Summer School in Discrete Mathematics

Eötvös Loránd University, Budapest

23-27 June, 2014

Special thanks go to Wiebke Höhn and Max Klimm for providing the amazing template of this guide book.

Contents

Welcome to Summer School in Discrete Mathematics 2014!

In this short guide we would like to provide you with some basic information about practical issues as well as a rather incomplete list of sights, museums, restaurants, bars and pubs.

If you have any question concerning the summer school or your stay in Budapest, do not hesitate to ask us!

We wish you a pleasant stay in Budapest!

The Organizers

Venue

The summer school takes place in the Southern building ("Déli tömb") at the Lágymányosi Campus of the Eötvös Loránd University. The lectures are given in room 4-710, refreshments are provided in room 4-713.

Internet access

Wireless internet connection is available in the lecture room. Participants also get an account to access the university's network at the computer labs which is on the 3rd floor of the building (rooms 3-107, 3-111 and 3-114). Your username and password are included in the package received upon arrival.

If you need a scanner or a printer, please contact the organizers.

Lunch

The registration fee also covers lunches. We will have lunch together at the cafeteria of the university. The cafeteria can be found on the ground floor of the Northern building. In case you are planning to have lunch somewhere else, please notify the organizers.

Contact

If you have any questions or problems, please contact the organizers or the lecturers.

István Ágoston
Email: agoston@cs.elte.hu
Tel.: +36 1 372 2500 / 8422
Room: 3-708

Kristóf Bérczi
Email: berkri@cs.elte.hu
Tel.: +36 1 372 2500 / 8582
Room: 3-516

Tamás Héger
Email: hetamas@cs.elte.hu
Tel.: +36 1 372 2500 / 8597
Room: 3-609

Tibor Jordán
Email: jordan@cs.elte.hu
Tel.: +36 1 372 2500 / 8579
Room: 3-504

Gyula Károlyi
Email: karolyi@cs.elte.hu
Tel.: +36 1 372 2500 / 8426
Room: 3-711

György Kiss
Email: kissgy@cs.elte.hu
Tel.: +36 1 372 2500 / 8550
Room: 4-723

Márton Naszódi
Email: nmarci@cs.elte.hu
Tel.: +36 1 372 2500 / 8548
Room: 4-721

Dömötör Pálvölgyi
Email: dom@cs.elte.hu
Tel.: +36 1 372 2500 / 8604
Room: 3-616

Péter Sziklai
Email: sziklai@cs.elte.hu
Tel.: +36 1 372 2500 / 8602
Room: 3-615

Schedule

	Monday	*Tuesday*	*Wednesday*	*Thursday*	*Friday*
9.00-10.30	**Jordán**	**Bérczi**	**Károlyi**	**Héger / Sziklai**	**Bérczi**
10.30-11.00	*refreshment*	*refreshment*	*refreshment*	*refreshment*	*refreshment*
11.00-12.30	**Naszódi**	**Pálvölgyi**	**Jordán**	**Naszódi**	**Pálvölgyi**
12.30-14.00	*lunch*	*lunch*	*lunch*	*lunch*	*lunch*
14.00-15.30	**Héger / Sziklai**	**Kiss**	*free afternoon*	**Kiss**	**Károlyi**
16.00-19.00		*Get together party*			

Getting around

Getting around by taxi

The taxi fares are uniformly calculated as follows: the base fee is 450 HUF with an additional 280 HUF/km or 70 HUF/min. Altogether, getting around by taxi is rather expensive and you might want to consider using public transport instead.

Getting around by bike

You can rent bikes for 2500 - 3500 HUF per day at many places. Here are some possibilities:

- **Budapestbike** - `http://www.budapestbike.hu/`
- **Yellow Zebra Bikes and Segways** - `http://www.yellowzebrabikes.com/`
- **Bikebase** - `http://bikebase.hu/home`
- **Bestbikerental** - `http://www.bestbikerental.hu/`

Getting around by public transport

The public transportation system in Budapest is a favourite internal travel option for a number of Budapest visitors. The system is efficient, inexpensive and runs throughout all of the major tourist areas of Budapest. The system consists of a combination of the bus, trolley-bus, tram, metro, and train lines and is streamlined so that tickets for all of them can generally be purchased at the same locations.

All regular transportation services stop around midnight (varies by route). However, night buses (blue coloured buses, marked with black in the schedule, numbered 900-999 and tram line 6) replace the metro lines, major tram and bus routes and run through the night until normal service resumes in the morning. Separate schedules for night and day buses are posted at every stop. In the inner areas buses run very frequently (appr. 10-15 min.) Please note: there's front door boarding-only on most lines, except tram 6 and articulated buses.

Budapest currently has four metro lines - M1 (Yellow), M2 (Red), M3 (Blue) and M4 (Green). The Yellow line is the oldest underground transportation line in continental Europe and retains much of its old-fashioned charm. Lines M1, M2 and M3 meet at Deák Ferenc Tér in central Pest. Line M4 opened on 28 March 2014 and connects the Kelenföld railway station located in Buda, and the eastern Keleti station in Pest. Trains run very frequently (3-5 minutes between rush hours and weekend, 1-3 minutes in rush-hours, 10 minutes late night).

Budapest has an extensive system of above-ground trams. The most useful lines for tourists are the famous 4 and 6, which follows the large ring road that encircles the Budapest city center and crosses Margaret bridge before terminating at Széll Kálmán Tér on the Buda side on the North - and Petőfi bridge before it terminates at Móricz Zsigmond Körtér, also on the Buda side; no 47 and 49, which runs through central Pest and across the river to Hotel Gellért; no. 2, which follows the Danube River on the Pest side; and no. 19, which follows the Danube River on the Buda Side.

Bus lines of use to most tourists are the 7 and 107 which connect the busy Keleti railway station and the area around the Kelenföld railway station on the Buda side. Some other notable places that they stop along is Blaha Lujza ter (connection to the red M2 metro line, also trams 4 and 6), Ferenciek tere (very near Váci utca), and in front of the Rudas bath and the Gellért Hotel both on the Buda side. Bus 86 is also very useful as it has a stop near the Gellért Hotel and runs along the river bank on the Buda side.

All public transport in Budapest is run by the company BKK. Connections can be easily checked at `http://www.bkk.hu/en/timetables/` or by using the convenient smart phone apps available for **Android** or **iPhone**.

On the metro lines, **tickets need to be bought and validated before boarding** while on buses and trams you have to validate your ticket on the spot. For a complete list of tickets and conditions see `http://www.bkk.hu/en/prices/`.

Single ticket 350 HUF
Valid for one uninterrupted trip without change. On the metro lines the ticket must be validated before the start of the trip; on other vehicles immediately after boarding or after the vehicle has departed. Validity period is 60 minutes after stamping; it is 120 minutes on night services.

Block of 10 tickets 3000 HUF
You can buy 10 tickets in a block with some discount compared to buying 10 single tickets separately.

Transfer ticket 530 HUF
Valid for one trip with one change. Trip interruptions - other than changes - and return trips are not permitted. The ticket must be validated at the printed number grids at either end: first when starting a trip at one end and at the other end when changing, with the exception of changes between metro lines.

Short section metro ticket for up to 3 stops 300 HUF
Valid on the metro for one short trip of up to 3 stops for 30 minutes after validation. Trip interruptions and return trips are not permitted.

Single ticket for public transport boat 750 HUF
24-hour, 72-hour, 7-day travelcards and Monthly Budapest pass is valid on weekdays.

Budapest 24-hour travelcard 1650 HUF
Valid for 24 hours from the indicated date and time (month, day, hour, minute) for an unlimited number of trips.

5/30 BKK 24-hour travelcard 4550 HUF
The 5/30 BKK travelcard consists of 5 slips, each with a validity period of 24 hours. The block can be purchased with any starting day with a validity period of 30 days from the starting day.

Budapest 72-hour travelcard 4150 HUF
Valid for 72 hours from the indicated date and time (month, day, hour, minute) for unlimited number of trips on the public transport services ordered by BKK on tram, trolleybus, underground, metro, cogwheel railway on the whole length of the lines on all days; on the whole length of boat services but only on working days.

Budapest 7-day travelcard 4950 HUF
Valid from 00:00 on the indicated starting day until 02:00

on the following 7th day for an unlimited number of trips. The travelcard is to be used only by one person; it is non-transferable as it is issued specifically for the holder.

Monthly Budapest pass for students 3450 HUF

Valid from 0:00 of the indicated optional starting day to 2:00 of the same day of the following month. Valid for students in higher education together with a Hungarian or EU or ISIC student ID.

Giraffe Hop On Hop Off

Giraffe Hop On Hop Off tours offer 2 bus, 1 boat and 1 walking tour in Budapest for tourists. They pass several sights on their way; the RED and YELLOW lines are audio guided in 20 languages and the BLUE boat line is audio guided in English and German. The ticket is valid on the day of the first departure while the next day is free.

Things to do in Budapest

- Walk along the Danube on the Pest side between Elisabeth Bridge and Chain Bridge. Then cross the Danube and continue towards Margareth Bridge to see the Parliament Building from Batthyány tér.
- Take Metro 1 from Vörösmarty tér to Hősök tere and see the monument there. You may take a walk in the City Park (Városliget) or go to the Zoo or the Museum of Fine Arts.
- After 7:00pm take a short walk along Ráday utca from Kálvin tér. You may want to enter one of the cafés or restaurants.
- Go to the Great Market Hall on Vámház körút, close to Liberty Bridge (Szabadság híd). After all, this is one of the few things Margareth Thatcher did when she visited Budapest in the 1980's (she bought garlic :)).
- Go to Gellért Hill to get a glimpse of the city from above.
- Go to a concert in one of the major or smaller concert halls, churches or open air locations. Some of them are free.
- Go to Margaret Island and see the fountain on the southern end or the music tower on the northern end of the island. In between you will find a garden of roses and a small zoo.
- Go at night to the Palace of Arts (across Eötvös University, close to Rákóczi Bridge on the Pest side) and enjoy the view of the National Theatre or of the glass walls of the Palace of Arts.
- See the bridges at night. You get a good view from Castle Hill.
- Go to some of the baths in Budapest (Széchenyi bath in the City park, Rudas bath at the Buda side of Elisabeth Bridge or Gellért bath in Hotel Gellért at the Buda side of Liberty Bridge).

Words of caution

- Don't go to a restaurant or café without checking the price list first. A reasonable dinner should not cost you more than 20 Euros (6000 HUF). (Of course you may be willing to pay more but you should know in advance.)
- Don't leave your valuables unattended, especially not in places frequently visited by tourists. Be aware of pickpockets on crowded buses or trams.
- Budapest is relatively safe even at night, nevertheless if possible, try to avoid being alone on empty streets at night. Some pubs should also be avoided.
- Don't carry too much cash with you: direct payment banking cards are widely accepted, although credit cards are not so wide spread. If you withdraw money from a banking machine, be careful and try to do it in a public place.
- If you get on a bus, tram, trolley or metro, usually you have to have a pass or a prepaid ticket which you have to validate upon boarding (or when entering the metro station). Most of the tickets are valid for a single trip only (even if it is only for a short distance). If you get a pass for a week, you have to enter on the ticket the number of a photo id (passport, id card) which you have to carry with you when using the pass. - On some buses you may get a single ticket from the driver but be prepared to have change with you. Even tickets bought from the driver have to be validated.

Shopping centres

Mammut. 1024 Budapest, II. district, Lövőház utca 2, +36 1 345 8000 `www.mammut.hu`

Westend City Center. 1062 Budapest, VI. district, Váci út 1-3, +36 1 238 7777 `www.westend.hu`

Corvin Plaza. 1083 Budapest, VIII. district, Futó utca 37, +36 1 301 0160 `corvinplaza.hu`

Arena Plaza. 1087 Budapest, VIII. district, Kerepesi út 9, +36 1 880 7000 `www.arenaplaza.hu`

Allee. 1117 Budapest, XI. district, Október huszonharmadika utca 8-10, +36 1 372 7208 `www.allee.hu`

Market halls

Batthyány téri Vásárcsarnok. 1011 Budapest, I. district, Batthyány tér 5

Rákóczi téri Vásárcsarnok. 1084 Budapest, VIII. district, Rákóczi tér 7-9, Mon–Fri 6:00am–4:00pm, Sat 6:00am–1:00pm

Vámház körúti Vásárcsarnok. 1093 Budapest, IX. district, Vámház körút 1-3, Sun–Fri 6:00am–5:00pm, Sat 6:00am–3:00pm

Fehérvári úti Vásárcsarnok. 1117 Budapest, XI. district, Kőrösi J. utca 7-9, Mon–Fri 6:30am–5:00pm, Sat 6:30am–3:00pm

Supermarkets

CBA `www.cba.hu/uzletek`

CBA Déli ABC Nagyáruház. 1013 Budapest, I. district, Krisztina krt. 37, Mon–Fri 6:00am–10:00pm, Sat 6:00am–8:00pm, Sun 7:00am–6:00pm

CBA Ferenciek Tere. 1053 Budapest, V. district, Ferenciek tere 2, Mon–Fri 6:00am–10:00pm, Sat 7:00am–10:00pm, Sun 8:00am–8:00pm

CBA Élelmiszer. 1051 Budapest, V. district, József Attila u. 16, Mon–Fri 6:30am–9:00pm, Sat 7:00am–8:00pm

CBA élelmiszer. 1061 Budapest, VI. district, Anker köz 1, Mon–Fri 6:00am–10:00pm, Sat 6:00am–11:00pm, Sun 7:00am–10:00pm

CBA Millenium Príma élelmiszerüzlet. 1061 Budapest, VI. district, Andrássy út 30, Mon–Fri 7:00am–10:00pm, Sat 8:00am–10:00pm, Sun 9:00am–10:00pm

CBA Rákóczi út. 1074 Budapest, VII. district, Rákóczi u. 48-50, Mon–Fri 6:30–9:00pm, Sat 7:00am–8:00pm, Sun 8:00am–7:00pm

Körúti Élelmiszer. 1075 Budapest, VII. district, Károly krt. 9, Mon–Fri 6:30am–10:00pm, Sat 8:00am–10:00pm, Sun 8:00am–7:00pm

Görög Csemege. 1085 Budapest, VIII. district, József krt. 31, Mon–Fri 6:00am–10:00pm, Sat–Sun 8:00am–8:00pm

WADI Kft. 1085 Budapest, VIII. district, József krt 84, Mon–Sat 6:00am–9:00pm

Corvin Átrium CBA. 1085 Budapest, VIII. district, Futó utca 37, Mon–Fri 7:00am-10:00pm, Sun 8:00am–8:00pm

Bakáts csemege. 1092 Budapest, IX. district, Bakáts tér 3, Mon–Fri 6:00am–7:30pm, Sat 7:00am–1:00pm

Szatócs delikátesz. 1111 Budapest, XI. district, Bartók Béla út 32, Mon–Fri 6:30am-8:00pm, Sat 6:30am–1:00pm, Sun 7:00am–1:00pm

Csemege a Karinthyn. 1111 Budapest, XI. district, Karinthy Frigyes u. 30, Mon–Fri 6:30am–8:00pm, Sat 6:30am–2:00pm

Spar/Interspar	www.spar.hu

SPAR. 1011 Budapest, I. district, Batthyány tér 5-6, Mon–Fri 6:30am–9:00pm, Sat 8:00am–8:00pm, Sun 8:00am–5:00pm

SPAR. 1024 Budapest, II. district Lövőház utca 2-6, Mon–Sat 6:30am–10:00pm, Sun 8:00am–7:00pm

City SPAR. 1052 Budapest, V. district, Károly körút 22-24, Mon–Fri 7:00am–10:00pm, Sat 8:00am–10:00pm, Sun 8:00am–8:00pm

SPAR. 1066 Budapest, VI. district, Teréz körút 28, Mon–Fri 7:00am-10:00pm, Sun 8:00am–8:00pm

SPAR. 1073 Budapest, VII. district, Erzsébet körút 24, Mon–Fri 7:00am–10:00pm, Sat 7:00am–7:00pm, Sun 8:00am–7:00pm

City SPAR. 1076 Budapest, VII. district, Thököly út 8, Mon–Fri 6:30am–10:00pm, Sat 7:30am–7:30pm, Sun 8:00am–7:00pm

SPAR. 1085 Budapest, VIII. district, Blaha Lujza tér 1, Mon–Fri 7:00am–10:00pm, Sat 7:00am–9:00pm, Sun 8:00am–6:00pm

City SPAR. 1092 BUDAPEST, IX. district, Mester utca 1, Mon–Fri 6:30am–10:00pm, Sat 7:00am–8:00pm, Sun 8:00am–8:00pm

City SPAR. 1092 Budapest, IX. district, Ráday utca 32, Mon–Fri 7:00am–10:00pm, Sat 7:00am–8:00pm, Sun 8:00am–7:00pm

SPAR. 1095 Budapest, IX. district, Soroksári út 1, Mon–Fri 6:30am–9:00pm, Sat 6:30am–5:00pm, Sun 8:00am–1:00pm

SPAR Market. 1111 Budapest, XI. district, Bartók Béla út 14, Mon–Sun 0:00am–0:00am

SPAR. 1117 Budapest, XI. district, Irinyi József utca 34, Mon–Fri 7:00am–9:00pm, Sat 7:00am–5:00pm, Sun 7:00am–4:00pm

INTERSPAR. 1117 Budapest, XI. district, Október 23-a utca 8-10 (in the basement of the Allee shopping center), Mon–Sat 7:00am–10:00pm, Sun 8:30am–8:00pm

SPAR. 1123 Budapest, XII. district, Alkotás utca 53, Mon–Sat 7:30am–10:00pm, Sun 8:00am–8:00pm

Penny Market	www.penny.hu

Penny Market. 1085 Budapest, VIII. district, József Körút 45, Mon–Sat 6:00am–9:00pm, Sun 7:00am–6:00pm

ALDI	www.aldi.hu

ALDI. 1054 Budapest, V. district, Báthory utca 8, Mon–Sun 7:00am–9:00pm

ALDI. 1081 Budapest, VIII. district, Rákóczi út 65, Mon–Sun 7:00am–10:00pm

ALDI. 1093 Budapest, IX. district, Vámház körút 1-3, Mon–Sat 6:00am–9:00pm, Sun 8:00am–9:00pm

ALDI. 1094 Budapest, IX. district, Tűzoltó utca 10-16, Mon–Sun 7:00am–10:00pm

Lidl	www.lidl.hu

Lidl. 1060 Budapest, VI. district, Bajcsy-Zsilinszky út 61, Mon–Sun 7:00am–9:00pm

Lidl. 1082 Budapest, VIII. district, Leonardo Da Vinci utca 23, Mon–Sun 7:00am–9:00pm

Tesco	www.tesco.hu

Tesco Astoria Szupermaket. 1088 Budapest, VIII. district, Rákóczi út 1-3, Mon–Sun 6:00am–0:00am

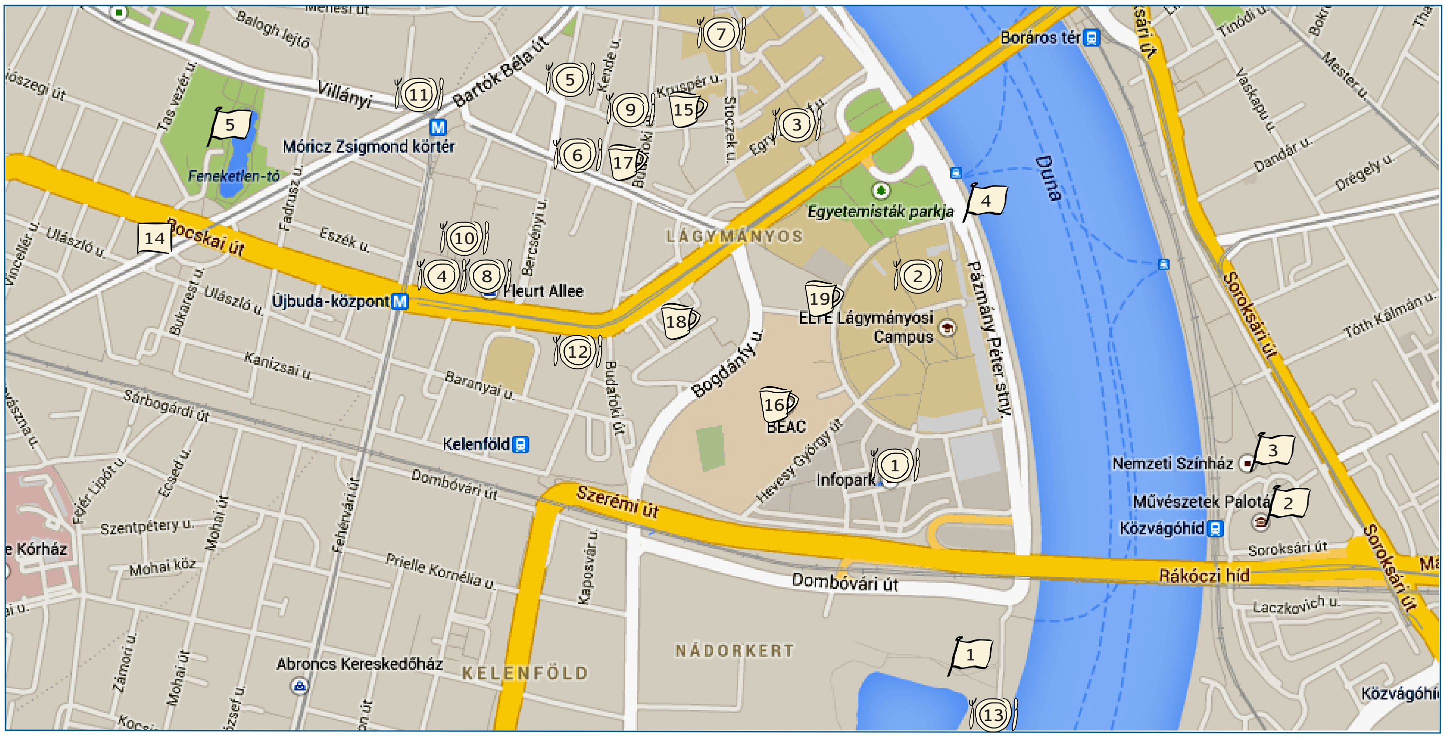
Tesco Expressz. 1088 Budapest, VIII. district, Rákóczi út 20, Mon–Sun 6:30am–10:00am

Tesco Expressz. 1088 Budapest, IX. district, Kálvin tér 7, Mon–Sun 6:30am–11:00am

Tesco Arena Plaza Hipermarket. 1087 Budapest, VIII. district, Kerepesi út 9-11

Tesco Soroksári út. 1097 Budapest, IX. district, Koppány utca 2-4, Mon–Sun 6:00am–11:00pm

Tesco Új Buda Center. 1117 Budapest, XI. district, Hengermalom út 19, Mon–Sun 0:00am-0:00am



Sights

Kopaszi-gát. *1117 Budapest, Kopaszi gát 5, Bus 103, 6:00am–10:00pm.*

Kopaszi-gát is a beautifully landscaped narrow peninsula in south Buda, next to Rákóczi Bridge. Nested in between the Danube on one side and a protected bay, it has a lovely beach feel. Kopaszi-gát is also a favourite picnic spot and the park offers lots of outdoor activities from biking to ball games. The sign in the park says it all: "Fűre lépni szabad!", which means "Stepping on the grass is permitted!"

Palace of Arts. *1095 Budapest, Komor Marcell utca 1, Suburban railway 7.*

The Palace of Arts in Budapest, also known as MÜPA for short (Művészetek Palotája), is located within the Millennium Quarter of the city, between Petőfi and Lágymányosi bridges. It is one of the most buzzing cultural and musical centres in Budapest, and as such one of the liveliest Budapest attractions. Think of the Palace of Arts as a cultural complex, which includes the Festival Theatre, the Béla Bartók National Concert Hall and the Ludwig Museum.

National Theatre. *1095 Budapest, Bajor Gizi park 1, Suburban railway 7.*

The building lies on the bank of the Danube, in the Ferencváros district, between the Soroksári road, the Grand Boulevard and the Lágymányosi bridge, and is a five-minute walk from the Csepel HÉV (Suburban railway 7). The area of the theatre can be functionally separated into three parts. The central part is the nearly round building of the auditorium and stage, surrounded by corridors and public areas. The second is the U-shaped industrial section around the main stage. The third section is the park that surrounds the area, containing numerous memorials commemorating the Hungarian drama and film industry.

A38 Ship. *1117 Budapest, a little South from Petőfi bridge, Buda side, Trams 4 and 6 (Petőfi híd, budai hídfő), Mon–Sun 11:00am–4:00am.*

The world's most famous repurposed Ukrainian cargo ship, A38 is a concert hall, cultural center and restaurant floating on the Danube near the abutment of Petőfi Bridge on the Buda-side with a beautiful panorama. Since its opening it has become one of Budapest's most important venues, and according to artists' feedback, one of Europe's coolest clubs.

Feneketlen-tó. *1114 Budapest, Bus 86, Tram 19, 49.*

Feneketlen-tó, which means bottomless lake, is surrounded by a beautiful park filled with paths, statues and children's playgrounds. The lake is not as deep as its name suggests. In the 19th century there was a brickyard in its place and the large hole dug by the workers filled with water when they accidentally hit a spring. Ever since, locals cherish the park and they come to feed the ducks, relax on the benches or take a stroll around the lake. The lake's water quality in the 1980s began to deteriorate, until a water circulation device was built. The lake today is a popular urban place for fishing.

Restaurants & Eateries

Infopark. *Next to the university campus, Mon–Fri 8:00am–6:00pm.*

Infopark is the first innovation and technology park of Central and Eastern Europe. There are several cafeterias and smaller sandwich bars hidden in the buildings, most of them are really crowded between 12:00am-2:00pm.

University Cafeteria. *University campus, Northern building, Mon–Fri 8:00am–4:00pm.*

The university has a cafeteria on the ground floor of the Northern building. You can also buy sandwiches, bakeries, etc here.

Goldmann restaurant. *1111 Budapest, Goldmann György tér 1, Mon–Fri 11:00am–3:00pm.*

Goldmann is a cafeteria of the Technical University, popular among students for its reasonable offers. Soups are usually quite good.

4 **Fehérvári úti vásárcsarnok.** *1117 Budapest, Kőrösi József utca 7–9, Mon 6:30am–5:00pm, Tue–Fri 6:30am–6:00pm, Sat 6:30am–3:00pm.*
A farmers market with lots of cheap and fairly good native canteens (e.g. Marika Étkezde) on the upper floors. You can also find cheese, cakes, fruits, vegetables etc.

