopping criterion of the conjugate gradient method, using the best preconditioning matrix, is only 4-5, and the speed of convergence is growing together with the refinement of the grid.

2.10.3 Multigrid method

In the second case the multigrid method is used to calculate A_h^{-1}. We describe the restriction operator and optimize the prolongation operator with numerical experiments. We compare several pre- and post-smoothing iterations. The optimal iteration parameter of the damped Jacobi iteration - as smoothing iteration - is also calculated. The results show that the Gauss-Seidel iteration combined with block Gauss-Seidel iteration - as smoothing iteration - is the most effective: the multigrid method using this iteration is on average 5-8 times faster than using the Gauss-Seidel iteration and this advantage is growing with the refinement of the grid. The less effective smoothing iteration is the Jacobi iteration, it results on average three times slower convergence than the Gauss-Seidel iteration.

3 Publications

STRAUBER, GY.: *Discrete Crouzeix-Velte decompositions on nonequidistant rectangular grids.* Annales Univ. Sci. Budapest.,44 (2002), 63-82.

STOYAN, G., STRAUBER, GY., BARAN, A.: *Generalizations to discrete and analytical Crouzeix-Velte decompositions.* Numer. Lin. Algebra with Appls., 11, (2004), 565–590.

STRAUBER, GY.: *Discrete Crouzeix-Velte decomposition for the disk domain.* Miskolc Mathematical Notes, Vol.6 (2005), No. 1, pp. 129-143.