5 **Anyu.** *1111 Budapest, Bercsényi utca 8, Mon–Fri 8:00am–8:00pm.*
Tiny bistro selling home-made soup, sandwiches and cakes.

6 **Turkish restaurant.** *1111 Budapest, Karinthy Frigyes út 26, Mon–Sun 10:00am–0:00am.*
This tiny Turkish restaurant offers gyros, baklava and salads at a reasonable price.

7 **Stoczek.** *1118 Budapest, Stoczek utca 1-3, Mon–Fri 11:00am–3:00pm.*
Stoczek is a cafeteria of the Technical University. It offers decent portions for good price. There are two floors, a café can be found downstairs.

8 **Allee.** *1117 Budapest, Október huszonharmadika utca 8-10, Mon–Sun 10:00am–10:00pm.*
A nearby mall with several restaurants on its 2nd floor.

9 **Íz-lelő étkezde.** *1111 Budapest, Lágymányosi utca 17, Mon–Fri 11:00am–5:00pm.*
Decent lunch for low price, and student friendly atmosphere. Only open from Monday to Friday!

10 **Cserpes Milk Bar.** *1117 Budapest, Október Huszonharmadika utca 8-10, Mon–Sat 7:30am–10:00pm, Sun 9:00am–8:00pm.*
A milk bar just next to the shopping center Allee. Great place for having a breakfast or a quick lunch.

11 **Wikinger Bistro.** *1114 Budapest, Móricz Zsigmond körtér 4, Mon–Sun 10:00am–21:00pm.*
If you are up for hamburgers, Wikinger Bistro offers a huge selection of different burgers.

12 **Hai Nam Bistro.** *1117 Budapest, Október huszonharmadika utca 27, Mon–Sun 10:00am–9:00pm.*
If you like Vietnamese cuisine and Pho, this may be the best place in the city. It is a small place, so be careful, it is totally full around 1:00pm.

13 **Vakvarjú.** *1117 Budapest, Kopaszi gát 2, Mon–Sun 11:30am–11:30pm.*
Vakvarjú can be found on the Kopaszi gát. It is a nice open-air restaurant where you can have lunch and relax next to the Danube for a reasonable price.

Others

14 **Gondola.** *1115 Budapest, Bartók Béla út 69-71, Mon–Sun 10:00am–8:00pm.*
This is a nice little ice cream shop right next to the Feneketlen-tó (Bottomless Lake).

Pubs

15 **Bölcső.** *1111 Budapest, Lágymányosi utca 19, 11:30am–11:00pm.*
Bölcső has a nice selection of Hungarian and Czech craft beers and one of the best all-organic homemade burger of the city. Other than burgers, the menu contains homemade beer snacks such as pickled cheese, hermelin (a typical Czech bar snack), and breadsticks. Bölcső also boasts a weekly menu that makes a perfect lunch or dinner.

16 **Szertár.** *1117 Budapest, Bogdánfy utca 10.*
Szertár is a small pub close to the university campus. It is located at the BEAC Sports Center and offers sandwiches and hamburgers as well. A perfect place to relax after a long day at the university where you can also play kicker.

17 **Pinyó.** *1117 Budapest, Karinty Frigyes út 26, Mon 10:00am–0:00am, Tue–Sat 10:00am–1:00am, Sun 4:00pm–0:00am.*
Squeezed to a basement, Pinyó looks like being after a tornado: old armchairs, kicker table, tennis racket on the wall, ugly chairs and tables. It does not promise a lot, but from the bright side, it is completely foolproof. Popular meeting place among students.

18 **Lusta Macska.** *1117 Budapest, Irinyi József utca 38, Mon–Sat 2:00pm–0:00am.*
Lusta Macska is a cheap pub for students close to the Schönherz dormitory of the Technical University. It is a tiny place with very simple furniture.

19 **Kocka.** *1111 Budapest, Warga László út 1, Mon–Fri 6:30am–7:50pm.*
Nearby the campus, the Kocka Pub is cool during the summer with its benches.

Southern part of the city center

Sights

1 **Great Market Hall.** *1093 Budapest, Vámház körút 1-3, Metro 4, Mon 6:00am–5:00pm, Tue–Fri 6:00am–6:00pm, Sat 6:00am–3:00pm.*
Central Market Hall is the largest and oldest indoor market in Budapest. Though the building is a sight in itself with its huge interior and its colourful Zsolnay tiling, it is also a perfect place for shopping. Most of the stalls sell fruits and vegetables but you can also find bakery products, meat, dairy products and souvenir shops. In the basement there is a supermarket.

2 **Károlyi Garden.** *1053 Budapest, Károlyi Mihály utca 16, Metro 2, Tram 47, 49.*
Károlyi Garden is maybe the most beautiful park in the center of Budapest. It was once the garden of the castle next to it (Károlyi Castle, now houses the Petőfi Literature Museum). In 1932 it was opened as a public garden. In the nearby Ferenczy utca you can see a fragment of Budapest's old town wall (if you walk in the direction of Múzeum körút).

3 **Gellért Hill and the Citadel.** *1118 Budapest, Metro 4, Bus 7, 86, 173.*
The Gellért Hill is a 235 m high hill overlooking the Danube. It received its name after St. Gellért who came to Hungary as a missionary bishop upon the invitation of King St. Stephen I. around 1000 a.d. If you approach the hill from Gellért square, you can visit the Gellért Hill Cave, which is a little chapel. The fortress of the Citadel was built by the Habsburgs in 1851 to demonstrate their control over the Hungarians. Though it was equipped with 60 cannons, it was used as threat rather than a working fortification.
From the panorama terraces one can have a stunning view of the city, especially at night. By a short walk one can reach the Liberation Monument.

4 **National Museum.** *1088 Budapest, Múzeum körút 14-16, Metro 3, 4, Tram 47, 49, Tue–Sun 10:00am–6:00pm, 1100 HUF.*
An 500-metre long route in a cave with narrow, canyon-like corridors, large level differences, astonishing stone formations, drip stones, glittering calcium-crystals and prints of primeval shells. Even with the 120 steps and the ladder that have to be mounted, the whole tour can easily be fulfilled in normal clothes and comfortable shoes.

5 **Rudas Gyógyfürdő és Uszoda.** *1013 Budapest, Döbrentei tér 9 (a little South from Erzsébet bridge, Buda side), Buses 5, 7, 8, Trams 18, 19, Mon–Sun 6:00am–8:00pm; Night bath: Fri–Sat 10:00pm–4:00am, 1500–4200 HUF.*
Centered around the famous Turkish bath built in the 16th century, Rudas Spa offers you several thermal baths and swimming pools with water temperatures varying from 16 to 42 Celsius degrees.

6 **Gellért Gyógyfürdő és Uszoda.** *1118 Budapest, Kelenhegyi út 4 (at Gellért tér), Metro 4, Buses 7, 86, Trams 18, 19, 47, 49, Mon–Sun 6:00am–8:00pm, 4900–5500 HUF.*
Gellért Thermal Bath and Swimming Pool is a nice spa in the center of the city.

7 **Sziklatemplom (Cave Church).** *1111 Budapest, Szent Gellért tér, Metro 4, Tram 18, 19, 41, 47, 49, Bus 7, 86, Mon–Sat 9:30am–7:30pm, 500 HUF.*
The Cave Church, located inside Gellért Hill, isn't your typical church with high ceilings and gilded interior. It has a unique setting inside a natural cave system formed by thermal springs.

Restaurants & Eateries

1 **Pagony.** *1114 Budapest, Kemenes utca 10, Mon–Sun 10:00am–1:00am.*

If you are looking for a cool spot in the blazing summer heat of Budapest, look no further. This joint was created by its resourceful proprietor by converting an unused toddler's pool section of the Gellért bath into a trendy pub. While there is no water (yet) in the pools, you can definitely find a table with comfy chairs which are actually in a wading pool.

2 **Hummus Bar.** *1225 Budapest, Bartók Béla út 6, Mon–Fri 10:00am–10:00pm, Sat–Sun 12:00am–10:00pm.*
The famous homemade Hummus can be enjoyed in variety of different dishes. The menu offers everything from a wide variety of quality salads, soups, desserts, meats and vegetarian dishes. The food is prepared with great care using only high quality products, and focusing on the simplicity of preparation - thus allowing affordable pricing.

3 **Főzelékfaló Ételbár.** *1114 Budapest, Bartók Béla út 43–47, Mon–Fri 10:00am–9:30pm, Sat 12:00–8:00pm.*
Főzelékfaló Ételbár boasts a selection centered on főzelék, a Hungarian vegetable dish that is the transition between a soup and a stew, but you can get fried meats, several side dishes, and desserts as well.

4 **Főzelékfaló Ételbár.** *1053 Budapest, Petőfi Sándor utca 1 (Ferenciek tere), Mon–Fri 10:00am–8:00pm, Sat 12:00–8:00pm.*
Főzelékfaló Ételbár boasts a selection centered on főzelék, a Hungarian vegetable dish that is the transition between a soup and a stew, but you can get fried meats, several side dishes, and desserts as well.

5 **Púder.** *1091 Budapest, Ráday utca 8, Sun–Thu 12:00am–1:00am, Fri–Sat 12:00am–2:00am.*
Restaurant and bar with a progressive, eclectic interior that was created by Hungarian wizards of visual arts. Its back room gives home to a studio theatre. Many indoor and outdoor cafés, bars, restaurants and galleries are located in the same street, the bustling neighborhood of Ráday Street is often referred to as "Budapest Soho".

Cafés

6 **CD-fű.** *1053 Budapest, Fejér György utca 1, Mon–Sat 4:00pm–12:00pm.*
As the third teahouse of Budapest, CD-fű is located in a slightly labyrinth-like basement. With its five rooms it is a bit larger than usual, and also gives place for several cultural events.

7 **Hadik kávéház.** *1111 Budapest, Bartók Béla út 36, Mon–Sat 9:00am–11:00pm.*
A lovely place to relax and soak up the atmosphere of prewar years in Budapest. Hadik is a pleasant, old-fashioned café serving excellent food.

8 **Sirius Teaház.** *1088 Budapest, Bródy Sándor utca 13, Mon–Sun 12:00am–10:00pm.*
Sirius teahouse has the perfect atmosphere to have a cup of tea with your friends, but it is better to pay attention to the street numbers, this teahouse is very hard to find, there is no banner above the entrance. Customers can choose from 80 different types of tea.

Pubs

9 **Mélypont Pub.** *1053 Budapest, Magyar utca 23, Mon–Tue 6:00pm–1:00am, Wed–Sat 6:00pm–2:00am.*
Basement pub in the old city center. Homey atmosphere with old furniture.

10 **Trapéz.** *1093 Budapest, Imre utca 2, Mon–Tue 10:00am–0:00am, Wed–Fri 10:00am–2:00am, Sat 12:00am–2:00am.*
Nice ruin pub in an old house behind the Great Market Hall which also has an open-air area. You can watch sports events and play kicker on the upper floor.

11 **Élesztő.** *1094 Budapest, Tűzoltó utca 22, Mon–Sun 3:00pm–3:00am.*
Élesztő is the Gettysburg battlefield of the Hungarian craft beer revolution; it's a like a mixture of a pilgrimage site for beer lovers, and a ruin-pub with 17 beer taps, a home brew bar, a theater, a hostel, a craft pálinka bar, a restaurant and a café.

12 **Mr. & Mrs. Columbo.** *1013 Budapest, Szarvas tér 1, Mon–Sat 4:00pm–11:00pm.*
A nice pub with excellent food and czech beers. Their hermelin is really good.

13 **Aréna Corner.** *1114, Budapest, Bartók Béla út 76, Sun–Thu 12:00am–0:00am, Fri–Sat 12:00am-2:00am.*
A nice place to watch World Cup matches while drinking Czech beer.

Others

14 **Mikszáth square.** *1088 Budapest, Mikszáth Kálmán tér.*
Mikszáth tér and the surrounding streets are home to many cafés, pubs and restaurants usually with nice outdoor terraces. Many places there provide big screens to watch World Cup matches.

Sights

1 **St. Stephen's Basilica.** *1051 Budapest, Szent István tér 1, Tram 2, Guided tours Mon–Fri 10:00am–3:00pm, 1600 HUF, student 1200 HUF.*
This Roman Chatolic Basilica is the most important church building in Hungary, one of the most significant tourist attractions and the third highest building in Hungary. Equal with the Hungarian Parliament Building, it is one of the two tallest buildings in Budapest at 96 metres (315 ft) - this equation symbolises that worldly and spiritual thinking have the same importance. According to current regulations there cannot be taller building in Budapest than 96 metres (315 ft). Visitors may access the dome by elevators or by climbing 364 stairs for a 360° view overlooking Budapest.

2 **Opera.** *1061 Budapest, Andrássy út 22, Metro 1, Tours start at 3:00pm and 4:00pm, 2900 HUF, Students: 1900 HUF.*
The Opera House was opened in 1884 among great splendour in the presence of King Franz Joseph. The building was planned and constructed by Miklós Ybl, who won the tender among other famous contemporary architects.

3 **Kossuth Lajos Square.** *1055 Budapest, Metro 2, Tram 2.*
The history of Kossuth Lajos Square goes back to the first half of the 19th century. Besides the Parliament, other attractions in the square refer to the Museum of Ethnography (which borders the square on the side facing the Parliament) and to several monuments and statues. The square is easily accessible, since the namesake metro station is located on the south side of the square.

4 **Parliament.** *1055 Budapest, Kossuth Lajos tér 1-3, Metro 2, Tram 2, Mon–Fri 8:00am–6:00pm, Sat–Sun 8:00am–4:00pm, 3500 HUF, EU citizens and students 1750 HUF, EU students 875 HUF.*
The commanding building of Budapest Parliament stretches between Chain Bridge and Margaret Bridge on the Pest bank of the Danube. It draws your attention from almost every riverside point. The Gellért Hill and the Castle Hill on the opposite bank offer the best panorama of this huge edifice. The Hungarian Parliament building is splendid from the inside too. You can visit it on organised tours. Same-day tickets can be purchased in limited numbers at our ticket office in the Museum of Ethnography. Advance tickets are available online at `www.jegymester.hu/parlament`.

5 **Buda Castle and the National Gallery.** *1014 Budapest, Szent György tér 2, Bus 16, Funicular, Tue–Sun 10:00am–6:00pm, 900 HUF.*
Buda Castle is the old royal castle of Hungary, which was damaged and rebuilt several times, last time after World War II. Now it houses the Széchényi Library and the National Gallery, which exhibits Hungarian paintings from the middle ages up to now. The entrance to the castle court is free (except if there is some festival event inside). One of the highlights of the court is the Matthias fountain which shows a group of hunters, and the monument of Prince Eugene Savoy. From the terrace of the monument you have a very nice view of the city.

6 **Fishermen's Bastion.** *1014 Budapest, Hess Andras Square 1-3, Bus 16, 16A, 116, all day, tower: daily 9:00am–11:00pm, free, tower: 700 HUF, students: 350 HUF.*
On the top of the old fortress walls, the Fishermen's Bastion was only constructed between 1895-1902. It is named after the fishermen's guild because according to customs in the middle ages this guild was in charge of defending this part of the castle wall. As a matter of fact it has never had a defending function. The architect was Frigyes Schulek, who planned the building in neo-gothic style.

7 **Matthias Church.** *1014 Budapest, Szentháromság tér 2, Bus 16, 16A, 116, Mon–Fri 9:00am–5:00pm, Sat 9:00am–1:00pm, Sun 1:00pm–5:00pm, 1200 HUF, students: 800 HUF.*

Matthias Church (Mátyás-templom) is a Roman Catholic church located in front of the Fisherman's Bastion at the heart of Buda's Castle District. According to church tradition, it was originally built in Romanesque style in 1015. The current building was constructed in the florid late Gothic style in the second half of the 14th century and was extensively restored in the late 19th century. It was the second largest church of medieval Buda and the seventh largest church of medieval Hungarian Kingdom.

8 **Heroes Square.** *1146 Budapest, Hősök tere, Metro 1.*

The monumental square at the end of Andrássy Avenue sums up the history of Hungary. The millennium memorial commemorates the 1000th anniversary of the arrival of the Hungarians in the Carpathian Basin.

9 **Városliget.** *1146 Budapest, Városliget, Metro 1.*

Városliget (City Park) is a public park close to the centre of Budapest. It is the largest park in the city, the first trees and walkways were established here in 1751. Its main entrance is at Heroes Square, one of Hungary's World Heritage sites.

10 **Vajdahunyad vára.** *1146 Budapest, Városliget, Metro 1, Courtyard always open, Castle Tue–Sun 10:00am-5:00pm, Courtyard free, Castle 1100 HUF.*

Vajdahunyad Castle is one of the romantic castles in Budapest, Hungary, located in the City Park by the boating lake / skating rink. The castle, despite all appearances, was built in 1896, and is in fact a fantasy pastiche showcasing the architectural evolution through centuries and styles in Hungary. The castle is the home of several festivals, concerts and the exhibitions of the Hungarian Agricultural Museum.

11 **Museum of Fine Arts (Szépművészeti Múzeum).** *1146 Budapest, Dózsa György út 41, Metro 1, Tue–Sun 10:00am–6:00pm, 1800 HUF.*

The Museum of Fine Arts is a museum in Heroes' Square, Budapest, Hungary. The museum's collection is made up of international art (other than Hungarian), including all periods of European art, and comprises more than 100,000 pieces. The Museum's collection is made up of six departments: Egyptian, Antique, Old sculpture gallery, Old painter gallery, Modern collection, Graphics collection.

12 **Zoo Budapest (Fővárosi Állat és Növénykert).** *1146 Budapest, Állatkerti körút 6-12, +36 1 273 4900, Metro 1, Mon–Thu 9:00am–6:00pm, Fri–Sat 9:00am–7:00pm, 1900 HUF.*

The Budapest Zoo and Botanical Garden is one of the oldest in the world with its almost 150 years of history. Some of its old animal houses were designed by famous Hungarian architects. Nowadays it houses more than 1000 different species. Currently the greatest attraction is Asha, the child elephant.

13 **Holnemvolt Park.** *1146 Budapest, Állatkerti körút 6-12, Metro 1, Mon–Thu 10:00am–6:00pm, Fri–Sat 10:00am–7:00pm, 500 HUF.*

Holnemvolt Park is situated next to the Zoo. It opened recently in the place of the old amusement park. It can either be visited with a Zoo ticket, or separately. Besides some local and some exotic species, it offers entertainment rides, some of which are nearly a hundred years old, and have been inherited from the oldest amusement parks of the city (wooden roller coaster, traditional carousel, enchanted castle).

14 **Great Synagogue.** *, Metro 2, Bus 7,9, Tram 47,49, Sun–Thu 10:00am–4:00pm, 3000 HUF.*

The Great Synagogue in Dohány Street is the largest Synagogue in Europe and the second largest in the world. It can accommodate close to 3,000 worshipers. It was built between 1854 and 1859 in Neo-Moorish style. During World War II, the Great Synagogue was used as a stable and as a radio communication center by the Germans. Today, it's the main center for the Jewish community.

15 **Millenáris.** *1024 Budapest, Kis Rókus utca 16–20, Tram 4, 6 (Széna tér), Fri–Sun 6:00am–11:00pm.*

Located next to the Mammut mall, at the site of the onetime Ganz Electric Works, Millenáris is a nice park and venue for exhibitions, concerts, performances. You can also see a huge hyperbolic quadric and its two reguli.

16 **Batthyány tér.** *1011 Budapest, Metro M2, Tram 19, 41, Bus 86, Suburban railway 5.*

Batthyány square has a great view on the beautiful Hungarian Parliament, one of Europe's oldest legislative buildings, a notable landmark of Hungary.

17 **Erzsébet tér.** *1051 Budapest, Tram 47, 49, Metro 1, 2, 3.*

Erzsébet Square is the largest green area in Budapest's inner city. The square was named after Elisabeth, 'Sisi', wife of Habsburg Emperor Franz Joseph, in 1858. The square's main attraction is the Danubius Fountain, located in the middle of the square, symbolizing Hungary's rivers. One of the world's largest mobile Ferris wheels can be also found on the square. The giant wheel offers fantastic views over Budapest day and night. Standing 65 meters tall, the wheel with its 42 cars is Europe's largest mobile Ferris wheel.

18 **Hungarian Academy of Sciences.** *1051 Budapest, Széchenyi István tér 9, Tram 2.*

The Hungarian Academy of Sciences is the most important and prestigious learned society of Hungary. Its seat is at the bank of the Danube in Budapest.

19 **Playground for adults.** *1124 Budapest, Vérmező.*

A playground for adults? Yes, this indeed exists and can be found in a nice park on the Buda side, close to the castle.

Restaurants & Eateries

1 **Onyx restaurant.** *1051 Budapest, Vörösmarty tér 7, Tue–Fri 12:00am–2:30pm, 6:30pm–11pm; Sat 6:30pm–11:00pm.*

Exclusive atmosphere, excellent and expensive food – Onyx is a highly elegant restaurant with one Michlein Star. Do not forget to reserve a table.

2 **Pizza King.** *1072 Budapest, Akácfa utca 9, Mon–Sun 10:00am–9:00pm.*
During lunchtime on weekdays offers nice menus for 900 HUF, and you can buy cheap pizza there at any time of the day. Also runs pizza takeaways at many locations in the city.

Pubs

3 **Snaps Galéria.** *1077 Budapest, Király utca 95, Mon–Fri 2:00pm–0:00am, Sat 6:00pm–0:00am.*
Snaps is a tiny two-floor pub located in the sixth district. From the outside it is nothing special, but entering it has a calm atmosphere. The beers selection - Belgian and Czech beers- is quite extraordinary compared to other same level pubs.

4 **Noiret Pool and Darts Hall, Cocktail Bar and Pub.** *1066 Budapest, Dessewffy utca 8-10, Mon–Sun 10:00am–0:40am.*
A good place to have a drink and play pool, darts, snooker, or watch soccer.

5 **Szimpla Kert.** *1075 Budapest, Kazinczy utca 14, Mon–Sun 12:00am–3:00am.*
Szimpla Kert (Simple Garden) is the pioneer of Hungarian ruin pubs. It is really a cult place giving new trends. Undoubtedly the best known ruin pub among the locals and the tourists, as well.

Others

6 **Mammut Shopping and Entertainment Centre.** *1024 Budapest, Lövőház utca 2, Fri–Sat 10:00am–9:00pm, Sun 10:00am–6:00pm.*
A twin mall in the heart of Buda.

7 **WestEnd City Center.** *1062 Budapest, Váci út 1-3, Mon–Sun 8:00am–11:00pm.*
A big mall with stores, restaurants etc. and a roof garden.

8 **Corvintető.** *1085 Budapest, Blaha Lujza tér 1, Mon–Sun 6:00pm–6:00am.*
Situated on the rooftop of once-glorious Corvin Department Store, Corvintető offers world-class DJs and concerts every day of the week. Recommended by The New York Times. Do not mess it up with Corvin Negyed, another stop of trams 4 and 6.

Margaret Island

Margaret Island (Margitsziget) is the green heart of Budapest. It lies in the middle of the Danube between Margaret Bridge and Árpád Bridge. Apart from a couple of hotels and sport facilities, there are no buildings on the Island, it is a huge green park with promenades and benches, great for a date or a picnic. Everyone can find their own cup of tea here: there is the Hajós Alfréd National Sports Swimming Pool, the Palatinus and the running track for the sporty, the petting zoo, the music fountain and the Water Tower for families, and we recommend the Japanese Garden or a ride on a 4-wheel bike car for couples. If you're hungry for culture, check out the open-air stages and the medieval ruins of the Island.

Sights

1 **Entrance of Margit-sziget.** *Budapest, Margit híd, Trams 4 and 6 (Margit híd, Margit-sziget).*
Here you can enter the beautiful Margit-sziget (Margaret Island) on foot. However, you may also take bus 26 to get to the Island.

2 **Kiscelli Múzeum.** *1037 Budapest, Kiscelli utca 108, Bus 17, 160, 260, Tue–Sun 10:00am–6:00pm, 500 HUF.*
Kiscelli Múzeum is located in a beautiful baroque monastery in Old-Buda. It offers exhibitions on the history of Budapest between the 18–21. centuries.

3 **Görzenál.** *1036 Árpád fejedelem útja 125, Bus 29, Suburban railway 5, Mon–Fri 2:00pm–8:00pm, Sat–Sun 9:30am–8:00pm, Mon–Fri 500 HUF, Sat–Sun 900 HUF.*
Görzenál currently is the biggest outdoor roller skating rink in Europe. The skating surface of the Gorzenal Roller Skate and Recreational Park is 14,000 square meters. This rink, which is located in picturesque surroundings along the Danube and Margaret Island, has a skating track as well as park structures for aggressive roller sports and BMX.

4 **Pál-völgyi Cave.** *1025 Budapest, Szépvölgyi út 162, Bus 65, Tue–Sun 10:00am–4:00pm, 1300 HUF.*
An 500-metre long route in a cave with narrow, canyon-like corridors, large level differences, astonishing stone formations, drip stones, glittering calcium-crystals and prints of primeval shells. Even with the 120 steps and the ladder that have to be mounted, the whole tour can easily be fulfilled in normal clothes and comfortable shoes.

5 **Gül baba's türbe.** *1023 Budapest, Mecset utca 14 (entrance: Türbe tér 1), Tram 4, 6, Tue–Sun 10:00am–6:00pm, free.*
The tomb of Gül Baba, "the father of roses", who was a Turkish poet and companion of Sultan Suleiman the Magnificent. He died shortly after the Turkish occupation of Buda in 1541 and his tomb is said to be the northernmost pilgrimage site of the muslims in the world. It is located on a hilltop, surrounded by a beautiful garden which offers a nice view of the city.

Bars

1 **Holdudvar Courtyard.** *1138 Budapest, Margitsziget, Mon–Tue 11:00am–0:00am, Wed 11:00am–2:00am, Thu 11:00am–4:00am, Fri–Sat 11:00–5:00, Sun 11:00–0:00am.*
A great entertainment spot in Budapest where everybody finds something to do: an open-air cinema, café, bar. The gallery exhibits works of contemporary fine art. Holdudvar hosts fashion shows and various cultural events.

Lecture notes

1 Edmonds' theorems

Let $D = (V, A)$ be a directed graph with designated root-node r. An **arborescence** is a directed tree in which every node is reachable from a given root node. We sometimes identify an arborescence (U, F) with its edge-set F and will say that the arborescence F **spans** U. An arborescence F with root node r is called an r**-arborescence**. We call D **rooted** k**-edge-connected** if for each $v \in V$, there exist k edge-disjoint directed paths from r to v. By Menger's theorem, this is equivalent to $\varrho(X) \geq k$ whenever $\emptyset \subset X \subseteq V - r$. A fundamental theorem on packing arborescences is due to Edmonds who gave a characterization of the existence of k edge-disjoint spanning arborescences rooted at the same node [6].

Theorem 1.1 (Edmonds' theorem, weak form)**.** *Let $D = (V, A)$ be a digraph with root r. D has k edge-disjoint spanning r-arborescences if and only if D is rooted k-edge-connected.*

This result inspired great many extensions in the last three decades. Edmonds actually proved his theorem in a stronger form where the goal was packing k edge-disjoint branchings of given root-sets. A **branching** is a directed forest in which the in-degree of each node is at most one. The set of nodes of in-degree 0 is called the **root-set** of the branching. Note that a branching with root-set R is the union of $|R|$ node-disjoint arborescences (where an arborescence may consist of a single node and no edge but we always assume that an arborescence has at least one node). For a digraph $D = (V, A)$ and root-set $\emptyset \subset R \subseteq V$ a branching (V, B) is called a **spanning** R**-branching** of D if its root-set is R. In particular, if R is a singleton consisting of an element r, then a spanning branching is a spanning r-arborescence.

Theorem 1.2 (Edmonds' theorem, strong form I.)**.** *In a digraph $D = (V, A)$, let $\mathcal{R} = \{R_1, \ldots, R_k\}$ be a family of k non-empty (not necessarily disjoint or distinct) subsets of V. There are k edge-disjoint spanning branchings of D with root-sets $R_1, \ldots, R_k$, respectively, if and only if*

$$\varrho_D(X) \geq p(X) \text{ for all } \emptyset \subset X \subseteq V \tag{1}$$

where $p(X)$ denotes the number of root-sets R_i disjoint from X.

Observe that in the special case of Theorem 1.2 when each root-set R_i is a singleton consisting of the same node r, we are back at Theorem 1.1. Conversely, when the R_i's are singletons (which may or may not be distinct), then Theorem 1.2 easily follows from Theorem 1.1. Indeed, add a new node r and directed arcs rr_i to the digraph (where $R_i = \{r_i\}$) and simply apply Theorem 1.1. However, for general R_i's no reduction is known.

Theorem 1.2 can be reformulated in terms of extending k arborescences that are partially built up.

Theorem 1.3 (Edmonds' disjoint arborescences theorem, strong form II.)**.** *Let $D = (V, A)$ be a digraph with $r \in V$ and $\mathcal{F} = \{F'_1, \ldots, F'_k\}$ be a family of edge-disjoint –not necessarily spanning– arborescences rooted at r. There are edge-disjoint spanning r-arborescences $F_1, \ldots, F_k$ such that $F'_i \subseteq F_i$ if and only if*

$$\varrho(X) \geq p_{\mathcal{F}}(X) \text{ for all } \emptyset \neq X \subseteq V - r,$$

where $p_{\mathcal{F}}(X) = |\{i : V(F'_i) \cap X = \emptyset\}|$.

It is easy to see that Theorems 1.2 and 1.3 are equivalent. Indeed, Theorem 1.2 follows from Theorem 1.3 by taking $R_i := V(F'_i)$. The other direction can be shown by adding a new node r to the graph and taking arc sets $F'_i := \{rv : v \in R_i\}$ as starting arborescences.

Another possible way of stating Edmonds' strong theorem is as follows.

Theorem 1.4 (Edmonds' theorem, strong form III.)**.** *Let $D = (V, A)$ be a digraph whose node set is partitioned into a root-set $R = \{r_1, \ldots, r_k\}$ (of distinct roots) and a terminal set T. Suppose that no edge of D enters any node of R. There are k disjoint arborescences $F_1, \ldots, F_k$ in D so that F_i is rooted at r_i and spans $T + r_i$ for each $i = 1, \ldots, k$ if and only if $\varrho_D(X) \geq |R - X|$ for every subset $X \subseteq V$ for which $X \cap T \neq \emptyset$.*

This follows from Theorem 1.2 by applying it to the subgraph D' of D induced by T with choice $R_i = \{v :$ there is an edge $r_i v \in A\}$ $(i = 1, \dots, k)$. The same construction shows the reverse implication, too.

The proofs of Theorems 1.1-1.4 are omitted here as in Section 3 we prove an abstract generalization of these results.

2 Direct extensions

The following proper extension of Theorem 1.4 was derived in [2] with the help of a theorem of Frank and Tardos [10] on covering supermodular functions by digraphs.

Theorem 2.1 (Frank and Tardos). *Let $D = (V, A)$ be a digraph whose node set is partitioned into a root-set $R = \{r_1, \dots, r_q\}$ and a terminal set T. Suppose that no edge of D enters any node of R. Let $m : R \to \mathbb{Z}_+$ be a function and let $k = m(R)$. There are k disjoint arborescences in D so that $m(r)$ of them are rooted at r and spanning $T + r$ for each $r \in R$ if and only if*

$$\varrho_D(X) \geq m(R - X) \text{ for every subset } X \subseteq V \text{ for which } X \cap T \neq \emptyset. \tag{2}$$

One way to extend Edmonds' theorems is to decrease the size of the node sets spanned by the arborescences in question. However, it is not easy to find such a generalization as one can easily run into difficult questions. The next theorem shows that even an apparently slight weakening of the reachability conditions result in NP-complete problems.

Theorem 2.2. *Let $D = (V, A)$ be a digraph with $u_1, u_2, v_1, v_2 \in V$ and let $U_1 = V$, $U_2 = V - v_1$. The problem of finding two edge-disjoint arborescences rooted at u_1, u_2 and spanning U_1, U_2, respectively, is NP-complete.*

Proof. Let D' be a digraph with $u_1, u_2, v_1, v_2 \in V$. It is well-known that the problem of finding edge-disjoint $u_1 v_1$ and $u_2 v_2$ paths is NP-complete. We may suppose that the in-degree of v_1 and v_2 is one. Let D denote the graph arising from D' by adding arcs $v_1 v$ and $v_2 v$ to A for each $v \in V$ except for the arc $v_2 v_1$. Clearly, there are edge-disjoint directed $u_1 v_1$ and $u_2 v_2$ paths in D' if and only if there are two arborescences F_1, F_2 in D such that F_i is rooted at u_i and spans U_i. $\square$

In 2009, Kamiyama, Katoh and Takizawa [17] were able to find a surprising new proper extension of Edmonds' strong theorem which implies Theorem 2.1 as well.

Theorem 2.3 (Kamiyama, Katoh and Takizawa). *Let $D = (V, A)$ be a digraph and $R = \{r_1, \dots, r_k\} \subseteq V$ a list of k (possibly not distinct) root-nodes. Let S_i denote the set of nodes reachable from r_i. There are edge-disjoint r_i-arborescences F_i spanning S_i for $i = 1, \dots, k$ if and only if*

$$\varrho_D(Z) \geq p_1(Z) \text{ for every subset } Z \subseteq V \tag{3}$$

where $p_1(Z)$ denotes the number of sets S_i for which $S_i \cap Z \neq \emptyset$ and $r_i \notin Z$.

The original proof of Theorem 2.3 is more complicated than that of Theorem 1.2 due to the fact that the corresponding set function p_1 in the theorem is no more supermodular. Based on Theorem 2.3, Fujishige [11] found a further extension. For two disjoint subsets X and Y of V of a digraph $D = (V, A)$, we say that Y is **reachable** from X if there is a directed path in D whose first node is in X and last node is in Y. We call a subset U of nodes **convex** if there is no node v in $V \setminus U$ so that U is reachable from v and v is reachable from U.

Theorem 2.4 (Fujishige). *Let $D = (V, A)$ be a directed graph and let $R = \{r_1, \dots, r_k\} \subseteq V$ be a list of k (possibly not distinct) root-nodes. Let $U_i \subseteq V$ be convex sets with $r_i \in U_i$. There are edge-disjoint r_i-arborescences F_i spanning U_i for $i = 1, \dots, k$ if and only if*

$$\varrho_D(Z) \geq p_1(Z) \text{ for every subset } Z \subseteq V \tag{4}$$

where $p_1(Z)$ denotes the number of sets U_i's for which $U_i \cap Z \neq \emptyset$ and $r_i \notin Z$.

Note that the set of nodes reachable from an r_i form a convex set, hence Theorem 2.3 immediately follows from Theorem 2.4. It has been showed recently in [18] that these results are in fact equivalent.

3 Abstract extensions

There is another line of results which extends Edmonds' theorems in a different direction. Frank observed that Edmonds' weak theorem can be reformulated in terms of covering intersecting set families and thus gave an abstract extension of Edmonds' results [8]. Given a directed graph $D = (V, A)$, a family $\mathcal{F} \subseteq 2^V$ of subsets of V is called **intersecting** if $X, Y \in \mathcal{F}$ and $X \cap Y \neq \emptyset$ implies $X \cap Y, X \cup Y \in \mathcal{F}$. We say that an arc $a \in A$ **covers** a set $X \in \mathcal{F}$ if a enters X, that is, the tail of a is outside of X while the head of A is inside X. A subset of edges $A' \subseteq A$ **covers** an intersecting family $\mathcal{F}$ if each member of $\mathcal{F}$ is covered by at least one arc from A'.

Theorem 3.1 (Frank)**.** *Let $D = (V, A)$ be a digraph and $\mathcal{F} \subseteq 2^V$ be an intersecting family. Then there are pairwise disjoint arc-sets $A_1, \ldots, A_k$ such that A_i covers $\mathcal{F}$ for $i = 1, \ldots, k$ if and only if*

$$\varrho(X) \geq k \quad \text{for all } X \in \mathcal{F}.$$

By choosing $\mathcal{F} = 2^{V-r} - \emptyset$, we immediately obtain the weak form of Edmonds' disjoint arborescences theorem. However, a weakness of Frank's result is that it does not imply the strong form. This was overcome in [20] by Szegő, who introduced the notion of mixed intersection property. Given a digraph $D = (V, A)$ and k intersecting families $\mathcal{F}_1, \ldots, \mathcal{F}_k \subseteq 2^V$, we say that these families satisfy the **mixed intersection property** if $X \in \mathcal{F}_i, Y \in \mathcal{F}_j, X \cap Y \neq \emptyset$ implies $X \cap Y \in \mathcal{F}_i \cap \mathcal{F}_j$.

Theorem 3.2 (Szegő)**.** *Let $D = (V, A)$ be a digraph and $\mathcal{F}_1, \ldots, \mathcal{F}_k \subseteq 2^V$ be intersecting families satisfying the mixed intersection property. Then there are pairwise disjoint arc sets $A_1, \ldots, A_k \subseteq A$ such that A_i covers $\mathcal{F}_i$ if and only if*

$$\varrho(X) \geq p(X) \quad \text{for all } X \subseteq V, \tag{5}$$

where $p(X)$ denotes the number of $\mathcal{F}_i$'s containing X.

Proof. We will need the following preparatory lemma.

Lemma 3.3. *If $p(X) > 0$, $p(Y) > 0$ and $X \cap Y \neq \emptyset$, then $p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y)$. Moreover, if there is an $\mathcal{F}_i$ for which $X \cap Y \in \mathcal{F}_i$ and $X, Y \notin \mathcal{F}_i$, then strict inequality holds.*

Proof. Consider the contribution of one family $\mathcal{F}_i$ to the two sides of the claimed inequality. If this contribution to the left hand side is two, that is, if both X and Y are in $\mathcal{F}_i$, then so are $X \cap Y$ and $X \cup Y$ and hence the contribution to the right hand side is also two. Suppose now that X belongs to $\mathcal{F}_i$ but Y does not. Since $p(Y) > 0$ is assumed, Y belongs to an $\mathcal{F}_j$. But then $X \cap Y$ belongs to $\mathcal{F}_i$ due to the mixed intersection property, that is, in this case the contribution of $\mathcal{F}_i$ to the right hand side is at least one. An $\mathcal{F}_i$ with the properties in the second part contributes only to the right hand side ensuring this way the strict inequality. □

Condition (5) is clearly necessary. We prove the sufficiency by induction on $\sum_i |\mathcal{F}_i|$. There is nothing to prove if this sum is zero so we may assume that $\mathcal{F}_1$, say, is non-empty. Let U be a maximal member of $\mathcal{F}_1$. Call a set **tight** if $\varrho(X) = p(X) > 0$.

Claim 3.4. *There is an edge e entering U in such a way that each tight set covered by e is in $\mathcal{F}_1$.*

Proof. Suppose indirectly that no such an edge exists. Then each edge e entering U enters some tight set $M \notin \mathcal{F}_1$. By the mixed intersection property, we cannot have $M \subseteq U$. Select a minimal tight set $M \notin \mathcal{F}_1$ which intersects U. Since p is monotone non-increasing, we know that $p(U \cap M) \geq p(M)$. Here, in fact, strict inequality must hold since $U \cap M \in \mathcal{F}_1$ and $M \notin \mathcal{F}_1$. The inequality $p(U \cap M) > p(M)$ implies that D has an edge $f = uv$ for which $u \in M - U$, $v \in U \cap M$. By the indirect assumption, f enters some tight set $Z \notin \mathcal{F}_1$. Lemma 3.3 implies that the intersection of M and Z is tight. Since neither of M and Z is in $\mathcal{F}_1$, the second part of the lemma implies that $M \cap Z$ is not in $\mathcal{F}_1$ either, contradicting the minimal choice of M. □

Let e be an edge ensured by Claim 3.4. Let $\mathcal{F}_1' := \{X \in \mathcal{F}_1 : e \text{ does not enter } X\}$. Then $\mathcal{F}_1'$ is an intersecting family. We claim that the mixed intersection property holds for the families $\mathcal{F}_1', \mathcal{F}_2, \ldots, \mathcal{F}_k$. Indeed, let $X \in \mathcal{F}_1'$ and $Y \in \mathcal{F}_i$ be two intersecting sets for some $i = 2, \ldots, k$. Since $\mathcal{F}_1' \subseteq \mathcal{F}_1$, one has $X \cap Y \in \mathcal{F}_i$. If indirectly $X \cap Y$ is not in $\mathcal{F}_1'$, then e enters $X \cap Y$. Since e enters U and U was selected to be maximal in $\mathcal{F}_1$, it follows that $X \subseteq U$. But then e must enter X as well, contradicting the assumption $X \in \mathcal{F}_1'$.

Let $p'(X)$ denote the number of these families containing X (that is, $p'(X) = p(X) - 1$ if $X \in \mathcal{F}_1$ and e enters X and $p'(X) = p(X)$ otherwise). Let ρ' denote the in-degree function with respect to $D' := D - e$. The choice of e implies $\rho' \geq p'$. By induction, the edge set of D' can be partitioned into k parts $F_1', \ldots, F_k$ in such a way that F_1' covers $\mathcal{F}_1$ and F_i covers $\mathcal{F}_i$ for $i = 2, \ldots, k$. By letting $F_1 := F_1' + e$, we obtain a partition of A requested by the theorem. □

When the k families are identical, we are back at Theorem 3.1. When $\mathcal{F}_i = 2^{V-R_i} - \{\emptyset\}$, we obtain Edmonds' strong theorem (Theorem 1.2). The proof of Szegő is based on the observation that the mixed intersection property implies that p is positively intersecting supermodular and this is why the above approach works.

Although Szegő's theorem provides a common extension of Edmonds' and Frank's results, it does not seem to easily imply the result of Kamiyama et al. The problem with the natural choice $\mathcal{F}_i = 2^{U_i - r_i} - \emptyset$ $(i = 1, \ldots, k)$ is that an arbitrary edge set A_i covering $\mathcal{F}_i$ does not necessarily contain an r_i-arborescence spanning U_i. Indeed, it may happen that a set in $2^{U_i - r_i} - \emptyset$ is covered in A_i by an edge which has a tail outside of U_i and hence can not be added to such an arborescence.

To circumvent this problem, a bi-set counterpart of Szegő's theorem was proved in [2]. Given a digraph $D = (V, A)$, a **bi-set** is a pair $X = (X_I, X_O)$ such that $X_I \subseteq X_O \subseteq V$ where X_I and X_O are called the **inner** and the **outer set** of X, respectively. We will identify a bi-set $X = (X_O, X_I)$ for which $X_O = X_I$ with the simple set X_I and hence the following notation can be also interpreted for sets. The set of all bi-sets on ground-set V is denoted by $\mathcal{P}_2(V) = \mathcal{P}_2$. The **intersection** and **union** of bi-sets can be defined in a straightforward manner: for bi-sets X and Y, we define $X \cap Y = (X_I \cap Y_I, X_O \cap Y_O)$ and $X \cup Y = (X_I \cup Y_I, X_O \cup Y_O)$. An edge $a \in A$ **enters** or **covers** a bi-set X if its head is in X_I and its tail is outside X_O. A subset of edges $A' \subseteq A$ **covers** a bi-set family $\mathcal{F}$ if each member of $\mathcal{F}$ is covered by at least one arc from A'. The set of arcs entering a bi-set X is denoted by $\Delta^{in}(X)$, while the **number of arcs entering** X is denoted by $\rho(X)$. An arc is **contained** in bi-set X if its tail is in X_O and its head is in X_I. We say that $X \subseteq Y$ if $X_I \subseteq Y_I$ and $X_O \subseteq Y_O$. Two bi-sets are **intersecting** if $X_I \cap Y_I \neq \emptyset$. A family $\mathcal{F}$ of bi-sets is called **intersecting** if $X, Y \in \mathcal{F}, X_I \cap Y_I \neq \emptyset$ implies $X \cap Y, X \cup Y \in \mathcal{F}$.

A **bi-set function** is a function $p : \mathcal{P}_2 \to \mathbb{R}$. A bi-set function p is called **fully supermodular** (respectively, **intersecting supermodular**) if

$$p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y)$$

for $X, Y \in \mathcal{P}_2$ (respectively, for intersecting $X, Y \in \mathcal{P}_2$). If the reverse inequality holds, we call p **fully submodular**. A basic example for a submodular bi-set function is the in-degree function ρ. We call p **positively intersecting supermodular** or **positively intersecting submodular** if the corresponding inequality holds whenever X and Y are intersecting and $p(X), p(Y) > 0$.

We say that the bi-set families $\mathcal{F}_1, \ldots, \mathcal{F}_k$ satisfy the **mixed intersection property** if $X \in \mathcal{F}_i, Y \in \mathcal{F}_j, X_I \cap Y_I \neq \emptyset$ implies $X \cap Y \in \mathcal{F}_i \cap \mathcal{F}_j$. The following theorem extends the result of Szegő to bi-set families.

Theorem 3.5 (Bérczi and Frank). *Let $D = (V, A)$ be a digraph and $\mathcal{F}_1, \ldots, \mathcal{F}_k$ be intersecting bi-set families satisfying the mixed intersection property. Then there are pairwise disjoint arc-sets $A_1, \ldots, A_k \subseteq A$ such that A_i covers $\mathcal{F}_i$ if and only if*

$$\rho(X) \geq p_2(X) \quad \textit{for all } X \in \mathcal{P}_2,$$

where $p_2(X)$ denotes the number of $\mathcal{F}_i$'s containing X.

4 Arborescences with arbitrary root nodes

Let $D = (V, A)$ be a digraph. We call a vector $z : V \to \{0, 1, \ldots, k\}$ a **root-vector** if there are k edge-disjoint spanning arborescences in D so that each node v is the root of $z(v)$ arborescences. From Edmonds' theorem one easily gets the following characterization of root-vectors.

Theorem 4.1. *Given a digraph $D' = (V', A')$, a vector z is a root-vector if and only if $z(V') = k$ and $z(X) \geq k - \varrho_{D'}(X)$ for every non-empty subset $X \subseteq V'$.*

Proof. The necessity of both conditions is evident. For the sufficiency, extend D' with a node r and $z(v)$ parallel edges from r to v for each $v \in V'$. In the resulting digraph D the out-degree of r is exactly k and $\varrho_D(X) = z(X) + \varrho_{D'}(X) \geq k$ holds for every non-empty $X \subseteq V'$. By Edmonds' theorem, D contains k edge-disjoint spanning arborescences of root r. Since $\delta_D(r) = k$, each of these arborescences must have exactly one edge leaving r and therefore their restrictions to A' form k arborescences of D' of root-vector z. □

By combining Theorem 4.1 with an earlier result of Frank and Tardos [9], one arrives at the following result appeared in [4,7].

Theorem 4.2 (Cai, Frank)**.** *In a digraph $D = (V, A)$ there exist k edge-disjoint spanning arborescences so that*

1. *each node v is the root of at most $g(v)$ of them if and only if*

$$\sum_{i=1}^{t} \varrho_D(X_i) \geq k(t-1) \tag{6}$$

   holds for every subpartition $\{X_1, \dots, X_t\}$ of V, and

$$g(X) \geq k - \varrho_D(X) \tag{7}$$

   for every $\emptyset \subset X \subseteq V$;

2. *each node v is the root of at least $f(v)$ of them if and only if $f(V) \leq k$ and*

$$\sum_{i=1}^{t} \varrho_D(X_i) \geq k(t-1) + f(X_0) \tag{8}$$

   holds for every partition $\{X_0, X_1, \dots, X_t\}$ of V for which $t \geq 1$ and only X_0 may be empty;

3. *each node v is the root of at least $f(v)$ and at most $g(v)$ of them if and only if the lower bound problem and the upper bound problem have separately solutions.*

Two interesting special cases are as follows.

Corollary 4.3. *A digraph $D = (V, A)$ includes k edge-disjoint spanning arborescences (with no restriction on their roots) if and only if $\sum_{i=1}^{t} \varrho_D(X_i) \geq k(t-1)$ for every subpartition $\{X_1, \dots, X_t\}$ of V.*

Corollary 4.4. *A digraph $D = (V, A)$ includes k edge-disjoint spanning arborescences whose roots are distinct if and only if $|X| \geq k - \varrho_D(X)$ holds for every non-empty subset $X \subseteq V$ set and $\sum_{i=1}^{t} \varrho_D(X_i) \geq k(t-1)$ for every subpartition $\{X_1, \dots, X_t\}$ of V.*

Theorem 4.2 characterized root-vectors satisfying upper and lower bounds. One may be interested in a possible generalization for the framework described in Theorem 2.1. We show that this problem is NP-complete. Indeed, let $D = (V, A)$ be a digraph whose node set is partitioned into a root-set $R = \{r_1, \dots, r_q\}$ and a terminal set T. Suppose that no edge of D enters any node of R.

Theorem 4.5. *The problem of deciding whether there are k disjoint arborescences so that they are rooted at distinct nodes in R and each of them spans T is NP-complete.*

Proof. Let T be a set with even cardinality and let $\mathcal{R} = \{R_1, \dots, R_q\}$ be subsets of T so that $|R_i| \geq 2$ for $i = 1, \dots, q$. It is well-known that the problem of deciding whether T can be covered with k members of $\mathcal{R}$ is NP-complete. Let D_T be a directed graph on T with $\varrho_{D_T}(Z) = k-1$ for each $Z \subseteq T$, $|Z| = 1$ or $|Z| = |T|-1$ and $\varrho_{D_T}(Z) \geq k$ otherwise. Such a graph can be constructed easily as follows. Take the same directed Hamilton cycle on the nodes $k-2$ times, then add the arcs $v_i v_{i+\frac{|T|}{2}}$ to the graph for each $i = 0, \dots, |T|-1$ where $v_0, \dots, v_{|T|-1}$ denote the nodes according to their order around the cycle (the indices are meant modulo $|T|$). The arising digraph satisfies the in-degree conditions.

Extend the graph with $R = \{r_1, \dots, r_q\}$ and with a new arc $r_i v$ for each $v \in R_i$. Let $r_{i_1}, \dots, r_{i_k} \in R$ be a set of distinct root-nodes. Edmonds' disjoint branchings theorem implies that there are edge-disjoint r_i-arborescences F_i spanning $r_i + T$ for $i = i_1, \dots, i_k$ if and only if $\varrho_{D_T}(Z) \geq p(Z)$ for each $\emptyset \subset Z \subseteq T$ where $p(Z)$ denotes the number of R_i's (with $i \in \{i_1, \dots, i_k\}$) disjoint from Z. For a subset Z with $|Z| \geq 2$ the inequality holds automatically because of the structure of D_T and $|R_i| \geq 2$. Hence one only has to care about sets containing a single node and so the existence of the arborescences is equivalent to cover T with $R_{i_1}, \dots, R_{i_k}$.

The observation above means that T can be covered with k members of $\mathcal{R}$ if and only if the digraph includes k arborescences rooted at different nodes in R. $\square$

5 Other notions of connectivity

5.1 Rooted node-connectivity

A natural idea is to reformulate Edmonds' theorem to the node-connected case. Let D and r denote a digraph and a root-node as previously, then D is called **rooted k-node-connected** (or rooted k-connected, for short) if there exist k internally node-disjoint directed paths from r to v for each $v \in V$, that is, any two of the paths have only r and v in common. The **maximum number of node-disjoint $r-v$ paths** is denoted by $\kappa(r,v)$. For an r-arborescence F, a node u is an **F-ancestor** of another node v if there is a directed path from u to v in F. We denote this unique path by $F(u,v)$. For example, the root is the F-ancestor of all other nodes. The **maximum number of edge-disjoint $r-v$ paths** is denoted by $\lambda(r,v)$. We say that a node w **dominates** a node v if every path from r to v includes w. We denote the set of nodes dominating v by $\mathtt{dom}(v)$. Clearly, r and v are in $\mathtt{dom}(v)$.

Note that two r-arborescences F_1 and F_2 are edge-disjoint if and only if for each $v \in V$ the two paths $F_1(r,v)$ and $F_2(r,v)$ are edge-disjoint. That gives the idea of the following definition: we call two spanning r-arborescences F_1 and F_2 **independent** if $F_1(r,v)$ and $F_2(r,v)$ are internally node-disjoint for each $v \in V$.

As a node-disjoint counterpart of Edmonds' theorem, Frank conjectured that in a rooted k-connected graph there exist k independent arborescences (see eg. [19]). The case $k=2$ was verified by Whitty [23], but for $k \geq 3$ the statement does not hold as was shown by Huck [13]. However, Huck also proved that the conjecture is true for simple acyclic graphs [14] and verified it for planar multigraphs except for a few values of k [15].

Theorem 5.1.

1. *(Whitty) Let $D = (V,A)$ be a digraph with root r. D has two independent spanning r-arborescences if and only if D is rooted 2-connected.*

2. *(Huck) Let $D = (V,A)$ be an acyclic digraph with root r such that $D - r$ is simple. D has k independent spanning r-arborescences if and only if D is rooted k-connected.*

3. *(Huck) Let $D = (V,A)$ be a directed multigraph with root r and $k \in \{1,2\} \cup \{6,7,8,\dots\}$ such that D is planar if $k \geq 6$. D has k independent spanning r-arborescences if and only if D is rooted k-connected.*

5.2 Strongly arc-disjoint arborescences

In [5], Colussi, Conforti and Zambelli introduced another type of disjointness concerning arborescences, which put slightly stronger restrictions on the paths than edge-disjointness. In a digraph we call two arcs **symmetric** if they share the same endnodes but have opposite orientations. Two edge-disjoint arborescences F_1, F_2 rooted at r are called **strongly edge-disjoint** if the paths $F_1(r,v), F_2(r,v)$ do not contain a pair of symmetric arcs. In [5], the following strengthening of Edmonds' theorem was proposed.

Conjecture 5.2 (Colussi, Conforti, Zambelli)**.** *Let $D = (V,A)$ be a digraph with root r. D has k strongly edge-disjoint spanning r-arborescences if and only if D is rooted k-edge-connected.*

For $k = 2$, the conjecture was verified in [5]. As Colussi et al. note, the motivation of the problem is the following. It is easy to see that a similar statement holds for strongly edge-disjoint directed $s-t$ paths. Hence the conjecture, if it were true, could be considered as a common generalization of Edmonds' disjoint arborescences theorem and Menger's theorem. Note that the arborescences in the conjecture are allowed to contain pairs of symmetric arcs, only the paths in question are required not to do so.

In what follows, we give a generalization of the case $k = 2$ and show that the conjecture does not hold for $k \geq 3$. As a side result, we get a new proof of a theorem of Georgiadis and Tarjan.

Disjoint Steiner-arborescences

For a digraph $D = (V+r, A)$ with root r and terminal set $T \subseteq V$, an r-arborescence spanning T is called a **Steiner-arborescence**. Two Steiner-arborescences F_1 and F_2 are called **edge-independent** if the paths $F_1(r,t), F_2(r,t)$ are edge-disjoint for every terminal $t \in T$. **Independent** Steiner-arborescences can be defined in a straightforward manner. Note that paths corresponding to non-terminal nodes are allowed to violate the disjointness condition hence the arborescences are not necessarily edge-disjoint.

Z. Király asked whether the existence of k edge-independent Steiner-arborescences is ensured by $\lambda(r,t) \geq k$ for each $t \in T$. As Frank's conjecture on independent arborescences would follow from such a result, Huck's counterexample shows that $k = 2$ is the only case when this statement may hold. The following example shows that even acyclicity is not satisfactory for the existence of edge-independent Steiner-arborescences [16].

Theorem 5.3 (Kovács)**.** *There is an acyclic graph for which there are three internally node-disjoint paths to all of the terminals but there are no three edge-independent Steiner-arborescences.*

Proof. The terminal set of the example consists of two nodes t_1, t_2 (see Figure 1). It can be easily checked that three edge-disjoint paths can be chosen only one way for both terminals but these cannot be partitioned into three arborescences. □

Concerning the case when $k = 2$, the following theorem appeared in [16].

Theorem 5.4 (Kovács)**.** *Let $D = (V+r, A)$ be a digraph with root r, terminal set $T \subseteq V$ and $\lambda(r,t) \geq 2$ for each $t \in T$. Then there exist two edge-independent Steiner-arborescences.*

The node-independent version of the theorem is also of interest. The following result of Georgiadis and Tarjan in [12] is a generalization of Theorem 5.1 (1).

Theorem 5.5 (Georgiadis and Tarjan)**.** *Let $D = (V+r, A)$ be a digraph with root r, terminal set $T \subseteq V$ and $\kappa(r,t) \geq 2$ for each $t \in T$. Then there exists two independent Steiner-arborescences.*

In fact, it can be showed that Theorems 5.4 and 5.5 are equivalent. By following the train of thoughts of both [12] and [16] one can prove the next theorem.

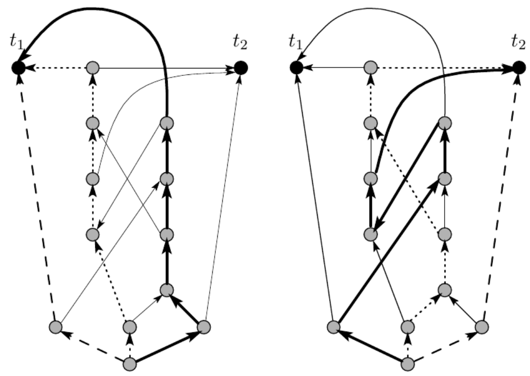


Figure 1: An example without three edge-independent Steiner-arborescences

Theorem 5.6 (Georgiadis and Tarjan, Kovács)**.** *Let $D = (V, A)$ be a digraph with root r. There exist two arborescences F_1 and F_2 such that for each $v \in V - r$, the paths $F_1(r, v)$ and $F_2(r, v)$ intersect only at the nodes of $\mathrm{dom}(v)$.*

This theorem is the base of our proof for a slight generalization of Conjecture 5.2 when $k = 2$.

A generalization

Note that a pair of symmetric arcs can be considered as a directed cycle. This gives the idea of the following definition. Let $D = (V + r, A)$ be a digraph with root r and terminal set $T \subseteq V$. We call two edge-independent Steiner-arborescences F_1 and F_2 **dicycle-disjoint** if for each $t \in T$ the union $F_1(r, t) \cup F_2(r, t)$ does not contain a directed cycle. The motivation of this definition is the following: if $T = V$ and the arborescences are dicycle-disjoint then they are also strongly edge-disjoint.

The following theorem generalizes the theorem of Colussi, Conforti and Zambelli for $k = 2$.

Theorem 5.7. *Let $D = (V, A)$ be a directed graph with root r and terminal set T. There exist two dicycle-disjoint Steiner-arborescences if and only if $\lambda(r, t) \geq 2$ for each $t \in T$.*

Proof. The necessity is clear, we prove sufficiency. Consider the arborescences provided by Theorem 5.6. We claim that these arborescences are dicycle-disjoint.

Assume indirectly that there is a node $t \in T$ such that the union of the paths $F_1(r, t)$ and $F_2(r, t)$ contains a directed cycle. Let $r = x_1, x_2, \ldots, x_p = t$ and $r = y_1, y_2, \ldots, y_q = t$ denote the nodes along these paths. As the union of the paths contains a cycle, there are indices i_1, i_2, j_1, j_2 such that $x_{i_1} = y_{j_2}$, $x_{i_2} = y_{j_1}$ and $i_1 < i_2, j_1 < j_2$. Let $x_{i_1} = y_{j_2} = w$ and $x_{i_2} = y_{j_1} = z$. The choice of F_1 and F_2 implies $w, z \in \mathrm{dom}(t)$. Now consider the graph $G - z$. Then the union $F_1(r, w) \cup F_2(w, t)$ contains a path from r to t, which contradicts to $z \in \mathrm{dom}(t)$. □

Disproof of Conjecture 5.2 for $k \geq 3$

We give a counterexample for $k = 3$ based on a graph given by Huck [13], for other values a similar construction works. Let D be the graph of Figure 2. It is easy to check that D is rooted 3-edge-connected. The set of nodes in $V - r$ is partitioned into three blocks B_1, B_2 and B_3. There is one arc from r to B_i, and there are two arcs from B_i to B_{i+1} for each i (the indices are meant modulo 3 plus 1) such that together they form two directed cycles of length three. The edges of these triangles are denoted by e_{12}, e_{23}, e_{31} and f_{12}, f_{23}, f_{31}, respectively (see Figure 2).

Assume that there exist three strongly edge-disjoint arborescences F_1, F_2 and F_3. Clearly, each F_i contains an edge from r to one of the blocks, say F_i contains the one that goes to B_i, and it uses exactly one of e_{ii+1} and f_{ii+1} and the same holds for e_{i+1i+2} and f_{i+1i+2}. Also, at least one of the arborescences has to use the pair e_{ii+1}, f_{i+1i+2}

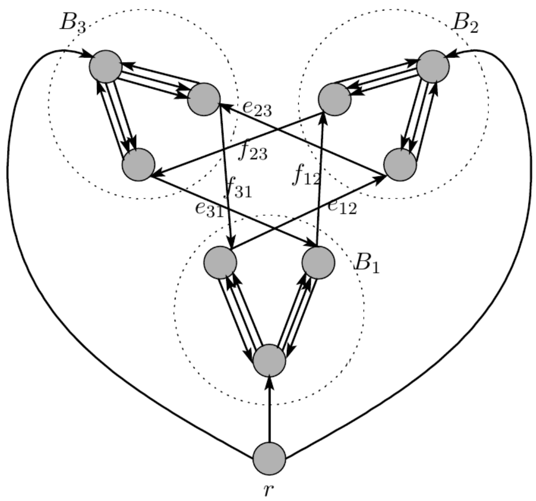


Figure 2: Counterexample for Conjecture 5.2

or f_{ii+1}, e_{i+1i+2}. Assume that F_1 does so. But that implies that F_1 and F_2 can not be strongly edge-disjoint as they have to share a symmetric pair in B_2 that they use when going to B_3, so for any node $v \in B_3$ the paths $F_1(r, v)$ and $F_2(r, v)$ contain a pair of symmetric arcs.

5.3 Further remarks

Edmonds' theorem gives a characterization of the existence of k edge-disjoint arborescences. On the other hand, we have seen that the analogue statement about independent arborescences does not hold. The notion of strongly edge-disjointness somehow lies between these two types of disjointness, but, as we showed, the conditions of Edmonds' theorem do not ensure the existence of such arborescences. So a natural idea is to turn to the other 'extremity' concerning the necessary conditions, and formulate the following conjecture.

Conjecture 5.8. *Let $D = (V + r, A)$ be a digraph with root r and assume that $\kappa(r, v) \geq k$ for each $v \in V$. Then there exist k dicycle-disjoint arborescences.*

6 In- and out-arborescences

Let now $D = (V, A)$ be a digraph without loops, but D may have parallel arcs. We assume that D is weakly connected, that is, the underlying undirected graph is connected (which also implies $|V| - 1 \leq |A|$). For each $a \in A$, we denote by $t(a)$ and $h(a)$ the tail and the head of a, respectively. From now on we distinguish two types of arborescences: **in-** and **out-arborescences**. An r-out-arborescence is just the same as an r-arborescence defined earlier, that is, it is a directed tree in which the edges are directed away from the root node r. An r-in-arborescence is a directed tree in which the edges are directed toward the root node r, so the reversal of its edges results in an out-arborescence.

The problem of finding k arc-disjoint spanning r-out-arborescences for a given root $r \in V$ is very important not only from the theoretical viewpoint but also from practical viewpoints, and it has been extensively studied. It is known that this problem can be solved in polynomial time, and several extensions have been considered. However, in many situations, we have to simultaneously consider not only an in-arborescence but also an out-arborescence. For example, in evacuation situations, an in-arborescence represents roads which refugees use. On the other hand, an out-arborescence represents roads used by emergency vehicles. Unfortunately, it is known [1] that the problem of finding a pair of arc-disjoint spanning r_1-in-arborescence and r_2-out-arborescence for given roots $r_1, r_2 \in V$ is NP-complete even if $r_1 = r_2$. As a special case, it is only known [1] that this problem in a tournament can be solved in polynomial time. In [3], the following results was proved.

Theorem 6.1. *Given a directed acyclic graph $D = (V, A)$ with roots $r_1, r_2 \in V$, we can discern the existence of a pair of arc-disjoint spanning r_1-in-arborescence and r_2-out-arborescence, and find such arborescences if they exist, in $O(|A|)$ time.*

The main idea of the proof of Theorem 6.1 is the definition of an associated bipartite graph, given in the next section.

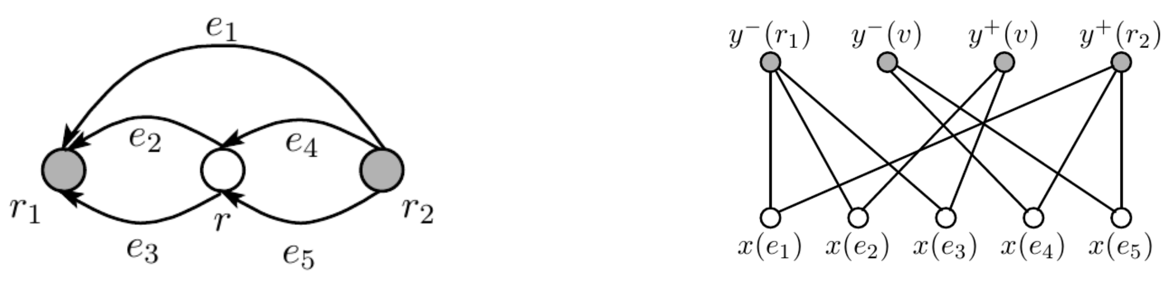
6.1 An associated bipartite graph

We define a bipartite graph $G_D = (X, Y; E)$ associated with our problem for a directed acyclic graph $D = (V, A)$, and we show that our problem in D is equivalent to the problem of finding a matching that covers all nodes of Y in G_D. In the sequel, we assume without loss of generality that $\delta_D(r_1) = 0$ and $\varrho_D(r_2) = 0$ holds. Note that if $\delta_D(r_1) \neq 0$ or $\varrho_D(r_2) \neq 0$ holds, there exists no feasible solution since D is acyclic.

Define a bipartite graph $G_D = (X, Y; E)$ with two node sets X and Y and an edge set E between X and Y as follows.

(i) Node set X is given by $X = \{x(a) \mid a \in A\}$, where $|X| = |A|$.
(ii) Node set Y consists of two disjoint sets Y^+ and Y^- given by $Y^+ = \{y^+(v) \mid v \in V \setminus \{r_1\}\}$ and $Y^- = \{y^-(v) \mid v \in V \setminus \{r_2\}\}$.
(iii) For each $a \in A$, we have two edges in E: one connects $x(a)$ and $y^+(t(a))$ and the other connects $x(a)$ and $y^-(h(a))$. That is, $E = \{(x(a), y^+(t(a))) \mid a \in A\} \cup \{(x(a), y^-(h(a))) \mid a \in A\}$.

For example, for a directed graph D in Figure 3 (a) the bipartite graph G_D becomes the one as illustrated in Figure 3 (b).



(a) (b)

Figure 3: (a) An input directed graph D. (b) The bipartite graph G_D associated with D.

Now we are ready to show the equivalence between our problem for D and the problem of finding a matching in G_D which covers all nodes of Y.

Lemma 6.2. *Given a directed acyclic graph $D = (V, A)$ with roots $r_1, r_2 \in V$, there exists a pair of arc-disjoint spanning r_1-in-arborescence F_1 and r_2-out-arborescence F_2 if and only if there exists a matching M in $G_D = (X, Y; E)$ which covers all nodes of Y. Furthermore, we can construct a pair of such F_1 and F_2 from a matching M in $O(|A|)$ time.*

Proof. Since it is not difficult to see the 'only if' part of the lemma, we show the 'if' part. Let M be a matching in G_D which covers all nodes of Y. Let A^+ (resp. A^-) be the set of arcs $a \in A$ such that $x(a)$ is connected with some node of Y^+ (resp. Y^-) by an edge of M. Let T_1 (resp. T_2) be the subgraph (V, A^+) (resp. (V, A^-)) of D. Since M covers all nodes of Y, $|\delta_{T_1}(v)| = 1$ (resp. $|\varrho_{T_2}(v)| = 1$) holds for each $v \in V \setminus \{r_1\}$ (resp. $V \setminus \{r_2\}$). Thus, since

D is acyclic, T_1 and T_2 are a spanning r_1-in-arborescence and a spanning r_2-out-arborescence, respectively. Furthermore, since M is a matching, A^+ and A^- are disjoint, which implies T_1 and T_2 are arc-disjoint. This completes the proof of the 'if' part.

The latter half of the lemma immediately follows from the proof of the 'if' part. □

By Lemma 6.2, we can discern the existence of a pair of arc-disjoint spanning r_1-in-arborescence and r_2-out-arborescence, and find such arborescences if they exist, by computing a maximum matching of G_D. Hence, we can solve our problem in polynomial time by using bipartite-matching algorithms. However, it can be shown that we can discern the existence of a matching of G_D which covers all nodes of Y and find such a matching if one exists, in $O(|A|)$ time.

6.2 An extension to multiple roots

Now we consider the case where we have multiple roots for in-arborescences and out-arborescences, respectively. Suppose that we are given a directed acyclic graph $D=(V,A)$, two disjoint finite index sets I_1 and I_2, and a root $r_i \in V$ for each $i \in I_1 \cup I_2$, where we allow $r_i = r_j$ for distinct i,j. We assume without loss of generality that $\delta_D(r_i) = 0$ (resp. $\rho_D(r_i) = 0$) holds for each $i \in I_1$ (resp. $i \in I_2$). Let R_1 (resp. R_2) be the set $\{r_i \mid i \in I_1\}$ (resp. $\{r_i \mid i \in I_2\}$). Then we consider the problem of discerning the existence of a set of arc-disjoint r_i-in-arborescences F_i $(i \in I_1)$ and r_i-out-arborescences F_i $(i \in I_2)$ such that for each $i \in I_1$ (resp. $i \in I_2$) the node set of F_i is $(V \setminus R_1) \cup \{r_i\}$ (resp. $(V \setminus R_2) \cup \{r_i\}$).

In the same manner as in Section 6.1, we can see that there exist desired arborescences if and only if there exists a matching which covers all nodes of Y in a bipartite graph $G_D = (X, Y; E)$ defined as follows.

(i') Node set $|X|$ is given by $X = \{x(a) \mid a \in A\}$, where $|X| = |A|$.

(ii') Node set Y consists of disjoint sets Y_i^+ $(i \in I_1)$ and Y_i^- $(i \in I_2)$. For each $i \in I_1$ (resp. $i \in I_2$) , Y_i^+ (resp. Y_i^-) is given by $\{y_i^+(v) \mid v \in V \setminus R_1\}$ (reps., $\{y_i^-(v) \mid v \in V \setminus R_2\}$).

(iii') The edge set E consists of two sets E^+ and E^-. For each $a \in A$ with $h(a) \notin R_1$ (resp. $t(a) \notin R_2$) and $i \in I_1$ (resp. $i \in I_2$), we connect $x(a)$ and $y_i^+(t(a))$ (resp. $y_i^-(h(a))$) by an edge in E^+ (resp. E^-). For each $a \in A$ with $h(a) \in R_1$ (resp. $t(a) \in R_2$), we connect $x(a)$ and $y_i^+(t(a))$ (resp. $y_i^-(h(a))$) for $i \in I_1$ with $h(a) = r_i$ (resp. $i \in I_2$ with $t(a) = r_i$). The edge sets E^+ and E^- contain no other edge.

We can discern the existence of desired arborescences and find them if they exist, by computing a maximum matching in G_D. However, notice that $d_{G_D}(x) \geq 3$ may hold for each $x \in X$, which is different from the case of the problem of finding a pair of an in-arborescence and an out-arborescence. It is left open whether we can find desired arborescences more efficiently than by using existing bipartite matching algorithms.

6.3 Thomassen's conjecture

As we have already mentioned, the problem of finding disjoint in- and out-arborescences for a given root node is NP-complete. The following conjecture was proposed by Thomassen [21]. Recall that a digraph D is k-edge-connected if $\lambda(u,v) \geq k$ for each $u, v \in V$.

Conjecture 6.3 (Thomassen). *There exists a value k so that in every k-edge-connected directed graph $D = (V,A)$ and for every node $v \in V$, there are disjoint spanning in- and out-arborescences rooted at v.*

It is known that Conjecture 6.3 is not true for $k = 2$, but it is still open for $k = 3$. Assume that $D = (V, A')$ is a directed graph and $r \in V$ is a designated root-node for which $D - r$ is acyclic. Then the existence of disjoint spanning in- and out-arborescences rooted at r can be decided easily with a slight modification of the bipartite graph defined in 6.1.

Define a bipartite graph $G=(V^+\cup V^-,A;E)$ where V^+ and V^- are two copies of $V-r$, each node in A corresponds to an arc of D and E contains the edges av^+ and au^- for each $uv=a\in A'$ (if $u,v\neq r$, in other case one of the edges is missing from E). Since $D-r$ is acyclic, a matching covering $V^+\cup V^-$ corresponds to a pair of disjoint spanning in- and out-arborescences, hence Hall's theorem gives a necessary and sufficient condition. However, as each node in A has degree at most 2, it is easy to see that -for example- $\rho(v),\delta(v)\geq 2\ \forall v\in V-r$ ensures the existence of such arborescences in this very special case.

7 Covering by arborescences

When can a digraph $D=(V,A)$ be covered by k spanning arborescences of root r? For any subset X of nodes, let $\Gamma^-(X)=\{v\in X$: there is an edge $uv\in A$ for which $u\in V\setminus X\}$ and call this set the **entrance** of X. That is, the entrance consists of the head nodes of edges entering X. The following result of [22] may be considered as a covering counterpart of Edmonds' disjoint arborescences theorem.

Theorem 7.1 (Vidyasankar)**.** *Let r be a root node of a digraph $D=(V,A)$ so that no edge enters r. It is possible to cover the edge set of D by k r-arborescences if and only if*

$$\rho(v)\leq k \text{ for every } v\in V-r \tag{9}$$

and

$$k-\rho(X)\leq \sum[k-\rho(v): v\in\Gamma^-(X)] \tag{10}$$

for every $\emptyset\subset X\subseteq V-r$, where $\Gamma^-(X)$ is the entrance of X.

Proof. As each arborescence contains one arc entering a node v, (9) is clearly necessary. Assume that there exist k r-arborescences covering A. Let $z(e)$ denote the number of arborescences containing edge e minus 1. Then $z\geq 0$, $\rho_z(X)+\rho(X)\geq k$ for each $\emptyset\neq X\subseteq V-r$ and $\rho_z(v)+\rho(v)=k$ for each $v\in V-r$. As each edge entering X has its head in $\Gamma^-(X)$, we get $\rho_z(X)\leq\sum[\rho_z(v):\ v\in\Gamma^-(X)]$ and so $k-\rho(X)\leq\rho_z(X)\leq\sum[\rho_z(v):\ v\in\Gamma^-(X)]=\sum[k-\rho(v):\ v\in\Gamma^-(X)]$, showing the necessity of (10).

To see sufficiency, we will use Edmonds' weak theorem. For each node $v\in V_r$ add a copy of v to D denoted by v'. Add k parallel arcs from v to v' and $k-\rho(v)$ parallel arcs from v' to v. Moreover, for each arc $uv\in A$ add k parallel arcs from u to v'. Let $D'=(V',A')$ denote the digraph thus obtained.

If there are k edge-disjoint r-arborescences in D' then the corresponding arborescences in D covers A. Otherwise, by Edmonds' weak theorem, there is a set $X'\subseteq V'-r$ such that $\rho_{D'}(X')<k$. Let $X=\{v\in V:\ v\in X'\}$, $Z=\{v\in V:\ v\in X',v'\notin X'\}$. The construction of D' implies that if $v'\in X'$ then $v\in X'$. Also, if uv enters X then $v\in Z$ and so $\Gamma^-(X)\subseteq Z$. Hence we have

$$k>\rho_{D'}(X')=\rho(X)+\sum[k-\rho(v):\ v\in Z]\geq\rho(X)+\sum[k-\rho(v):\ v\in\Gamma^-(X)],$$

contradicting (10). □

One may be interested in a similar covering counterpart of Theorems 2.3 and 2.4 as well. The following theorem shows that such a generalization of Theorem 7.1 is indeed valid.

Theorem 7.2. *Let $D=(V,A)$ be a digraph and $\{r_1,\dots,r_k\}=R\subseteq V$ be a set of (not necessary distinct) root-nodes. Let $U_i\subseteq V$ be convex sets with $r_i\in U_i$. The edge set A can be covered by r_i-arborescences F_i not leaving U_i if and only if*

$$\rho(v)\leq p_1(v) \text{ for each } v\in V \tag{11}$$

and

$$p_1(X)-\rho(X)\leq\sum[p_1(v)-\rho(v):\ v\in\Gamma^-(X)] \tag{12}$$

for every $\emptyset\subset X\subseteq V$, where $\Gamma^-(X)$ denotes the entrance of X and $p_1(X)$ denotes the number of sets U_i's for which $U_i\cap X\neq\emptyset$ and $r_i\notin X$.

The proof goes along the same way as that of Theorem 7.1 and uses Theorem 2.4 when proving sufficiency.

[1] J. Bang-Jensen, Edge-disjoint in-and out-branchings in tournaments and related path problems. Journal of Combinatorial Theory, Series B 51, 1 (1991): 1-23.

[2] K. Bérczi, A. Frank, Variations for Lovász' submodular ideas. Building Bridges. Springer Berlin Heidelberg, 2008. 137-164.

[3] K. Bérczi, N. Kamiyama, S. Fujishige, A linear-time algorithm to find a pair of arc-disjoint spanning in-arborescence and out-arborescence in a directed acyclic graph. Information Processing Letters 109, 23 (2009): 1227-1231.

[4] M. Cai, Arc-disjoint arborescences of digraphs. Journal of Graph Theory 7, 2 (1983): 235-240.

[5] L. Colussi, M. Conforti, G. Zambelli, Disjoint paths in arborescences. Discrete mathematics 292, 1 (2005): 187-191.

[6] J. Emonds, Edge-disjoint branchings. Combinatorial algorithms 9 (1973): 91-96.

[7] A. Frank, Covering branchings. Acta Scientiarum Mathematicarum (Szeged) 41, 77-81 (1979): 3.

[8] A. Frank, Kernel systems of directed graphs. Acta Scientiarum Mathematicarum 41, 1-2 (1979): 63-76.

[9] A. Frank, Generalized polymatroids and submodular flows. Mathematical Programming 42, 1-3 (1988): 489-563.

[10] A. Frank, É. Tardos, An application of submodular flows. Linear algebra and its applications 114 (1989): 329-348.

[11] S. Fujishige, A note on disjoint arborescences. Combinatorica 30, 2 (2010): 247-252.

[12] L. Georgiadis, R.E. Tarjan, Dominator tree verification and vertex-disjoint paths. Proceedings of the sixteenth annual ACM-SIAM symposium on Discrete algorithms. Society for Industrial and Applied Mathematics, 2005.

[13] A. Huck, Disproof of a conjecture about independent branchings in k-connected directed graphs. Journal of Graph Theory 20, 2 (1995): 235-239.

[14] A. Huck, Independent branchings in acyclic diagraphs. Discrete mathematics 199, 1 (1999): 245-249.

[15] A. Huck, Independent trees and branchings in planar multigraphs. Graphs and Combinatorics 15, 2 (1999): 211-220.

[16] E. Kovács, Network coding (in Hungarian). Master thesis, Eötvös Loránd University (2007).

[17] N. Kamiyama, N. Katoh, A. Takizawa, Arc-disjoint in-trees in directed graphs. Combinatorica 29, 2 (2009): 197-214.

[18] Cs. Király, On maximal independent arborescence-packing. 8th Japanese-Hungarian Symposium on Discrete Mathematics and Its Applications June 4-7, 2013, Veszprém, Hungary. 2013.

[19] A. Schrijver, Fractional packing and covering. Packing and Covering in Combinatorics (1979): 201-274.

[20] L. Szegő, On covering intersecting set-systems by digraphs. Discrete Mathematics 234, 1 (2001): 187-189.

[21] C. Thomassen, Configurations in graphs of large minimum degree, connectivity, or chromatic number. Annals of the New York Academy of Sciences 555, 1 (1989): 402-412.

[22] K. Vidyasankar, Covering the edge set of a directed graph with trees. Discrete Mathematics 24, 1 (1978): 79-85.

[23] R.W. Whitty, Vertex-disjoint paths and edge-disjoint branchings in directed graphs. Journal of Graph Theory 11, 3 (1987): 349-358.

1 Strongly regular graphs

Throughout this lecture $G=(V,E)$ denotes a simple, undirected graph (so there are no multiple edges or loops in G) with vertex set V and edge set E. The *complete graph* (where every pair of vertices are connected by an edge) on n vertices is denoted by K_n. A graph is *empty* if it has no edges. If two vertices, u and v are connected by an edge, we call them *adjacent* or *neighbors*, and we may write $u \sim v$. A graph G is called k-*regular*, if every vertex of G has precisely k neighbors.

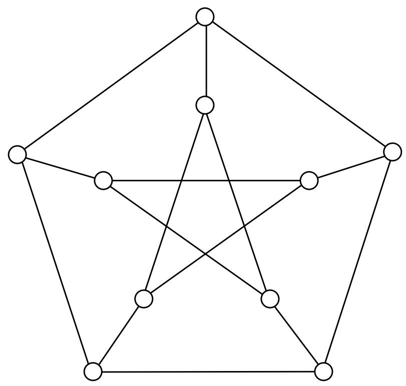


Figure 4: This is the famous Petersen-graph. It is 3-regular on 10 vertices. Moreover, if two vertices are adjacent, then they have no common neighbors; if two vertices are not adjacent, then they have exactly one common neighbor.

Definition 1.1. A graph G is called a *strongly regular graph* with parameters (n,k,λ,μ) (in notation: $\mathrm{SRG}(n,k,\lambda,\mu)$) if the following properties hold:

1. G has n vertices and G is k-regular;
2. if two distinct vertices are adjacent, then they have λ common neighbors;
3. if two distinct vertices are nonadjacent, then they have μ common neighbors;
4. G is not complete, nor empty (that is, $1 \leq k \leq n-2$).

For example, the Petersen-graph is an $\mathrm{SRG}(10,3,0,1)$. Sometimes the 4th point is omitted from the definition. Note that if we did not require this property, the parameters λ and μ would not be well defined; for example, the complete graph K_n would be an $\mathrm{SRG}(n,n-1,n-2,\mu)$ for arbitrary μ, since there are no non-adjacent vertices in K_n.

Exercise 1.2. *Determine which cycles are strongly regular, and determine their parameters.*

Exercise 1.3. *Show that if a k-regular bipartite graph is strongly regular, then either $k=1$ (so the graph consists of independent edges) or it is isomorphic to $K_{k,k}$. ($K_{k,k}$ is the complete bipartite graph on $k+k$ vertices; that is, both vertex classes have k vertices and any two vertices from different classes are adjacent.)*

Exercise 1.4. *Show that the Petersen-graph is the unique* $\mathrm{SRG}(10,3,0,1)$.

Exercise 1.5. *Construct an* $\mathrm{SRG}(16,5,0,2)$ *and show that it is unique. (This graph is called the Clebsch-graph. Hint: the Petersen-graph is a subgraph of it.)*

Exercise 1.6. *Construct an* $\mathrm{SRG}(16,6,2,2)$.

Clearly there are some restrictions on the parameters of a strongly regular graph; for example, one must have $\lambda \leq k-1$ and $\mu \leq k$. Next we establish a connection among the parameters, which shows that any three of them determines the fourth.

Theorem 1.7. *Suppose that an* $\mathrm{SRG}(n,k,\lambda,\mu)$ *exists. Then*

$$k(k-1-\lambda)=(n-1-k)\mu.$$

Proof. Fix a vertex u and count the triplets $\{(u,v,w)\colon uv\in E, vw\in E, uw\notin E, u\neq w\}$ (which we may call cherries). We may choose v in k different ways, and after that there are $k-1-\lambda$ suitable neighbors of v for the choice of w. Thus the number of such triplets is $k(k-1-\lambda)$. On the other hand, if we choose w first ($n-1-k$ possibilities), then we have μ choices for v. □

Recall that the complement of a graph G has the same vertex set as G, and two vertices are adjacent in it if and only if they are not adjacent in G.

Theorem 1.8. *If G is an* $\mathrm{SRG}(n,k,\lambda,\mu)$*, then its complement, denoted by* $\overline{G}$*, is an* $\mathrm{SRG}(n,\bar{k},\bar{\lambda},\bar{\mu})$*, where* $\bar{k}=n-k-1$*,* $\bar{\lambda}=n-2k+\mu-2$*,* $\bar{\mu}=n-2k+\lambda$*.*

Proof. It is clear that $\overline{G}$ is $(n-k-1)$-regular. Let u and v be two adjacent vertices in $\overline{G}$. Then the number of vertices not adjacent to u nor v in G is $n-2k+\mu-2$, which is just the number of common neighbors of u and v in $\overline{G}$. Now suppose that u and v are non-adjacent in $\overline{G}$. Then, similarly, they have $n-2k+\lambda$ common neighbors in $\overline{G}$. □

Note that the above theorem yields further restrictions on the parameters: by $\bar{\lambda}\geq 0$ and $\bar{\mu}\geq 0$ we obtain $\mu\geq 2k-n+2$ and $\lambda\geq 2k-n$. Next we show that disconnected strongly regular graphs are not too interesting.

Theorem 1.9. *Suppose that G is a disconnected strongly regular graph. Then it is the union of some complete graphs of the same size.*

Proof. Let G be an $\mathrm{SRG}(n,k,\lambda,\mu)$ that is disconnected. Take two vertices from two distinct components. Then they cannot have a common neighbor, thus $\mu=0$. Consider a connected component. If there were two vertices in it at distance at least two, then we found easily two vertices at distance exactly two, in contradiction with $\mu=0$. Hence every component is a complete graph, namely K_{k+1}. We remark that $n=c\cdot(k+1)$ for some integer $c\geq 2$, and $\lambda=k-1$. □

Example 1.10. *Consider the graph on $2n$ vertices that consists of n independent edges (that is, the graph is the union of n disjoint K_2-s). This is called the* ladder *graph. Its complement (also strongly regular) is called the* cocktail party *graph.*

By Theorem 1.9, we see that it is enough to treat connected strongly regular graphs whose complement is also connected.

Exercise 1.11. *Consider the two element subsets of $\{1;2;3;4;5\}$ as vertices, and join two of them if and only if they are disjoint. Do you know this graph? (You do.)*

Example 1.12. *The lattice-graph $L(m)$ is defined as follows. Consider an $m\times m$ grid, whose m^2 points are the vertices of $L(m)$, and two vertices are adjacent if and only if they are in the same row or column. Formally, let $V=\{1,2,\dots,m\}\times\{1,2,\dots,m\}$, and (i,j) is adjacent to (i',j') if and only if $i=i'$ or $j=j'$. $L(m)$ is an* $\mathrm{SRG}(m^2,2(m-1),m-2,2)$*.*

As we have seen in the case of the Petersen-graph, sometimes the parameters of a strongly regular graph uniquely determine the graph, but this is not true in general.

Exercise 1.13. *Prove that for every $4\neq m\geq 2$, the only* $\mathrm{SRG}(m^2,2(m-1),m-2,2)$ *is the lattice graph $L(m)$.*

The next exercise shows that Exercise 1.13 does not hold if $m=4$.

Exercise 1.14. *Let V be the vertex set of the lattice graph $L(4)$, and let S be the set of the four diagonal vertices. We define a new graph G (the* Shrikhande-graph*) on the set V. Let u, v be two distinct vertices of V. If $u \notin S$ and $v \notin S$, then uv is an edge in G if and only if uv is an edge in $L(4)$. If $u \in S$ and $v \notin S$, then uv is an edge in G if and only if uv is not an edge in $L(4)$. No two vertices of S are adjacent. Show that G is a strongly regular graph with the same parameter set as $L(4)$, but G is not isomorphic to $L(4)$.*

Example 1.15. *The triangular graph $T(m)$ is defined as follows. Let the vertex set V of $T(m)$ be the set of two-element subsets of $\{1,2,\dots,m\}$, and let two of them be adjacent if their intersection is of size one. Then $T(m)$ is an $\mathrm{SRG}(\frac{m(m-1)}{2}, 2(m-2), m-2, 4)$.*

Note that in Exercise 1.11 we have already encountered the complement of $T(5)$.

Exercise 1.16. *Consider the even element subsets of $\{1,2,3,4,5\}$ (including the empty set) and let two be adjacent if their symmetric difference has four elements. Prove that the arising graph is strongly regular. Do you know this graph?*

Exercise 1.17. *Construct an $\mathrm{SRG}(35,18,9,9)$.*

Exercise 1.18. *Construct an $\mathrm{SRG}(120,56,32,28)$.*

Exercise 1.19. *Let p be a prime such that $p \equiv 1 \pmod 4$. The Paley-graph $P(p)$ is defined in the following way: its vertex set is $\{0,1,\dots,p-1\}$, and two distinct vertices u and v are connected if and only if $u-v$ is a quadratic residue modulo p. (A number n is a quadratic residue modulo p if $n \equiv x^2 \pmod p$ for some integer x.) Prove that $P(p)$ is an $\mathrm{SRG}(p, \frac{p-1}{2}, \frac{p-5}{4}, \frac{p-1}{4})$. (Hint: use automorphisms; consider also $\overline{P(p)}$.)*

Exercise 1.20. *Is it possible to color the edges of K_{10} with three colors so that the edges of each color form a Petersen-graph?*

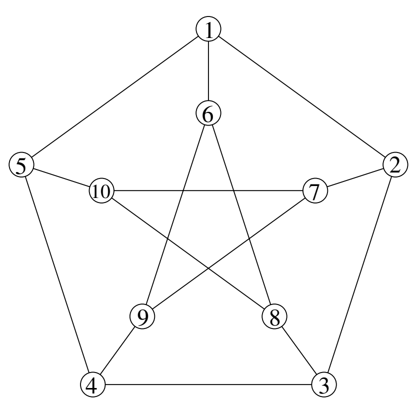
2 Linear algebraic techniques for graphs

Next we associate a matrix to a graph, which allows us to use linear algebraic techniques and results. Throughout **1** denotes the all-one vector (of suitable dimension), I is the identity matrix, J is the all-one matrix.

Definition 2.1. Let $G=(V,E)$ be a graph, and suppose that V has some ordering, $V=\{v_1, v_2, \dots, v_n\}$. The *adjacency matrix* of G is the matrix $A \in \mathbb{R}^{n\times n}$, where $A_{ij}=1$ if v_i and v_j are adjacent, and zero otherwise.

Note that the adjacency matrix of a graph is symmetric, and it has zeros in the diagonal.

Example 2.2.



$$\begin{pmatrix}
0&1&0&0&1&1&0&0&0&0\\
1&0&1&0&0&0&1&0&0&0\\
0&1&0&1&0&0&0&1&0&0\\
0&0&1&0&1&0&0&0&1&0\\
1&0&0&1&0&0&0&0&0&1\\
1&0&0&0&0&0&0&1&1&0\\
0&1&0&0&0&0&0&0&1&1\\
0&0&1&0&0&1&0&0&0&1\\
0&0&0&1&0&1&1&0&0&0\\
0&0&0&0&1&0&1&1&0&0
\end{pmatrix}$$

The Petersen-graph and its adjacency matrix.

A graph is completely described by its adjacency matrix, so information on one of them gives information on the other one. We will examine the adjacency matrix of graphs, in particular the eigenvalues and the eigenvectors of it. First we consider some facts from linear algebra. Recall that the trace of a (square) matrix A is the sum of the entries on its diagonal.

Theorem 2.3. *Let $A \in \mathbb{R}^{n\times n}$ with eigenvalues $\lambda_1,\dots,\lambda_n$. Then $\prod_{i=1}^n \lambda_i = \det(A)$ and $\sum_{i=1}^n \lambda_i = \mathrm{trace}(A)$.*

We remark that the trace of the adjacency matrix of a (loopless) graph is zero.

Theorem 2.4. *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix. Then there is an orthonormal eigenbasis $v_1,\dots,v_n$ of $\mathbb{R}^n$ with respect to A; that is,*

- *$v_1,\dots,v_n$ is a basis of $\mathbb{R}^n$;*
- *$Av_i = \lambda_i v_i$ for some $\lambda_i \in \mathbb{R}$ ($1 \le i \le n$);*
- *$v_i^T v_j = 0$ for all $1 \le i < j \le n$;*
- *$v_i^T v_i = 1$ for all $1 \le i \le n$.*

Note that the above theorem implies that a real symmetric matrix has real eigenvalues.

Definition 2.5. The *spectrum* of a matrix is the multiset of its eigenvalues. The *spectrum of a graph* is that of its adjacency matrix. If the matrix is of dimension $n \times n$, we usually order its eigenvalues as $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$. We may indicate the multiset of eigenvalues as a set in which the elements have an exponent, which refers to the multiplicity of the eigenvalue.

Exercise 2.6. *Show that the spectrum of a graph is the union of the spectra of its connected components.*

One may think of an eigenvector and the corresponding eigenvalue of a graph in the following way. Let A be the adjacency matrix of the graph G on n vertices and let v be an eigenvector of A with eigenvalue λ; that is, $Av = \lambda v$. For any $1 \le i \le n$ the ith coordinate of the left-hand-side is $(Av)_i = \sum_{v_k \in V:\, v_k \sim v_i} v_k$, while the ith coordinate of the right-hand-side is λv_i. So if we write the entries of the eigenvector v on the corresponding vertices of G, and then replace every entry by the sum of the entries on the neighboring vertices (in the same time), then we get the original value multiplied by λ *on all vertices*. For an illustration, see Figure 5.

Theorem 2.7. *A graph is regular if and only if **1** is an eigenvector of its adjacency matrix. The eigenvalue of **1** is the common degree of the vertices.*

Proof. Trivial. □

Theorem 2.8. *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix, and let λ_1 be its largest eigenvalue. Then for all $u \in \mathbb{R}^n$ we have*

$$u^T A u \le \lambda_1 |u|^2.$$

Equality holds if and only if u is an eigenvector of A with eigenvalue λ_1.

Proof. Let $v_1,\dots,v_n$ be an orthonormal eigenbasis as in Theorem 2.4. Then $u = \sum_{i=1}^n \alpha_i v_i$ for some $\alpha_i \in \mathbb{R}$, and $|u|^2 = u^T u = \sum_{i,j} \alpha_i \alpha_j v_i^T v_j = \sum_{i=1}^n \alpha_i^2$. Thus

$$u^T A u = \left(\sum_{i=1}^n \alpha_i v_i^T\right) A \left(\sum_{j=1}^n \alpha_j v_j\right) = \sum_{i=1}^n \alpha_i v_i^T \sum_{j=1}^n \alpha_j A v_j =$$

$$\sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j \lambda_j v_i^T v_j = \sum_{i=1}^n \alpha_i^2 \lambda_i \le \lambda_1 |u|^2.$$

Equality holds if and only if $\lambda_i < \lambda_1$ implies $\alpha_i = 0$, thus u is in the subspace generated by the eigenvectors with eigenvalue λ_1. □

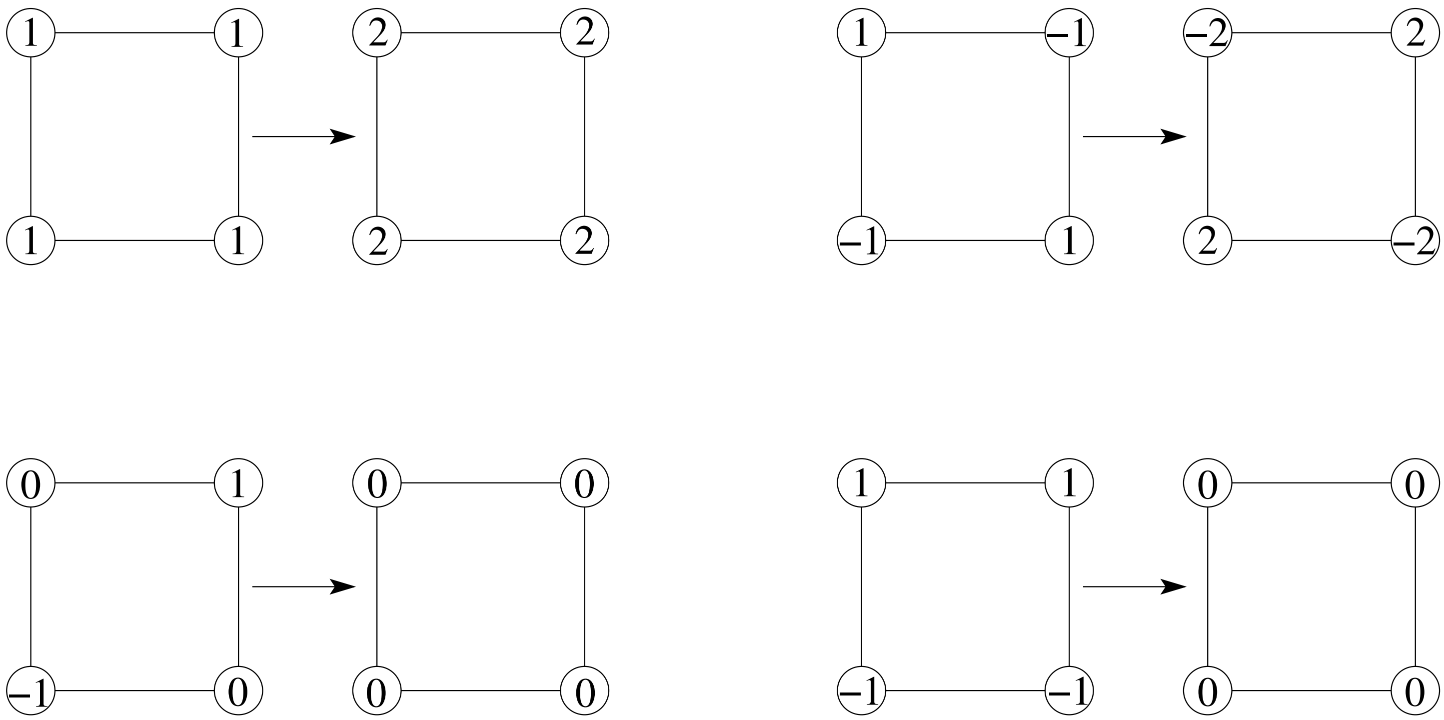


Figure 5: The cycle of length four has spectrum $\{2^1, 0^2, -2^1\}$. On the left part we depicted the eigenvector, on the right part we depicted the result after adding up the entries of the neighbors. Ordering the vertices from the top-left corner clockwise, the four eigenvectors are $(1;1;1;1)$, $(1;-1;1;-1)$, $(0;1;0;-1)$, $(1;1;-1;-1)$.

Theorem 2.9. *Let G be a graph with average degree $\bar{d}$ and maximum degree Δ. Then $\bar{d} \le \lambda_1 \le \Delta$.*

Proof. Let e denote the number of edges in G. Then $\bar{d} = 2e/n$. Suppose that $Av = \lambda v$, $v = (v_1, \ldots, v_n) \neq \mathbf{0}$. We may assume that $v_1 \ge v_2 \ge \cdots \ge v_n$ and $v_1 > 0$ (as $-v$ is also an eigenvector). Then $\lambda v_1 = (Av)_1 \le \Delta v_1$. On the other hand, Theorem 2.8 yields $2e = \mathbf{1}^T A \mathbf{1} \le \lambda_1 n$. $\square$

The next theorem establishes the connection of the structure of the graph and the powers of its adjacency matrix.

Theorem 2.10. *$(A^m)_{ij}$ is the number of walks of length m from v_i to v_j.*

Proof. By induction. The cases $m = 0, 1$ are trivial. (Recall that $A^0 = I$ by definition.) We prove the theorem by induction on m. Now

$$(A^m)_{ij} = (A^{m-1}A)_{ij} = \sum_{k=1}^{m} (A^{m-1})_{ik} A_{kj} = \sum_{k:\, v_k \in N(v_j)} (A^{m-1})_{ik},$$

which (by the inductive hypothesis) is the number of walks of length $m-1$ from v_i to some neighbor of v_j, which is just the number of walks of length m from v_i to v_j. $\square$

Next we give a characterization of bipartite graphs in terms of their spectrum. Note that if v is an eigenvector of A with eigenvalue λ, then v is also an eigenvector of A^k with eigenvalue λ^k (as $A^k v = \lambda A^{k-1} v$ etc.). We say that the spectrum of a matrix A is symmetric, if whenever λ is an eigenvalue of A with multiplicity m, then $-\lambda$ is also an eigenvalue of A with multiplicity m.

Theorem 2.11. *Let G be a graph on n vertices, and let A be its adjacency matrix. Then G is bipartite if and only if the spectrum of A is symmetric.*

Proof. Let G have n vertices. G is bipartite if and only if the number of closed walks of length m in G is zero for all odd integer m, equivalently, if trace$(A^m) = 0$ for all odd integer m. This holds if and only if $s_m := \sum_{i=1}^{n} \lambda_i^m = 0$ for all odd integer m. If the spectrum is symmetric, this holds trivially. Now suppose that, say, $\lambda_1 > -\lambda_n$. Then $\lim_{k\to\infty} s_{2k+1} = \infty$. Thus $\lambda_1 = -\lambda_n$. After that, $\lambda_2 = -\lambda_{n-1}$ also follows etc. □

The following theorem is a consequence of the more general Frobenius–Perron theorem. We only formulate the results for adjacency matrices of graphs.

Theorem 2.12 (Frobenius–Perron)**.** *Let A be the adjacency matrix of a connected, undirected graph G. Then*

- *the largest eigenvalue λ_1 of A has multiplicity one;*
- *the components of an eigenvector of A with eigenvalue λ_1 are either all positive or all negative;*
- *for the smallest eigenvalue λ_n, we have $|\lambda_n| \le \lambda_1$.*

As an illustration, we prove a stronger version Theorem 2.11. Recall that $v^T A v \le \lambda_1 |v|^2$ for all vectors v.

Theorem 2.13. *Let G be a connected graph on n vertices, and let A be its adjacency matrix. Then G is bipartite if and only if $\lambda_1 = -\lambda_n$.*

Proof. As $\lambda_1 = 0$ if and only if G has no edges, we may assume that this is not the case. Suppose that G is bipartite on $n+m$ vertices, where the two classes have n and m vertices, respectively, and let A be its adjacency matrix. Let $v = (v_1, \ldots, v_{n+m})$ be an eigenvector of A with eigenvalue λ. By a proper ordering we may assume that the first n coordinates correspond to the vertices of first vertex class. Let $\bar{v} = (-v_1, \ldots, -v_n, v_{n+1}, \ldots, v_{n+m})$. Then $\bar{v}$ is also an eigenvector of A with eigenvalue $-\lambda$, hence the spectrum of A is symmetric, and, in particular, $\lambda_1 = -\lambda_n$.
Now suppose that $\lambda_1 = -\lambda_n$. Let v be an eigenvector of length $|v| = 1$ with eigenvalue λ_n, and let the vector u be defined by $u_i = |v_i|$ $(1 \le i \le n+m)$. Then also $|u| = 1$. As

$$\lambda_n = v^T A v = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} v_i v_j,$$

we have

$$\lambda_1 = |\lambda_n| = \left| \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} v_i v_j \right| \le \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} |v_i||v_j| = u^T A u \le \lambda_1.$$

It the second estimate equality holds if and only if u is an eigenvector with eigenvalue λ_1 (Theorem 2.8). By the Frobenius–Perron theorem we have that all components of u are positive. As equality holds in the first estimate (triangle-inequality), either $v_i v_j = |v_i||v_j|$ or $v_i v_j = -|v_i||v_j|$ for all pairs i and j such that the corresponding vertices are adjacent. As $\lambda_n < 0$, the second option holds. Thus two vertices may be adjacent only if the corresponding components of v have different signs; that is, the signs of the components of v yield a bipartition. □

3 The spectrum of strongly regular graphs.

Theorem 3.1. *Let G be a graph, and let A be its adjacency matrix. Then the following are equivalent:*

1. *G is an* $\mathrm{SRG}(n, k, \lambda, \mu)$*;*
2. *$A^2 + (\mu - \lambda)A - (k - \mu)I = \mu J$.*

Proof. The entry $(A^2)_{ij}$ is the inner product of the vectors corresponding to v_i and v_j, which is just the number of common neighbors of v_i and v_j. Thus G is an $\mathrm{SRG}(n,k,\lambda,\mu)$ if and only if this quantity is k if $i = j$; λ if v_i and v_j are adjacent; and μ otherwise. In other words, $A^2 = kI + \lambda A + \mu(J - I - A)$, which is equivalent to the formula stated. □

Theorem 3.2. *Let G be an $\mathrm{SRG}(n,k,\lambda,\mu)$, and let A be its adjacency matrix. Then*

1. *the spectrum of A is $\{k^1, r^f, s^g\}$, where $r > s$ ($r = k$ may occur);*
2. *$rs = \mu - k$ and $r + s = \lambda - \mu$;*
3. *$f, g = \frac{1}{2}\left(n - 1 \pm \frac{(n-1)(\mu-\lambda) - 2k}{\sqrt{(\mu-\lambda)^2 + 4(k-\mu)}}\right)$ are non-negative integers.*

Proof. It is clear that $\mathbf{1}$ is an eigenvector with eigenvalue k. Let v be an eigenvector of A with eigenvalue x and $\mathbf{1}^T v = 0$. Then $Jv = 0$. As $A^2 + (\mu - \lambda)A - (k - \mu)I = \mu J$, we obtain

$$x^2 v + (\mu - \lambda)xv - (k - \mu)v = 0,$$

thus $x^2 + (\mu - \lambda)x - (k - \mu) = 0$. Thus

$$x = \frac{\lambda - \mu \pm \sqrt{((\mu-\lambda)^2 + 4(k-\mu))}}{2}.$$

The two roots r and s are different as $(\mu - \lambda)^2 = 4(\mu - k)$ would contradict $\mu \le k$ and $\lambda \le k - 1$. Thus the first two points follow. As $f + g = n - 1$ and $\mathrm{trace}(A) = k + fr + gs = 0$, the last assertion can also be obtained easily. □

The third point of the above theorem is a strong restriction on the parameters of strongly regular graphs, and it is called the integrality or rationality condition.

Exercise 3.3. *Let G be an $\mathrm{SRG}(n,k,\lambda,\mu)$ with three distinct eigenvalues, $k > r > s$. Show that $(k - r)(k - s) = n\mu$.*

Exercise 3.4. *Let G be an $\mathrm{SRG}(n,k,\lambda,\mu)$. Show that either $(n,k,\lambda,\mu) = (4t+1, 2t, t-1, t)$ for some integer t or the eigenvalues of G are integral. (An $\mathrm{SRG}(4t+1, 2t, t-1, t)$ is called a* conference graph.*)*

Exercise 3.5. *Let G be an $\mathrm{SRG}(n,k,\lambda,\mu)$, where $n = p$ is a prime. Show that G is a conference-graph.*

Exercise 3.6. *We are about to show that the edges of K_{10} cannot be partitioned into three Petersen-graphs in terms of their adjacency matrices: the adjacency matrix of K_{10} is $J - I$, and our aim is to show that it cannot be expressed as $A + B + C$, where A, B and C are adjacency matrices of Petersen-graphs.*

- *Show that the eigenvalue 1 of a Petersen-graph has multiplicity five.*
- *Show that the eigensubspaces belonging to the eigenvalue 1 in two edge-disjoint Petersen-graphs intersect nontrivially. (Hint: there is a 9-dimensional subspace containing both.)*
- *Show that if A and B are the adjacency matrices of two edge-disjoint Petersen-graphs, then -3 is an eigenvalue of C, so C is not the adjacency matrix of a Petersen-graph.*

3.1 The Hoffman–Singleton theorem

In the sequel we treat the famous Hoffman–Singleton theorem on strongly regular graphs of girth five, that is, SRGs with $\lambda = 0$ and $\mu = 1$. (The girth of a graph is the length of the shortest cycle in it.) Note that Theorem 1.7 yields $n = k^2 + 1$ for this case. First let us see the background of this theorem.

Exercise 3.7. • *Let G be a graph of diameter two and maxmimal degree at most k. Show that G has at most $k^2 + 1$ vertices, and in case of equality G is an $\mathrm{SRG}(k^2+1, k, 0, 1)$.*

- *Let G be a k-regular graph of girth five. Show that G has at least k^2+1 vertices, and in case of equality G is an* $\mathrm{SRG}(k^2+1,k,0,1)$.

Theorem 3.8 (Hoffman–Singleton)**.** *Let G be an* $\mathrm{SRG}(n,k,0,1)$*. Then $k=2,3,7$ or 57.*

Proof. By the integrality condition we have that

$$\frac{1}{2}\left(n-1\pm\frac{(n-1)(\mu-\lambda)-2k}{\sqrt{(\mu-\lambda)^2+4(k-\mu)}}\right)=\frac{1}{2}\left(k^2\pm\frac{k^2-2k}{\sqrt{4k-3}}\right)$$

are non-negative integers. Then either $k^2-2k=0$, thus $k=2$, or $\sqrt{4k-3}$ is an integer dividing $k(k-2)$. Then $4k-3$ divides $k^2(k-2)^2$, so it also divides $256k^2(k-2)^2-(64k^3-208k^2+100k+75)(4k-3)=225=3^2\cdot 5^2$. As $4k-3$ is a square, $4k-3\in\{9;25;225\}$ follows, which proves the assertion. $\square$

For $k=2$ and 3, the unique $\mathrm{SRG}(k^2+1,k,0,1)$ graphs are the pentagon and the Petersen-graph. For $k=7$ we will show a construction of an $\mathrm{SRG}(50,7,0,1)$, which is called the Hoffman–Singleton-graph. The existence of an $\mathrm{SRG}(3250,57,0,1)$ is still an open question.

3.2 The Hoffman–Singleton-graph

The next construction is due to Robertson. Let P_m be a pentagon, and let Q_x be a pentagram as seen in Figure 6, $0\le m\le 4$, $0\le x\le 4$. Let the vertex labelled b of P_m be denoted by the pair $[m,b]$, and let the vertex labeled y of Q_x be denoted by (x,y). Besides the edges of the pentagons and the pentagrams, add an edge between (x,y) and $[m,b]$ if and only if $y\equiv mx+b \pmod 5$.

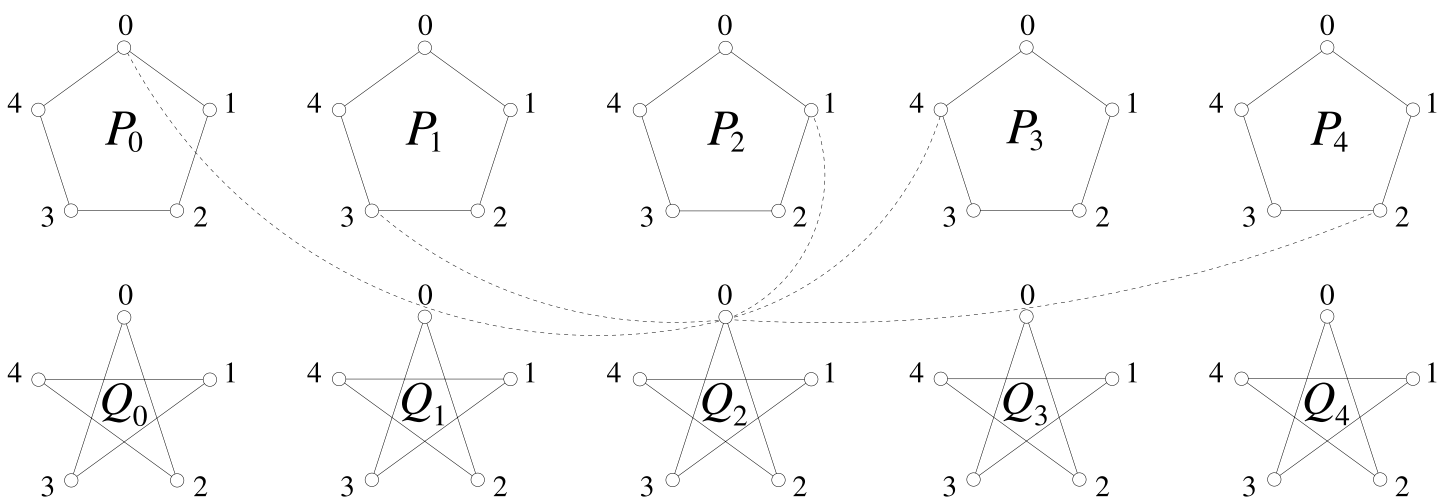


Figure 6: The pentagons and the pentagrams in Robertson's construction for the Hoffman–Singleton-graph.

It is clear that there is precisely one edge between any pentagon and pentagram, so the resulting graph is 7-regular. It is also clear that the graph does not contain any triangle. Suppose that we have a quadrangle. Then its four vertices are of form $[m_1,b_1]$, $[m_2,b_2]$, $m_1\ne m_2$, (x_1,y_1), (x_2,y_2), $x_1\ne x_2$, where

$$y_1 \equiv m_1x_1+b_1 \pmod 5 \tag{13}$$
$$y_2 \equiv m_1x_2+b_1 \pmod 5 \tag{14}$$
$$y_1 \equiv m_2x_1+b_2 \pmod 5 \tag{15}$$
$$y_2 \equiv m_2x_2+b_2 \pmod 5. \tag{16}$$

Then $(13)-(14)-(15)+(16)\equiv 0 \pmod 5$, thus

$$(m_1-m_2)(x_1-x_2)\equiv 0 \pmod 5,$$

a contradiction.

Exercise 3.9. *Let P be* any *subgraph of the Hofmann–Singleton-graph isomorphic to the Petersen-graph. Show that each vertex not in P has exactly one neighbor in P.*

Exercise 3.10. *Let F be a subset of the vertices of the Hoffman–Singleton-graph that span an empty graph. Show that $|F| \leq 15$ and if $|F| = 15$ then each vertex not in F has precisely three neighbors in F.*

Acknowledgement.

Most of the exercises are taken from Péter Csikvári's course on discrete mathematics. Thank you, Péter.

Bibliography

[1] A. E. Brouwer, A. M. Cohen, A. Neumaier: *Distance-Regular Graphs*. Springer-Verlag, New York, 1989, 485 pp., ISBN 3-540-50619-5

[2] A. E. Brouwer, W. H. Haemers: *Spectra of graphs*. Springer-Verlag, New York, 2012, ISBN 978-1-4614-1938-9.

[3] C. Godsil: *Algebraic combinatorics*. Chapman and Hall Mathematics Series. New York: Chapman and Hall, 1993. pp. xvi+362. ISBN 0-412-04131-6.

[4] C. Godsil, G. F. Royle: *Algebraic graph theory*. Springer-Verlag, New York, 2001, 462 pp., ISBN 0-387-95220-9

Throughout these notes $\mathbb{F}$ denotes an arbitrary field, which we later specify, according to the nature of the application, as the real number field $\mathbb{R}$, the complex number field $\mathbb{C}$ or a finite field $\mathbb{F}_p$ for some prime number p, obtained from the ring of integers by modulo p arithmetic. According to Fermat's little theorem, every $0 \neq a \in \mathbb{F}_p$ satisfies $a^{p-1} = 1$. In a more abstract way, it follows from Lagrange's theorem, applied to the multiplicative group of $\mathbb{F}_p$.

1 The Combinatorial Nullstellensatz

The *Combinatorial Nullstellensatz*, formulated by Noga Alon [1] in the late nineties, describes, in an efficient way, the structure of multivariate polynomials whose zero-set includes a Cartesian product over $\mathbb{F}$.

Theorem 1.1. *Let $f = f(x_1,\dots,x_n)$ be a polynomial in $\mathbb{F}[x_1,\dots,x_n]$. Let $S_1,\dots,S_n$ be nonempty subsets of $\mathbb{F}$ and define $g_i(x_i) = \prod_{s\in S_i}(x_i - s)$. If $f(s_1,\dots,s_n) = 0$ for all $s_i \in S_i$, then there exist polynomials $h_1,\dots,h_n \in \mathbb{F}[x_1,\dots,x_n]$ satisfying $\deg(h_i) \le \deg(f) - \deg(g_i)$ such that $f = \sum_{i=1}^n h_i g_i$.*

Example. Here $n = 2$ and for simplicity we denote the variables x_1, x_2 by x and y. Take $\mathbb{F} = \mathbb{R}$ and consider the points $(0,0),(0,1),(1,0),(1,1)$ in the euclidean plane; they form the vertex set of a square. More formally, they are the points of the Cartesian product $S_1 \times S_2$, where $S_1 = S_2 = \{0,1\}$. The polynomial $f(x,y) = (y-x)(y+x-1)$ attains the value 0 at each point $(x,y) \in S_1 \times S_2$. Thus, we define $g_1(x) = x(x-1)$ and $g_2(y) = y(y-1)$. Here each polynomial is of degree 2, and $f = h_1g_1 + h_2g_2$ indeed holds with the constant polynomials $h_1 = 1$, $h_2 = -1$.

Theorem 1.1 immediately implies the following non-vanishing criterion that we will refer to as the *Polynomial Lemma*. Informally, it is a strong multivariate analogue of the well-known fact that a univariate polynomial of degree d over a field cannot have more than d roots.

Theorem 1.2. *Let $f = f(x_1,\dots,x_n)$ be a polynomial in $\mathbb{F}[x_1,\dots,x_n]$. Suppose that there is a monomial $\prod_{i=1}^n x_i^{d_i}$ such that $\sum_{i=1}^n d_i$ equals the degree of f and whose coefficient in f is nonzero. If $S_1,\dots,S_n$ are subsets of F with $|S_i| > d_i$, then there are $s_1 \in S_1,\dots,s_n \in S_n$ such that $f(s_1,\dots,s_n) \neq 0$.*

A standard application of the polynomial method to prove a combinatorial hypothesis works as follows. Assuming the falsity of the hypothesis, build a polynomial whose values are all zero over a large Cartesian product, then compute the coefficient of the appropriate leading term. If that coefficient is not zero, the criterion leads to the desired contradiction. The difficulty often lies in the computation of that coefficient. This is where the power of the following *Coefficient Lemma*, formulated independently by Lasoń [17] and by Karasev and Petrov [12], comes into play.

Theorem 1.3. *Let $f \in \mathbb{F}[x_1,\dots,x_n]$ be a polynomial of degree $\deg(f) \le d_1 + \dots + d_n$. For arbitrary subsets $S_1,\dots,S_n$ of $\mathbb{F}$ with $|S_i| = d_i + 1$, the coefficient of $\prod x_i^{d_i}$ in f can be written as*

$$\sum_{s_1\in S_1}\sum_{s_2\in S_2}\cdots\sum_{s_n\in S_n}\frac{f(s_1,s_2,\dots,s_n)}{g_1'(s_1)g_2'(s_2)\dots g_n'(s_n)},$$

where $g_i(x) = \prod_{s\in S_i}(x-s)$.

A close relative of the Combinatorial Nullstellensatz, this result also implies the Polynomial Lemma: If this coefficient is nonzero, then one of the summands must be nonzero, hence the existence of $s_i \in S_i$ with $f(s_1,\dots,s_n) \neq 0$. Thus we have

$$CN \Rightarrow PL \Leftarrow CL.$$

Both the CN and the CL are relatively easy consequences of the multivariate extension of the Lagrange interpolation formula. Simple as stated, they provide a powerful algebraic tool to attack various problems in discrete mathematics. They lead to proofs of sheer beauty and elegance which we intend to demonstrate through a set of diverse examples.

2 Additive Combinatorics I

Given two nonempty sets of integers $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_l\}$, their sumset is defined as

$$A + B = \{a_i + b_j \mid 1 \le i \le k, 1 \le j \le l\}.$$

Assuming $a_1 < \cdots < a_k$ and $b_1 < \cdots < b_l$ one can argue that

$$a_1 + b_1 < a_2 + b_1 < \ldots < a_k + b_1 < a_k + b_2 < \ldots < a_k + b_l,$$

thus $A + B$ has at least $k + l - 1$ different elements. In the particular case when

$$A = \{1, 2, \ldots, k\}, \quad B = \{1, 2, \ldots, l\},$$

the sumset $A + B = \{2, 3, \ldots, k + l\}$ has exactly $k + l - 1$ elements.

What happens, if instead of integers we consider the integers modulo a prime p? Thus, we are working in the cyclic group $\mathbb{Z}/p\mathbb{Z}$. Such groups, having no proper nontrivial subgroups have the simplest possible structure among all nontrivial groups. They are exactly the additive structures underlying the finite fields $\mathbb{F}_p$. The problem is that working modulo p we lose the natural ordering of the integers, so the above simple argument does not work. There is of course no way to establish the same result, as the simple example $A = B = A + B = \mathbb{Z}/p\mathbb{Z}$ indicates.

Nevertheless the lower bound remains valid if k, l are not too large, namely when $k + l - 1 \le p$. This was first established by Cauchy [5] in 1813 in relation to Lagrange's four-square theorem (every nonnegative integer can be represented as the sum of 4 perfect squares) and rediscovered by Davenport [7] more than 100 years later. Note that $A' \supseteq A$, $B' \supseteq B$ implies $A' + B' \supseteq A + B$, therefore it follows that $A + B = \mathbb{Z}/p\mathbb{Z}$ holds whenever $k + l - 1 \ge p$. The fact that $\mathbb{Z}/p\mathbb{Z}$ is the additive group of the field $\mathbb{F}_p$ opens up the possibility to apply Theorem 1.2 with $\mathbb{F} = \mathbb{F}_p$, which we indicate below; it is probably one of the most straightforward applications of the Polynomial Lemma.

Theorem 2.1. *Let A and B be nonempty subsets of $\mathbb{Z}/p\mathbb{Z}$, with $|A| = k$ and $|B| = l$. If $p \ge k+l-1$, then $|A+B| \ge k+l-1$.*

Proof. Assume for a contradiction that $|A + B| < k + l - 1$. Then $A + B$ is contained in a set $C \subsetneq \mathbb{Z}/p\mathbb{Z}$ of cardinality $|C| = k + l - 2 = (k - 1) + (l - 1)$. We will apply the Polynomial Lemma with $\mathbb{F} = \mathbb{F}_p$, $S_1 = A$, $S_2 = B$. As in the example in Section 1, we denote the variables x_1, x_2 by x and y, respectively. Consider the polynomial

$$f(x, y) = \prod_{c \in C} (x + y - c) \in \mathbb{F}_p[x, y];$$

then $f(a, b) = 0$ for every pair $a \in A$, $b \in B$. The degree of this polynomial is $(k - 1) + (l - 1)$ and it vanishes on $A \times B$, which is a $k \times l$ Cartesian product. But the coefficient of the leading term $x^{k-1}y^{l-1}$ is $\binom{k+l-2}{k-1}$, which is different from 0 in $\mathbb{F}_p$, given that $k + l - 2 < p$. This contradicts the Polynomial Lemma. $\square$

Of course the arguments of Cauchy and Davenport were entirely different, rather combinatorial in nature. You may try to reconstruct their way of thinking, but be aware, it is not that easy!

3 Geometry I

Let $C_n = \{0, 1\}^n$ be the vertex set of the unit cube in euclidean n-space. It is obvious that one can cover all the vertices by two hyperplanes: take for example those two whose equations are $x_1 = 0$ and $x_1 = 1$, respectively. Suppose next that we want to cover all vertices except the origin by a set of m hyperplanes $H_1, \ldots, H_m$. We have the conditions

$$C_n \setminus \{\mathbf{0}\} \subset \bigcup_{i=1}^{m} H_i, \quad \mathbf{0} \notin H_i \text{ for } i = 1, \ldots, m.$$

This is how to do it with $m = n$ hyperplanes: let H_i be the hyperplane whose equation is $x_i = 1$. An entirely different way to do it is letting H_i be the hyperplane of all points satisfying

$$x_1 + x_2 + \cdots + x_n = i.$$

It was conjectured by Komjáth that less than n hyperplanes will never suffice.

Theorem 3.1. *If the hyperplanes $H_1,\ldots,H_m$ satisfy the above conditions, then $m \geq n$.*

Proof. Here we follow a simplified version of Alon and Füredi [3], based on an application of Theorem 1.2 with $\mathbb{F} = \mathbb{R}$. Note that $C_n = S_1 \times S_2 \times \cdots \times S_n$ with $S_i = \{0,1\}$ for $1 \leq i \leq n$. Suppose that on the contrary, $m < n$. Since none of the hyperplanes passes through the origin, they are affine but not linear subspaces, hence are determined by linear equations in the normalized form

$$H_i: \quad \sum_{j=1}^{n} a_{ij} x_j = 1.$$

Consider the polynomial $f \in \mathbb{R}[x_1,\ldots,x_n]$ defined as

$$f(x_1,\ldots,x_n) = \prod_{i=1}^{m}\left(\sum_{j=1}^{n} a_{ij} x_j - 1\right) - (-1)^{m-n}\prod_{j=1}^{n}\left(x_j - 1\right).$$

The first term in the right-hand side vanishes at every point of $C_n \setminus \{\mathbf{0}\}$ because each such point satisfies the equation of at least one of the hyperplanes H_i. Its value at $\mathbf{0}$ is $(-1)^m$. The second term is also 0 at every point of $C_n \setminus \{\mathbf{0}\}$ for all of them have at least one coordinate equal to 1. Its value at 0 is also $(-1)^m$; the coefficient $(-1)^{m-n}$ was designed so as to achieve this. Therefore f, which is the difference of these two terms, vanishes at every point of C_n.

By the assumption $m < n$, the degree of f is n and the coefficient of the leading term $x_1 x_2 \ldots x_n$ is $-(-1)^{m-n}$, which is different from 0. According to the Polynomial Lemma, f cannot vanish on the whole Cartesian product $C_n = S_1 \times \cdots \times S_n$. This contradiction proves that indeed it must be $m \geq n$. □

Suppose that $m = n$ and the hyperplanes $H_1,\ldots,H_n$ do the job. Consider the $n \times n$ matrix $A = \left(a_{ij}\right)$. A closer inspection of the above proof reveals that the permanent of the matrix A must be equal to 1. This is a necessary, but alas!, not a sufficient condition; we do not have a complete description of these extremal structures.

4 Graph Theory

Let $G = G(V,E)$ be a simple graph, where V and E stands for the set of vertices and edges, respectively. It is k-regular, if each vertex has the same degree k:

$$|\{u \in V \mid uv \in E\}| = k$$

for every $v \in V$. A subgraph of G is a graph $G' = G'(V',E')$ with $V' \subseteq V$, $E' \subseteq E$. Note that we are not talking about induced subgraphs here. For example, the 4-regular complete graph K_5 on 5 vertices has a 3-regular subgraph isomorphic to K_4. In fact, every 4-regular simple graph has a 3-regular subgraph (see Tashkinov [19]). With the help of the Polynomial Lemma it is easy to prove a slightly weaker result: If G is obtained from a 4-regular graph by adding an extra edge to the set of edges, then it contains a 3-regular subgraph. It is an immediate consequence of a special case ($p = 3$) of the following more general theorem of Alon, Friedland and Kalai [2].

Theorem 4.1. *If p is a prime and G a simple graph of average degree $> 2p-2$ and maximum degree $\leq 2p-1$, then G contains a p-regular subgraph.*

Proof. The goal is to assign to each edge of G 0 or 1 such a way, that there is at least one edge assigned with 1 (to exclude the empty graph), and for every vertex, the number of edges starting at that vertex and having the value 1 is either 0 or p. In view of the assumption on the maximum degree we can rephrase the second condition: the number of such edges must be divisible by p for every vertex. In such an assignment, the edges assigned with 1 form a p-regular subgraph.

If $G = G(V,E)$, the incidence matrix of G is the 0–1 matrix $(a_{ve})_{v\in V, e\in E}$, where $a_{ve} = 1$ if and only if the vertex v is incident to the edge e. Thus, if we introduce a 0–1 variable x_e for every edge $e \in E$, the second condition can be formulated as

$$\sum_{e\in E} a_{ve}x_e = 0 \text{ in } \mathbb{F}_p. \tag{17}$$

Accordingly, we consider the following polynomial $f \in \mathbb{F}_p[x_e \mid e \in E]$:

$$f(x_e \mid e \in E) = \prod_{v\in V}\left(1 - \left(\sum_{e\in E} a_{ve}x_e\right)^{p-1}\right) - \prod_{e\in E}(1 - x_e).$$

Since the average degree of G is $> 2p-2$, that is, $(2p-2)|V| < 2|E|$, we have $(p-1)|V| < |E|$, thus $\deg f = |E|$. Then $\prod_{e\in E} x_e$ is a leading monomial with coefficient $-(-1)^{|E|} \neq 0$. By the Polynomial Lemma, f cannot vanish on $\{0,1\}^{|E|}$. This means that we have a choice $s_e \in \{0,1\}$ for each x_e such that $f(s_e \mid e \in E) \neq 0$. Were $s_e = 0$ for every $e \in E$, the value of the polynomial would also be 0. So this cannot be the case; at least one edge is assigned with 1. It also follows that $\prod_{e\in E}(1-s_e) = 0$, hence $\prod_{v\in V}\left(1 - (\sum_{e\in E} a_{ve}s_e)^{p-1}\right)$ must be different from 0. In view of Fermat's little theorem this means that condition (17) is satisfied for every vertex v. □

5 Algebraic Combinatorics

A Laurent polynomial is like a polynomial except that the exponents of the indeterminates may also be negative integers. For example,

$$x^2y^3z^4 - 2x^{-1}y^6z^{-4} + 13y^{-5}z - 21$$

is a Laurent polynomial in the variables x, y, z; its constant term, that is, the coefficient of $x^0y^0z^0$ being -21.

For Laurent polynomials given in a product form, the constant terms often have combinatorial interpretation. More surprisingly, their systematic study originates in statistical mechanics. Perhaps the most famous constant term identity is the one associated with the name of Freeman Dyson. In his seminal paper [8] dated back to 1962, Dyson proposed to replace Wigner's classical Gaussian-based random matrix models by what now is known as the circular ensembles. The study of their joint eigenvalue probability density functions led Dyson to the following conjecture. Consider the family of Laurent polynomials

$$\mathcal{D}(\boldsymbol{x};\boldsymbol{a}) := \prod_{1\le i\ne j\le n}\left(1 - \frac{x_i}{x_j}\right)^{a_i}$$

parameterized by a sequence $\boldsymbol{a} = (a_1,\dots,a_n)$ of nonnegative integers, where $\boldsymbol{x} = (x_1,\dots,x_n)$ is the sequence of indeterminates. Denoting by $\mathrm{CT}[\mathcal{L}(\boldsymbol{x})]$ the constant term of the Laurent polynomial $\mathcal{L} = \mathcal{L}(\boldsymbol{x})$, Dyson's hypothesis can be formulated as the identity

$$\mathrm{CT}[\mathcal{D}(\boldsymbol{x};\boldsymbol{a})] = \frac{(a_1 + a_2 + \cdots + a_n)!}{a_1!a_2!\dots a_n!}.$$

Using the shorthand notation $\mathcal{D}(\boldsymbol{x};k)$ for the equal parameter case $\boldsymbol{a} = (k,\dots,k)$, the constant term of $\mathcal{D}(\boldsymbol{x};k)$ for $k = 1,2,4$ corresponds to the normalization factor of the partition function for the circular orthogonal, unitary and symplectic ensemble, respectively.

Dyson's conjecture was confirmed independently by Gunson and Wilson (who 20 years later won the Nobel Prize for his work on the renormalization group) in the same year. The most elegant proof, based on Lagrange interpolation, is due to Good [10]. His proof exploits the recursive nature of the multinomial coefficients.

Theorem 1.3 leads to a short direct proof of the equal parameter case. Note that the $k = 0$ case is trivial.

Theorem 5.1. *For arbitrary integers $n \geq 2$ and $k \geq 1$,*

$$\mathrm{CT}[\mathcal{D}(x_1, \ldots, x_n; k)] = \frac{(nk)!}{(k!)^n}.$$

Proof. Multiplying $\mathcal{D}(\boldsymbol{x}; k)$ by $M = \prod x_i^{(n-1)k}$ one finds that the the constant term equals the coefficient of the monomial M in the homogeneous polynomial

$$\prod_{1 \leq i \neq j \leq n} (x_j - x_i)^k,$$

which is the same as the coefficient of M in

$$f(\boldsymbol{x}) = \prod_{1 \leq i < j \leq n} \left(\prod_{u=0}^{k-1} (x_j - x_i - u) \prod_{v=1}^{k} (x_i - x_j - v) \right).$$

We may apply the Coefficient Lemma with $d_i = (n-1)k$ and the choice

$$S_i = \{0, 1, \ldots, (n-1)k\}$$

for $i = 1, \ldots, n$. Suppose that $f(s_1, \ldots, s_n) \neq 0$ for some $s_i \in S_i$, then $|s_j - s_i| \geq k$ for every $i \neq j$, thus the numbers $s_1, s_2, \ldots, s_n$, in some order, must coincide with the numbers $0, k, 2k, \ldots, (n-1)k$. Moreover, it must be the natural order, for if $s_i > s_j$ for some $i < j$, then $s_i - s_j \geq k + 1$, otherwise $f(s_1, \ldots, s_n)$ would be zero. Accordingly, the complicated summation formula in Theorem 1.3 in this case reduces to one nonzero summand,

$$\frac{f(0, k, \ldots, (n-1)k)}{g'(0) g'(k) \ldots g'((n-1)k)},$$

where $g(x) = x(x-1) \ldots (x-(n-1)k)$. The actual substitution can be left to the reader. $\square$

The above idea is from Karasev and Petrov [12]. Can you reconstruct their proof of the full Dyson conjecture?

6 Additive Combinatorics II

According to a recent result of Preissmann and Mischler [18], seating n couples around the King's round table according to a certain royal protocol is always possible if $p = 2n+1$ is a prime number. The precise mathematical formulation is as follows.

Theorem 6.1. *Let $p = 2n + 1$ be an odd prime and let $t_1, \ldots, t_n$ be arbitrary nonzero elements of $\mathbb{F}_p$. Then the nonzero elements of $\mathbb{F}_p$ can be enumerated as $a_1, \ldots, a_n, b_1, \ldots, b_n$ such that $b_i - a_i = t_i$ holds for every $i = 1, \ldots, n$.*

For example, if $t_1 = \cdots = t_n = 1$, then $a_i = 2i - 1$, $b_i = 2i$ will do. The below argument, based on the Polynomial Lemma, is again due to Karasev and Petrov [12].

Proof. Consider the polynomial

$$f(\boldsymbol{x}) = \prod_{1 \leq i < j \leq n} (x_j - x_i)(x_j + t_j - x_i)(x_i + t_i - x_j)(x_i + t_i - x_j - t_j) \in \mathbb{F}_p(\boldsymbol{x});$$

it is homogeneous of degree $n(2n-2) = n(p-3)$. The coefficient of the monomial $M = \prod x_i^{p-3}$ is the same as in the polynomial

$$\prod_{1\le i\ne j\le n} (x_j - x_i)^2,$$

which, according to Theorem 5.1, is

$$\mathrm{CT}[\mathcal{D}(x_1,\dots,x_n;2)] = \frac{(2n)!}{(2!)^n} = \frac{(p-1)!}{2^n} \ne 0.$$

Consider the sets $S_i = \{1,2,\dots,p-1\} \setminus \{-t_i\}$, then $|S_i| > p-3$. According to the Polynomial Lemma, there are elements $s_i \in S_i$ such that $f(s_1,\dots,s_n) \ne 0$. Put $a_i = s_i$, $b_i = s_i + t_i$; then $b_i - a_i = t_i$ as required. By the choice of s_i, neither a_i nor b_i is zero. Since $f(s_1,\dots,s_n) \ne 0$, the elements $a_1,\dots,a_n,b_1\dots,b_n$ are pairwise different. □

7 Geometry II

As we have seen in Section 3, covering the vertex set C_n of the hypercube with hyperplanes is simple: two hyperplanes suffice in every dimension, whereas it is never possible using only one hyperplane. In general we address the following problem. Recall that C_n is a $2\times 2\times\cdots\times 2$ Cartesian product. Now let $S_i = \{a_{i1},\dots,a_{ik}\}$ be k-element subsets of $\mathbb{R}$ for $i = 1,2,\dots,n$, and consider their Cartesian product $C = S_1\times\cdots\times S_n$ in euclidean n-space. It is easy to cover C with a set of k hyperplanes, all parallel to the same coordinate hyperplane. Indeed, for any $1\le i\le n$ one can take the system of hyperplanes

$$\mathcal{H}_i = \{H_{ij} : \; x_i = a_{ij} \quad (j = 1,2,\dots,k)\}.$$

Less than k hyperplanes will never do, for $|C| = k^n$, and $|H\cap C| \le k^{n-1}$ for any hyperplane H, as one can prove it by induction on the dimension n. (How?)

Now consider a system $\mathcal{H}$ of k hyperplanes which cover C. What can we say about its structure? Is it true that $\mathcal{H}$ necessarily coincides with one of the n axis-parallel systems described above? Well, not exactly: the example in Section 1, which is easy to generalize to arbitrary dimensions, demonstrates that it is not true when $k = 2$, and the situation is even worse for $k = 1$. The Combinatorial Nullstellensatz reveals that there are no counterexamples for larger values of k.

Theorem 7.1. *Let $k\ge 3$ and let $\mathcal{H} = \{H_1,\dots,H_k\}$ be a system of hyperplanes in $\mathbb{R}^n$. If $C \subset H_1\cup\cdots\cup H_k$, then $\mathcal{H} = \mathcal{H}_i$ for some $i \in \{1,2,\dots,n\}$.*

Proof. For simplicity, we prove the result for $n = 2$ and write

$$S_1 = \{a_1,a_2,\dots,a_k\}, \quad S_2 = \{b_1,b_2,\dots,b_k\}.$$

As before, we denote the coordinates by x and y. If H_i is the line of equation $A_ix+B_iy+C_i = 0$, then the polynomial

$$f(x,y) = \prod_{i=1}^{k}(A_ix + B_iy + C_i)$$

vanishes on the $k\times k$ Cartesian product $C = S_1\times S_2$. Applying Theorem 1.1 for this situation, the degree condition implies that the polynomials h_1,h_2 have degree 0. Accordingly, there exist real numbers α,β such that

$$f(x,y) = \alpha\prod_{i=1}^{k}(x-a_i) + \beta\prod_{i=1}^{k}(y-b_i).$$

Comparing the degree k homogeneous parts of the two representations of f we find that

$$\prod_{i=1}^{k}(A_i x + B_i y) = \alpha x^k + \beta y^k.$$

By symmetry, we may assume that $\alpha \neq 0$. As $\alpha = \prod A_i$, it follows that $A_i \neq 0$. Thus, putting $y = 1$ we obtain

$$\prod_{i=1}^{k}\left(x + \frac{B_i}{A_i}\right) = x^k + \frac{\beta}{\alpha}.$$

The polynomial on the left-hand side splits into linear factors over $\mathbb{R}$. On the other hand, if $\beta \neq 0$, then the roots of the polynomial on the right-hand side form the vertex set of a regular k-gon on the plane of the complex numbers, of which at most 2 can lie on the real line. Since $k \geq 3$, the polynomial on the right-hand side splits into linear factors if and only if $\beta = 0$. It follows that $B_1 = \cdots = B_k = 0$, reducing the equation of H_i to $A_i x + C_i = 0$. That is, the lines H_i are all parallel to the y-axis and then it must be $\mathscr{H} = \mathscr{H}_1$. Similarly, the assumption $\beta \neq 0$ leads to $\mathscr{H} = \mathscr{H}_2$. □

From this proof we can extract the hidden algebraic reason: If $\alpha \neq 0$, then the polynomial $x^k + \alpha$ has at most two roots in $\mathbb{R}$. The following example shows that Theorem 7.1 fails if one replaces the real number field by the complex number field.

Example. Let $\varepsilon = e^{2\pi i/3} = \cos 120° + i \sin 120°$. Consider the Cartesian product

$$C = \{0, 1, -\varepsilon\} \times \{0, -1, \varepsilon\} \subset \mathbb{C}^2.$$

One readily checks that it is contained in the union of the following three lines:

$$H_1 : \ x + y = 0, \quad H_2 : \ x + \varepsilon y = -\varepsilon, \quad H_3 : \ x + \varepsilon^2 y = 1.$$

Getting back to $\mathbb{R}^n$ and analyzing the previous proof one finds that the 'hidden algebraic reason' can be generalized to a great extent.

Lemma 7.2. Consider a polynomial $f(x) = c_0 x^d + c_1 x^{d-1} + \cdots + c_{d-1}x + c_d \in \mathbb{R}[x]$ with $c_0 \neq 0$. If $c_{e-2} = c_{e-1} = 0$ and $c_e \neq 0$ for some e, then the polynomial does not split into linear factors over $\mathbb{R}$.

Proof. Assume that on the contrary, f has d real roots (counted with the appropriate multiplicities). If α is a root of multiplicity $m > 1$, then α is also a root of f' with multiplicity $m-1$. Thus, it follows from Rolle's mean value theorem that f' has $d-1$ real roots. Iterating this $d-e$ times, after normalization we obtain a polynomial

$$g(x) = c_0^* x^e + c_1^* x^{e-1} + \cdots + c_{e-1}^* x + c_e^*$$

with $c_{e-2}^* = c_{e-1}^* = 0$ and $c_0^*, c_e^* \neq 0$, which has e nonzero real roots. The reciprocal polynomial

$$c_e^* x^e + c_{e-1}^* x^{e-1} + \cdots + c_1^* x + c_0^*$$

also has e nonzero roots $\alpha_1, \ldots \alpha_e \in \mathbb{R}$. According to Viète's formulas,

$$\sigma_1 = \alpha_1 + \alpha_2 + \cdots + \alpha_e = -\frac{c_{e-1}^*}{c_e^*} = 0$$

and

$$\sigma_2 = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \cdots + \alpha_{e-1}\alpha_e = \frac{c_{e-2}^*}{c_e^*} = 0.$$

Consequently,

$$\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_e^2 = \sigma_1^2 - 2\sigma_2 = 0,$$

implying $\alpha_1 = \alpha_2 = \cdots = \alpha_e = 0$, a contradiction. □

Alternatively, one may apply Descartes' rule of signs to prove the above lemma. This tool makes it possible to prove the following stability result, a very strong version of Theorem 7.1. Note that the condition on m cannot be improved upon.

Theorem 7.3. *Let $m \le 2k-3$ and let $\mathscr{H} = \{H_1,\dots,H_m\}$ be a system of hyperplanes in $\mathbb{R}^n$. If $C \subset H_1 \cup \cdots \cup H_m$, then $\mathscr{H}$ contains a system of k hyperplanes that already covers C. That is, $\mathscr{H} \supseteq \mathscr{H}_i$ for some $i \in \{1,2,\dots,n\}$.*

Proof. We may assume that the system $\mathscr{H}$ is irredundant, that is, no hyperplane H_u occurs twice, and each of them intersects C. Writing the equation of H_u in the form $\sum_{j=1}^n A_{uj}x_j + B_u = 0$, the polynomial

$$f(\boldsymbol{x}) = \prod_{u=1}^m \left(\sum_{j=1}^n A_{uj}x_j + B_u \right)$$

vanishes on $C = S_1 \times \cdots \times S_n$. According to Theorem 1.1 there exist polynomials $h_i \in \mathbb{R}[\boldsymbol{x}]$ with $\deg h_i \le m-k \le k-3$ such that

$$f(\boldsymbol{x}) = \sum_{i=1}^n h_i(\boldsymbol{x}) \prod_{j=1}^k (x_i - a_{ij}).$$

From each h_i, collect the monomial terms of degree $m-k$ and denote the resulting polynomials by $\widetilde{h}_i$; each of them is either homogeneous of degree $m-k$ or identically zero. Thus we have

$$\prod_{u=1}^m \left(\sum_{j=1}^n A_{uj}x_j \right) = \sum_{i=1}^n x_i^k \widetilde{h}_i(\boldsymbol{x}).$$

Without any loss of generality we may assume that $\widetilde{h}_1 \ne 0$. Rewrite the above equation in the form

$$\prod_{u=1}^m \left(\sum_{j=1}^n A_{uj}x_j \right) = \widetilde{h}(\boldsymbol{x}) + \widetilde{g}(\boldsymbol{x}), \tag{18}$$

where

$$\widetilde{h}(\boldsymbol{x}) = x_1^k \widetilde{h}_1(\boldsymbol{x}) = \sum_{i=k}^m \widetilde{g}_i(x_2 \dots, x_n) x_1^i$$

and

$$\widetilde{g}(\boldsymbol{x}) = \sum_{i=2}^n x_i^k \widetilde{h}_i(\boldsymbol{x}) = \sum_{i=0}^{k-3} \widetilde{g}_i(x_2 \dots, x_n) x_1^i$$

with some polynomials $\widetilde{g}_i \in \mathbb{R}[x_2,\dots,x_n]$. By the assumption $\widetilde{h}_1 \ne 0$, there is a largest d with $k \le d \le m$ such that $\widetilde{g}_d \ne 0$. We will argue that $\widetilde{g} = 0$.

Suppose that, on the contrary, there is a smallest e with $d \ge e \ge d-k+3$ such that $\widetilde{g}_{d-e} \ne 0$. Then there exist $s_2,\dots,s_n \in \mathbb{R}$ such that

$$\widetilde{g}_d(s_2,\dots,s_n) \ne 0 \quad \text{and} \quad \widetilde{g}_{d-e}(s_2,\dots,s_n) \ne 0.$$

Writing $g_i(s_2,\dots,s_n) = c_{d-i}$, specializing at $x_j = s_j$ for $2 \le j \le n$, equation (18) reads as

$$\prod_{u=1}^m \left(A_{u1}x_1 + \sum_{j=2}^n A_{uj}s_j \right) = c_0 x_1^d + \cdots + c_{d-k}x_1^k + c_e x_1^{d-e} + \cdots + c_d.$$

In particular, exactly d of the coefficients A_{u1} are different from zero. According to Lemma 7.2, the polynomial on the right-hand side does not split into linear factors over $\mathbb{R}$, whereas the polynomial on the left-hand side obviously does. This contradiction proves that indeed $\widetilde{g} = 0$, as claimed.

All in all, we see from (18) that the nonzero polynomial

$$\prod_{u=1}^{m}\left(\sum_{j=1}^{n} A_{uj}x_j\right)$$

is divisible by x_1^k. By unique factorization in $\mathbb{R}[\boldsymbol{x}]$ it follows that (at least) k of the above factors are in the form $A_{u1}x_1$ with $A_{u1} \neq 0$. The corresponding hyperplanes H_u, having equation in the form $A_{u1}x_1 + B_u = 0$ are all orthogonal to the x_1-axis. By the irredundancy hypothesis, they must form the system $\mathscr{H}_1$. $\square$

The results of this last section are from [4]. Note that it is possible to prove the above theorems by purely elementary arguments. The point is: The application of the Combinatorial Nullstellensatz made it possible to discover these results in the first place.

8 Further Reading

For a proof of the Combinatorial Nullstellensatz as well as for a wealth of applications, see Alon's original paper [1]. To read more about constant term identities, their relevance in statistical physics and their connection to the Selberg integral, see [16] and the references therein; it contains many applications of the Coefficient Lemma, including additive combinatorics. For the latter subject, we also refer to the expository paper [13] and the recent monograph [11]. See [9,14,15] for more delicate applications of Theorem 1.1. The manuscript [6] is probably the most recent advance on the topic of these notes.

[1] N. Alon, Combinatorial Nullstellensatz, *Combin. Prob. Comput.* **8** (1999) 7–29

[2] N. Alon, S. Friedland, G. Kalai, Every 4-regular graph plus an edge contains a 3-regular subgraph, *J. Combin. Th. B* **37** (1984) 92–93

[3] N. Alon, Z. Füredi, Covering the cube by affine hyperplanes, *European J. Combin.* **14** (1993) 79–83

[4] J. Borbély, Gy. Károlyi, Covering Cartesian products by hyperplanes, *manuscript*

[5] A.L. Cauchy, Recherches sur les nombres, *J. École Polytech.* **9** (1813) 99–116

[6] P.L. Clark, A. Forrow, J.R. Schmitt, Warning's second theorem with restricted variables, *arXiv:1404.7793*

[7] H. Davenport, On the addition of residue classes, *J. London Math. Soc.* **10** (1935) 30–32

[8] F.J. Dyson, Statistical theory of energy levels of complex systems. I, *J. Math. Phys.* **3** (1962) 140–156

[9] A. Gács, T. Héger, Z.L. Nagy, D. Pálvölgyi, Permutations, hyperplanes and polynomials over finite fields, *Finite Fields Appl.* **16** (2010) 301–314

[10] I.J. Good, Short proof of a conjecture by Dyson, *J. Math. Phys.* **11** (1970) 1884

[11] D.J. Grynkiewicz, Structural Additive Theory, Springer, 2013

[12] R.N. Karasev, F.V. Petrov, Partitions of nonzero elements of a finite field into pairs, *Israel J. Math.* **192** (2012) 143–156

[13] Gy. Károlyi, A compactness argument in the additive theory and the polynomial method, *Discrete Math.* **302** (2005) 124–144

[14] Gy. Károlyi, An inverse theorem for the restricted set addition in Abelian groups, *J. Algebra* **290** (2005) 557–593

[15] Gy. Károlyi, Restricted set addition: The exceptional case of the Erdős–Heilbronn conjecture, *J. Combin. Th. A* **116** (2009) 741–746

[16] Gy. Károlyi, Z.L. Nagy, F.V. Petrov, V. Volkov, A new approach to constant term identities and Selberg-type integrals, *arXiv:1312.6369*

[17] M. Lasoń, A generalization of Combinatorial Nullstellensatz, *Electron. J. Combin.* **17** (2010) #N32, 6 pages

[18] E. Preissmann, M. Mischler, Seating couples around the King's table and a new characterization of prime numbers, *Amer. Math. Monthly* **116** (2009) 268–272

[19] V.A. Tashkinov, 3-regular subgraphs of 4-regular graphs, *Math. Notes* **36** (1984) 612–623

1 Introduction

In the network localization problem the locations of some nodes (called anchors) of a network as well as the distances between some pairs of nodes are known, and the goal is to determine the location of all nodes. This is one of the fundamental algorithmic problems in the theory of wireless sensor networks, see for example [1].

A natural additional question is whether a solution to the localization problem is unique. The network, with the given locations and distances, is said to be *uniquely localizable* if there is a unique set of locations consistent with the given data. The unique localizability of a two-dimensional network, whose nodes are 'in generic position', can be characterized by using results from graph rigidity theory. In this case unique localizability depends only on the combinatorial properties of the network and can be tested by efficient algorithms.

The goal of this series of lectures is to explore the combinatorial background of this characterization and the corresponding algorithms. After proving some of the classical results of combinatorial rigidity theory and discussing the necessary algorithmic tools, we shall investigate several versions and extension of the network localization problems and their solutions.

1.1 Basic definitions

In what follows we shall summarize the basic concepts and some of the key preliminary results. See the Appendix for more definitions concerning graphs and matroids.

As we shall see, unique localizability (in the 'generic case') is determined completely by the *distance graph* of the network and the set of anchors, or equivalently, by the *grounded graph* of the network and the number of anchors. The vertices of the distance and grounded graph correspond to the nodes of the network. In both graphs two vertices are connected by an edge if the corresponding distance is explicitly known. In the grounded graph we have additional edges: all pairs of vertices corresponding to anchor nodes are adjacent. The grounded graph represents all known distances, since the distance between two anchors can be obtained from their locations. Before stating the basic observation about unique localizability we need some additional terminology. It is convenient to investigate localization problems with distance information by using frameworks, the central objects of rigidity theory.

A d-dimensional *framework* (also called *geometric graph* or *formation*) is a pair (G,p), where $G=(V,E)$ is a graph and p is a map from V to $\mathbb{R}^d$. We consider the framework to be a straight line *realization* of G in $\mathbb{R}^d$. Two frameworks (G,p) and (G,q) are *equivalent* if corresponding edges have the same lengths, that is, if $\|p(u)-p(v)\|=\|q(u)-q(v)\|$ holds for all pairs u,v with $uv\in E$, where $\|.\|$ denotes the Euclidean norm in $\mathbb{R}^d$. Frameworks (G,p), (G,q) are *congruent* if $\|p(u)-p(v)\|=\|q(u)-q(v)\|$ holds for all pairs u,v with $u,v\in V$. This is the same as saying that (G,q) can be obtained from (G,p) by an isometry of $\mathbb{R}^d$. We shall say that (G,p) is *globally rigid*, or that (G,p) is a *unique realization* of G, if every framework which is equivalent to (G,p) is congruent to (G,p), see Figure 7.

The next observation shows that the theory of globally rigid frameworks is the mathematical background which is needed to investigate the unique localizability of networks.

Theorem 1.1. *Let N be a network in $\mathbb{R}^d$ consisting of m anchors located at positions $p_1,\dots,p_m$ and $n-m$ ordinary nodes located at $p_{m+1},\dots,p_n$. Suppose that there are at least $d+1$ anchors in general position. Let G be the grounded graph of N and let $p=(p_1,\dots,p_n)$. Then the network is uniquely localizable if and only if (G,p) is globally rigid.*

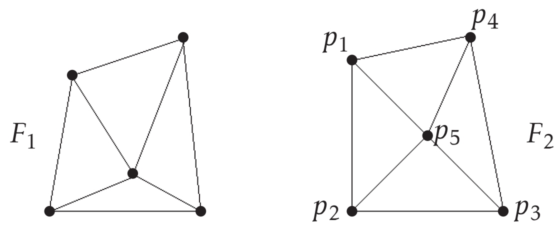


Figure 7: Two realizations of the same graph G in $\mathbb{R}^2$: F_1 is globally rigid; F_2 is not since we can obtain a realization of G which is equivalent but not congruent to F_2 by reflecting p_2 in the line through p_1, p_5, p_3.

1.2 Generic frameworks

It is a hard problem to decide if a given framework is globally rigid. Indeed Saxe [8] has shown that this problem is NP-hard even for 1-dimensional frameworks. The problem becomes more tractable, however, if we assume that there are no algebraic dependencies between the coordinates of the points of the framework.

A framework (G,p) is said to be *generic* if the set containing the coordinates of all its points is algebraically independent over the rationals. (Recall that a set $\{\alpha_1, \alpha_2, \ldots, \alpha_t\}$ of real numbers is *algebraically independent* over the rationals if, for all non-zero polynomials with rational coefficients $p(x_1, x_2, \ldots, x_t)$, we have $p(\alpha_1, \alpha_2, \ldots, \alpha_t) \neq 0$.) Restricting to generic frameworks gives us two important 'stability properties'. The first is that, if (G,p) is a globally rigid d-dimensional generic framework then there exists an $\epsilon > 0$ such that all frameworks (G,q) which satisfy $\|p(v) - q(v)\| < \epsilon$ for all $v \in V$ are also globally rigid. The second, which follows from a recent result of Gortler at al. [4], is that if some d-dimensional generic realization of a graph G is globally rigid, then all d-dimensional generic realizations of G are globally rigid.

2 Rigidity and global rigidity of graphs

Rigidity, which is a weaker property of frameworks than global rigidity, plays an important role in the exploration of the structural results of global rigidity as well as in the corresponding algorithmic problems. Intuitively, we can think of a d-dimensional framework (G,p) as a collection of bars and joints where vertices correspond to joints and each edge to a rigid bar joining its end-points. The framework is rigid if it has no continuous deformations. Equivalently, and more formally, a framework (G,p) is *rigid* if there exists an $\epsilon > 0$ such that, if (G,q) is equivalent to (G,p) and $\|p(u) - q(u)\| < \epsilon$ for all $v \in V$, then (G,q) is congruent to (G,p).

Rigidity, like global rigidity, is a generic property of frameworks, that is, the rigidity of a generic realization of a graph G depends only on the graph G and not the particular realization. We say that the graph G is *rigid*, respectively *globally rigid* or *uniquely realizable*, in $\mathbb{R}^d$ if every (or equivalently, if some) generic realization of G in $\mathbb{R}^d$ is rigid, respectively globally rigid.

The problem of characterizing when a graph is rigid in $\mathbb{R}^d$ has been solved for $d = 1, 2$. We refer the reader to [5, 10, 11] for a detailed survey of the rigidity of d-dimensional frameworks. A similar situation holds for global rigidity: the problem of characterizing when a generic framework is globally rigid in $\mathbb{R}^d$ has also been solved for $d = 1, 2$.

We shall state these characterizations and study their algorithmic implications. Here we only mention a general necessary condition, due to Hendrickson, which is valid in all dimensions. We say that G is *redundantly rigid* in $\mathbb{R}^d$ if $G - e$ is rigid in $\mathbb{R}^d$ for all edges e of G.

Theorem 2.1. *[6] Let (G,p) be a generic framework in $\mathbb{R}^d$. If (G,p) is globally rigid then either G is a complete graph with at most $d+1$ vertices, or G is $(d+1)$-connected and redundantly rigid in $\mathbb{R}^d$.*

3 Rigidity matrices and matroids

A matroid is an abstract structure which extends the notion of linear independence of vectors in a vector space. We will see that many of the rigidity properties of a generic framework (G,p) are determined by an associated matroid defined on the edge set of G. (See the Appendix for the basic definitions and [7,9] for more information on matroids.)

Let (G,p) be a d-dimensional realization of a graph $G=(V,E)$. The *rigidity matrix* of the framework (G,p) is the matrix $R(G,p)$ of size $|E|\times d|V|$, where, for each edge $e=v_iv_j\in E$, in the row corresponding to e, the entries in the two columns corresponding to vertices i and j contain the d coordinates of $(p(v_i)-p(v_j))$ and $(p(v_j)-p(v_i))$, respectively, and the remaining entries are zeros. See [5,10] for more details. The rigidity matrix of (G,p) defines the *rigidity matroid* of (G,p) on the ground set E where a set of edges $F\subseteq E$ is *independent* if and only if the rows of the rigidity matrix indexed by F are linearly independent. Any two generic d-dimensional frameworks (G,p) and (G,q) have the same rigidity matroid. We call this the *d-dimensional rigidity matroid* $\mathcal{R}_d(G)$ of the graph G. We denote the rank of $\mathcal{R}_d(G)$ by $r_d(G)$.

As an example, consider a 1-dimensional framework (G,p). In this case, the rows of $R(G,p)$ are just scalar multiples of a directed incidence matrix of G. It is well known that a set of rows in this matrix is independent if and only if the corresponding edges induce a forest in G. Thus $\mathcal{R}_1(G)$ is the cycle matroid of G.

Gluck characterized rigid graphs in terms of their rank.

Theorem 3.1. *[3] Let $G=(V,E)$ be a graph. Then G is rigid in $\mathbb{R}^d$ if and only if either $|V|\le d+1$ and G is complete, or $|V|\ge d+2$ and $r_d(G)=d|V|-\binom{d+1}{2}$.*

This characterization does not give rise to a polynomial algorithm for deciding whether a graph is rigid in $\mathbb{R}^d$. The problem is that to compute $r_d(G)$ we need to determine the rank of the rigidity matrix of a generic realization of G in $\mathbb{R}^d$. There is no known polynomial algorithm for calculating the rank of a matrix in which the entries are linear functions of algebraically independent numbers.

We say that a graph $G=(V,E)$ is *M-independent* in $\mathbb{R}^d$ if E is independent in $\mathcal{R}_d(G)$. Knowing when subgraphs of G are M-independent allows us to determine the rank of G (and hence determine whether G is rigid), since we can construct a base for $\mathcal{R}_d(G)$ by greedily constructing a maximal independent set of $\mathcal{R}_d(G)$. This follows from axiom (M3) which guarantees that an independent set which is maximal with respect to inclusion is also an independent set of maximum cardinality. For example, when $d=1$, we have seen that a subgraph is independent if and only if it is a forest. Thus we can determine the rank of G by greedily growing a maximal forest F in G. By Theorem 3.1, G is rigid if and only if F has $|V|-1$ edges, i.e. F is a spanning tree of G.

4 Warm up exercises

The following exercises may help warm up for these lectures.

Exercise 4.1. Show that a framework (G,p) is rigid in $\mathbb{R}^1$ if and only if G is connected.

Exercise 4.2. Characterize the redundantly rigid graphs in $\mathbb{R}^1$ and develop an efficient algorithm for testing whether a given graph has this property.

Exercise 4.3. Construct two-dimensional frameworks (G,p) on n vertices for all $n\ge 2$ which are rigid and have $2n-3$ edges. Define a family of graphs which contains a rigid graph in $\mathbb{R}^2$ on n vertices and with $2n-3$ edges for all $n\ge 2$.

Exercise 4.4. Construct two-dimensional frameworks (G,p) on n vertices for all $n\ge 4$ which are rigid and have $2n-4$ edges. Can you do that so that the framework is in generic (or general) position?

Exercise 4.5. Construct globally rigid graphs in $\mathbb{R}^2$ on n vertices for all $n\ge 2$. Try to do it so that the number of edges is as small as possible.

5 Appendix

In what follows we introduce the basic graph (and matroid) theoretical notions. For more details see for example [2].

A *graph* $G = (V, E)$ consists of two sets V and E. The elements of V are called *vertices* (or *nodes*). The elements of E are called *edges*. Each edge $e \in E$ joins two vertices from V, which are called the *endvertices* of e. The notations $V(G)$ and $E(G)$ are also used for the vertex- and edge-sets of a graph G. If vertex v is an endvertex of edge e then v is said to be *incident* with e and e is incident with v. A vertex v is *adjacent* to vertex u if they are joined by an edge. A graph is *simple* if the pairs of endvertices of its edges are pairwise distinct.

The *degree* of a vertex v in a graph G, denoted by $d_G(v)$, is the number of edges incident with v. A graph is *regular* if every vertex is of the same degree. It is k*-regular* if every vertex is of degree k.

A *path* in a graph G from vertex u to vertex v is an alternating sequence of vertices and edges, which starts and ends with u and v (which are its initial and final vertices, respectively), and for which consecutive elements are incident with each other and no internal vertex is repeated. A *cycle* is a path which contains at least one edge and for which the initial vertex is also the final vertex. A graph is *connected* if between every pair of vertices there is a path.

A *subgraph* of a graph G is a graph H with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. In a graph G the *induced subgraph* on a set X of vertices, denoted by $G[X]$, has X as its vertex set and it contains every edge of G whose endvertices are in X. A subgraph H is a *spanning subgraph* if $V(H) = V(G)$. A *component* of a graph G is a maximal connected subgraph. A k*-factor* of a graph G is a k-regular spanning subgraph.

The operation of *deleting a vertex set* $X \subseteq V(G)$ from a graph G removes the vertices in X from $V(G)$ and also removes every edge which has an endvertex in X from $E(G)$. The resulting graph is denoted by $G - X$ (or $G - x$, if $X = \{x\}$ is a single vertex). The operation of *deleting an edge set* $F \subseteq E(G)$ from a graph G removes the edges in F from $E(G)$. The resulting graph is denoted by $G - F$ (or $G - f$, if $F = \{f\}$ is a single edge).

A *forest* is a graph without cycles and a *tree* is a connected forest. A *spanning tree* of a graph G is a spanning subgraph which is a tree.

A graph is a *complete graph* if each pair of its vertices is joined by an edge. A complete graph on n vertices is denoted by K_n. A graph is *bipartite* if its vertices can be partitioned into two sets in such a way that no edge joins two vertices in the same set. A *complete bipartite graph* is a bipartite graph in which each vertex in one partite set is adjacent to all vertices in the other partite set. If the two partite sets have cardinalitites m and n, then this graph is denoted by $K_{m,n}$. A graph G on n vertices is a *wheel*, denoted by W_n, if it has an induced subgraph which is a cycle on $n-1$ vertices and the remaining vertex is joined to all vertices of this cycle.

A k*-vertex-cut* in a graph G is a set $X \subseteq V(G)$ of k vertices for which $G - X$ is not connected. A k*-edge-cut* is a set $F \subseteq E(G)$ of k edges for which $G - F$ is not connected. A graph is called k*-vertex-connected* (or k*-connected*) if it has at least $k+1$ vertices and contains no l-vertex-cut for $l \leq k-1$. A graph is k*-edge-connected* if it contains no l-edge-cuts for $l \leq k-1$.

Two paths are called *openly disjoint* if they have no common internal vertex. They are called *edge disjoint* if they have no common edge. A fundamental theorem of Menger states that if u and v are non-adjacent vertices in graph G then the smallest integer k for which there is a k-vertex-cut X in G such that u and v are in different components of $G - X$ is equal to the maximum number of pairwise openly disjoint paths from u to v. The edge disjoint version of Menger's theorem is as follows. For any pair of vertices u, v in G the smallest integer k for which there is a k-edge-cut F in G such that u and v are in different components of $G - F$ is equal to the maximum number of pairwise edge disjoint paths from u to v.

An *isomorphism* between two graphs G and H is a vertex bijection $\phi : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $\phi(u)\phi(v) \in E(H)$. A *graph automorphism* is an isomorphism of the graph to itself. The *orbit of a vertex* u of a graph G is the set of all vertices $v \in V(G)$ such that there is an automorphism ϕ such that $\phi(u) = v$. A graph is *vertex-transitive* if all the vertices are in the same orbit.

The *incidence matrix* of a graph $G = (V, E)$ is an $|E| \times |V|$ matrix I where the entry in the row of edge e and vertex v is equal to 1 if e is incident with v, and 0 otherwise. The *directed incidence matrix* of G is obtained from I by replacing exactly one of the two 1's in each row of I by -1.

5.1 Matroids

A *matroid* is an ordered pair $\mathcal{M} = (E, \mathcal{I})$ where E is a finite set, and $\mathcal{I}$ is a family of subsets of E, called *independent sets*, which satisfy the following three axioms.

(M1) $\emptyset \in \mathcal{I}$,
(M2) if $I \in \mathcal{I}$ and $D \subseteq I$ then $D \in \mathcal{I}$,
(M3) for all $F \subseteq E$, the maximal independent subsets of F have the same cardinality.

The fundamental example of a matroid is obtained by taking E to be a set of vectors in a vector space and $\mathcal{I}$ to be the family of all linearly independent subsets of E.

Given a matroid $\mathcal{M} = (E, \mathcal{I})$, the cardinality of a maximum independent subset of a set $F \subseteq E$ is defined to be the *rank* of F and denoted by $r(F)$. The rank of E is referred to as the *rank of* $\mathcal{M}$. A *base* of $\mathcal{M}$ is a maximum independent subset of E. A subset of E which is not independent is said to be *dependent*. A *circuit* of $\mathcal{M}$ is a minimal dependent subset of E. The matroid $\mathcal{M}$ is said to be *connected* if every pair of elements of E are contained in a circuit.

Given a graph $G = (V, E)$, we may define a matroid $\mathcal{M} = (E, \mathcal{I})$ by letting $\mathcal{I}$ be the family of all edge sets of forests in G. The rank of a set $F \subseteq E$ is given by $r(F) = |V| - k(F)$, where $k(F)$ denotes the number of connected components in the graph (V, F). A base of $\mathcal{M}$ is the edge set of a forest which has the same number of components as G. A circuit of $\mathcal{M}$ is the edge set of a cycle of G, and $\mathcal{M}$ is connected if and only if G is 2-connected. This matroid is called the *cycle matroid of* G.

[1] J. Aspnes, T. Eren, D.K. Goldenberg, A.S. Morse, W. Whiteley, Y.R. Yang, B.D.O. Anderson, P.N. Belhumeur, A theory of network localization, *IEEE Trans. on Mobile Computing*, vol. 5, issue 12, pp. 1663-1678, 2006.

[2] J.A. Bondy, U.S.R. Murty, *Graph theory*, Springer, 2008.

[3] H. Gluck, Almost all simply connected closed surfaces are rigid, *Geometric topology (Proc. Conf., Park City, Utah, 1974)*, pp. 225–239. Lecture Notes in Math., Vol. 438, Springer, Berlin, 1975.

[4] S.J. Gortler, A.D. Healy, and D.P. Thurston, Characterizing generic global rigidity, 2007, arXiv:0710.0926v3.

[5] J. Graver, B. Servatius, and H. Servatius, Combinatorial Rigidity, AMS Graduate Studies in Mathematics Vol. 2, 1993.

[6] B. Hendrickson, Conditions for unique graph realizations, *SIAM J. Comput.* **21** (1992), no. 1, 65-84.

[7] A. Recski, *Matroid theory and its applications in electric network theory and in statics*, Akadémiai Kiadó, Budapest, 1989.

[8] J.B. Saxe, Embeddability of weighted graphs in k-space is strongly NP-hard, Tech. Report, Computer Science Department, Carnegie-Mellon University, Pittsburgh, PA, 1979.

[9] A. Schrijver, *Combinatorial Optimization*, Springer, Berlin, 2003.

[10] W. Whiteley, Some matroids from discrete applied geometry. Matroid theory (Seattle, WA, 1995), 171–311, Contemp. Math., 197, Amer. Math. Soc., Providence, RI, 1996.

[11] W. Whiteley, Rigidity and scene analysis, in: Handbook of Discrete and Computational Geometry (J. E. Goodman and J. O'Rourke, eds.), CRC Press, Second Edition, pp. 1327-1354, 2004.

These are notes for my 2×90 minutes minicourse. I included all the definitions and theorems, but most of the examples and exercises will be presented at the lectures. A list of detailed reference books and papers for suggested further reading is given at the end of the notes.

1 Introduction

In this section we introduce finite affine and projective planes. Their basic combinatorial properties are presented and some interesting point-sets are studied.

1.1 Affine planes

Let us start with classical geometry what we know from high school. The points and lines of the classical euclidean plane can be described in the following way:

points:	(a,b)	$a,b \in \mathbb{R}$	
lines:	$[c]$, $[m,k]$	$c,m,k \in \mathbb{R}$	
incidence:	$(a,b)\,\mathrm{I}\,[c]$	$\iff$	$a = c,$
	$(a,b)\,\mathrm{I}\,[m,k]$	$\iff$	$b = ma + k.$

If we replace $\mathbb{R}$ by any (commutative) field $\mathbf{K}$, then we get the points and lines of the affine plane AG$(2,\mathbf{K})$:

points:	(a,b)	$a,b \in \mathbf{K}$	
lines:	$[c]$, $[m,k]$	$c,m,k \in \mathbf{K}$	
incidence:	$(a,b)\,\mathrm{I}\,[c]$	$\iff$	$a = c,$
	$(a,b)\,\mathrm{I}\,[m,k]$	$\iff$	$b = ma + k.$

We say, that the line $[m,k]$ has equation $Y = mX + k$, while the line $[c]$ has equation $X = c$. Two lines are said to be parallel, if they do not have any point in common, or if they coincide.

One can prove the following incidence properties by solving sets of linear equations.

- **E1.** For any two distinct points there is a unique line joining them.
- **E2.** For any non-incident point-line pair (P,e) there is a unique line f such that $P\,\mathrm{I}\,f$ and $e \cap f = \emptyset$.

In the case $\mathbf{K} = \mathrm{GF}(q)$, we get $\mathrm{AG}(2,q)$, the finite affine plane of order q. The points of $\mathrm{AG}(2,q)$ are the ordered pairs (a,b), where $a,b \in \mathrm{GF}(q)$, hence there are q^2 points on the plane. The lines of $\mathrm{AG}(2,q)$ are of two types: the non-vertical lines are the ordered pairs $[m,k]$, where $m,k \in \mathrm{GF}(q)$, in this case m is called the slope of the line; and the vertical lines, these are the elements $[c]$, where $c \in \mathrm{GF}(q)$. Hence there are $q^2 + q$ lines on the plane. The point (a,b) is on the line $[m,k]$ if and only if $b = ma + k$ holds in $\mathrm{GF}(q)$, and is on the line $[c]$ if and only if $a = c$.

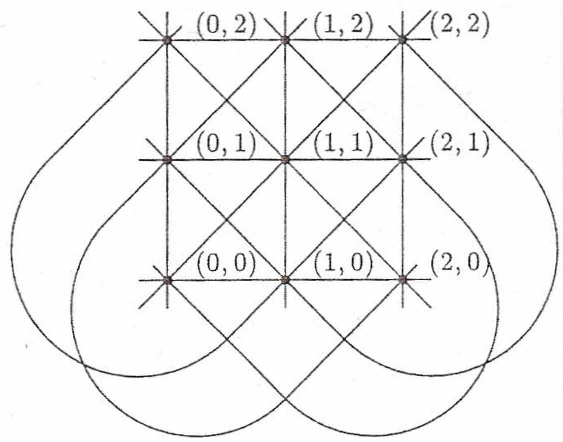


Figure 1, AG(2,3)

Using properties E1 and E2 we define the abstract affine plane.

Definition 1.1. Let $\mathcal{P}$ and $\mathcal{L}$ be two distinct sets, the elements of $\mathcal{P}$ are called points, the elements of $\mathcal{L}$ are called lines. Let $\mathrm{I} \subset (\mathcal{P} \times \mathcal{L}) \cup (\mathcal{L} \times \mathcal{P})$ be a symmetric relation, called incidence. The triple $(\mathcal{P}, \mathcal{L}, \mathrm{I})$ is called an *affine plane* if it satisfies the following axioms.

- **A1.** For any two distinct points there is a unique line joining them.
- **A2.** For any non-incident point-line pair (P, e) there is a unique line f such that $P \mathrm{I} f$ and $e \cap f = \emptyset$.
- **A3.** There exist three non-collinear points.

1.2 Projective planes

The classical projective plane is an extension of the euclidean plane. It contains all points and lines of the euclidean plane and some extra points, called *points at infinity* and an extra line, called the *line at infinity*. The points at infinity correspond to the classes of parallel lines of the euclidean plane. Each line of the euclidean plane is incident with exactly one point at infinity in such a way that two euclidean lines have the same point at infinity if and only if they are parallel, while the line at infinity contains all points at infinity and no euclidean point.

The classical projective plane has simpler incidence properties than the euclidean plane, namely:

- **C1.** For any two distinct points there exists a unique line incident with both of them.
- **C1.** For any two distinct lines there exists a unique point incident with both of them.

Starting from an abstract affine plane, $\mathcal{A}$, in the same way we can construct the projective closure of $\mathcal{A}$. Each projective closure satisfies C1 and C2. We define the abstract projective planes using these properties.

Definition 1.2. Let $\mathcal{P}$ and $\mathcal{L}$ be two distinct sets, the elements of $\mathcal{P}$ are called points, the elements of $\mathcal{L}$ are called lines. Let $\mathrm{I} \subset (\mathcal{P} \times \mathcal{L}) \cup (\mathcal{L} \times \mathcal{P})$ be a symmetric relation, called incidence. The triple $(\mathcal{P}, \mathcal{L}, \mathrm{I})$ is called a *projective plane* if it satisfies the following axioms.

- **P1.** For any two distinct points there is a unique line joining them.
- **P2.** For any two distinct lines there is a unique point of intersection.
- **P3.** Each line is incident with at least three points and each point is incident with at least three lines.

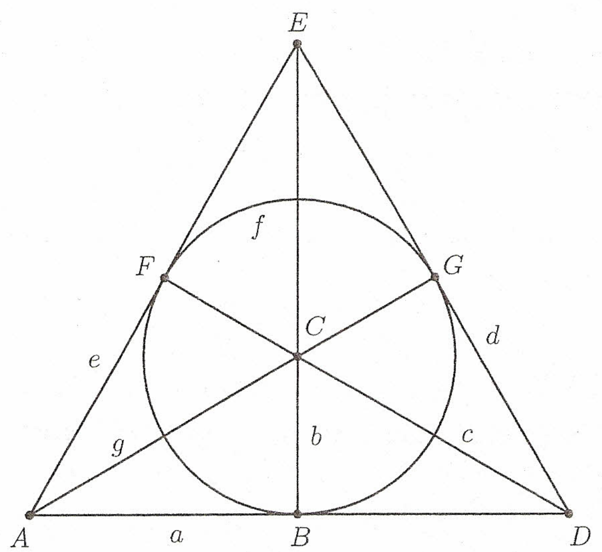


Figure 2, The Fano plane

1.3 Homogeneous coordinates, PG$(2,\mathbf{K})$

The standard example of projective planes is the vector space model. Let V_3 be a 3-dimensional vector space over the field $\mathbf{K}$. The points of the projective plane $\mathrm{PG}(2,\mathbf{K})$ are the 1-dimensional subspaces of V_3, the lines of $\mathrm{PG}(2,\mathbf{K})$ are the 2-dim subspaces of V_3 and the incidence is the set theoretical inclusion.

The relation $\sim$

$$\mathbf{x} \sim \mathbf{y} \Longleftrightarrow \exists\, 0 \neq \lambda \in \mathbf{K} : \mathbf{x} = \lambda \mathbf{y}$$

is an equivalence relation on the elements of V_3. The equivalence class of the vector $\mathbf{v} \in V_3$ is denoted by $[\mathbf{v}]$.

We introduce the *homogeneous coordinates* in the following way.

- Each 1-dimensional subspace can be represented by any of its generating vectors. If a point P is represented by the class of vectors $[\mathbf{v}]$ and $\mathbf{0} \neq \mathbf{v} = (v_0, v_1, v_2)$, then the homogeneous coordinates of P are $(v_0 : v_1 : v_2)$,
- Each 2-dimensional subspace can be represented by any generating vectors of its orthogonal complement. If a line ℓ is represented by the class of vectors $[\mathbf{u}]$ and $\mathbf{0} \neq \mathbf{u} = (u_0, u_1, u_2)$, then the homogeneous coordinates of ℓ are $[u_0 : u_1 : u_2]$.

The incidence is the set theoretical inclusion, hence

$$P \,\mathrm{I}\, \ell \Leftrightarrow \sum_{i=0}^{2} u_i v_i = 0.$$

It is easy to give the collinearity condition of points in PG$(2,\mathbf{K})$. Three distinct points $X = [\mathbf{x}]$, $Y = [\mathbf{y}]$ and $Z = [\mathbf{z}]$ are collinear if and only if their coordinate vectors are linearly dependent.

$$\exists\, \alpha, \beta \in \mathbf{K} : \mathbf{x} = \alpha \mathbf{y} + \beta \mathbf{z}.$$

It happens if and only if

$$\begin{vmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \\ z_0 & z_1 & z_2 \end{vmatrix} = 0.$$

It is easy to check that the projective closure of $\mathrm{AG}(2,\mathbf{K})$ is $\mathrm{PG}(2,\mathbf{K})$. The following table gives the correspondence.

Cartesian coordinates	homogeneous coordinates
(a,b)	$(1:a:b)$
(m)	$(0:1:m)$
(∞)	$(0:0:1)$
$[m,k]$	$[k:m:-1]$
$[c]$	$[c:-1:0]$
$[\infty]$	$[1:0:0]$

1.4 Basic combinatorial properties

Theorem 1.3. *Let Π be a projective plane. If Π has a line which is incident with exatly $n+1$ points, then*

1. *each line is incident with $n+1$ points,*
2. *each point is incident with $n+1$ lines,*
3. *the plane contains n^2+n+1 points,*
4. *the plane contains n^2+n+1 lines.*

The number n is called the *order* of the plane.

Proof. It follows from P1 and P2 that if (P,ℓ) is a non-incident point-line pair, then there is a bijection between the set of points on ℓ and the set of lines through P

$$F_i \, \mathrm{I}\, \ell \Longleftrightarrow PF_i.$$

Hence the number of points on ℓ is the same as the number of lines through P.

Let now ℓ be the line which has exatly $n+1$ points. If A is any point not on ℓ, then the number of lines through A is $n+1$. Let m be an arbitrary line of the plane different from ℓ. Let P be the point $\ell \cap m$. It follows from P3 that there is a third line, say k, through P which is different from both ℓ and m, and there is a point Q on k which is different from P. Q is neither on ℓ, nor on m, hence the number of lines through Q is the same as the number of points on ℓ and the same as the number of points on m. So m also has $n+1$ points, we proved (1).

Next we show (2). Let R be an arbitrary point of the plane. It follows from P3 that there exists a line m which is not incident with R. We have proved that there are $n+1$ points on m, thus R is incident with $n+1$ points.

Now we count the total number of points and lines of the plane. Let H be any point of the plane. By (2) there are $n+1$ lines through H. Since any two points of the plane are joined by a unique line, every point of the plane except H is on exactly one of these $n+1$ lines. By (1) each of these lines contains n points distinct from H. Thus the total number of points is $1+(n+1)n = n^2+n+1$.

Let h be any line of the plane. By (1) there are $n+1$ points on h. Since any two lines of the plane intersect in a unique point, every line of the plane except h is on exactly one of these $n+1$ points. By (2) each of these points is incident with n lines distinct from h. Thus the total number of lines is $1+(n+1)n = n^2+n+1$. □

The last two paragraphs of the previous proof illustrate an interesting and important property of the projective planes, the *Principle of Duality.* Let $\mathcal{T}$ be any theorem about the incidence of points and lines of projective planes. If $\mathcal{T}^*$ is the statement obtained by interchanging the words "point" and "line", then $\mathcal{T}^*$ is also a theorem about projective planes. This follows from the fact that the axioms P1 and P2 are duals of each other, while the axiom P3 is self–dual.

In general the existence problem of projective planes with a given order is an unsolved problem. (And it seems to be hopeless to solve.) If q is a power of a prime then the vector space model guarantees the existence of a projective plane of order q. There are some more partial results. The first number which is not a prime power is six, and we know that there is no projective plane of order six [2]. It was proved in 1989 that there is no projective plane of order 10 [13]. The next number which is not a prime power is 12, and we do not know whether a projective plane of 12 exists or does not exist.

For $n \leq 8$ the projective planes of order n are unique up to isomorphism. It is not too difficult to prove it for $n \leq 5$. The cases $n = 7$ and $n = 8$ are much more complicated, the proofs can be found in [14] and [5]. For $n = 9$ there are four non-isomorphic planes, their constructions can be found in [9].

1.5 Arcs, ovals and hyperovals

Definition 1.4. A k-*arc* is a set of k points no three of them are collinear.

A k-arc is *complete* if it is not contained in any $(k+1)$-arc.

Definition 1.5. Let $\mathcal{K}$ be a k-arc and ℓ be a line. Then ℓ is called

– a *secant* to $\mathcal{K}$ if $|\mathcal{K}\cap\ell| = 2$,

– a *tangent* to $\mathcal{K}$ if $|\mathcal{K}\cap\ell| = 1$,

– an *external* line to $\mathcal{K}$ if $|\mathcal{K}\cap\ell| = 0$.

Theorem 1.6 (Bose)**.** *If there exists a k-arc in a finite plane of order n, then*

$$k \leq \begin{cases} n+1 & \text{if } n \text{ odd,} \\ n+2 & \text{if } n \text{ even.} \end{cases}$$

Proof. If the points $P_1, P_2, \ldots, P_k$ form a k-arc, then the lines P_1P_i are distinct lines through P_1. But there are $n+1$ lines through P_1, hence $k \leq n+2$.

Assume that the points $P_1, P_2, \ldots, P_{n+2}$ form an $(n+2)$-arc $\mathcal{H}$. Then each line of the plane meets $\mathcal{H}$ in either 0 or 2 points, hence $\mathcal{H}$ contains an even number of points, so n must be even. □

Definition 1.7. An $(n+1)$-arc in a projective plane of order n is called *oval*.

An $(n+2)$-arc in a projective plane of order n is called *hyperoval*.

It follows from the Theorem of Bose that here are no hyperovals in planes of odd order.

Theorem 1.8. *There are ovals in* $\mathrm{PG}(2,q)$ *for all* q*. If* q *is even then* $\mathrm{PG}(2,q)$ *contains hyperovals, too.*

Proof. Consider the conic $\mathcal{C}$ having equation $X_1^2 = X_0X_2$. Then

$$\mathcal{C} = \{(1:t:t^2) : t \in \mathrm{GF}(q)\} \cup \{(0:0:1)\}.$$

We show that no three points of $\mathcal{C}$ are collinear. It is enough to prove that

$$\begin{vmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ 1 & t_3 & t_3^2 \end{vmatrix} \neq 0 \quad \text{and} \quad \begin{vmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ 0 & 0 & 1 \end{vmatrix} \neq 0$$

if $t_i \neq t_j$. The second condition obvious, while the first one follows from the properties of the Vandermonde determinants. Hence $\mathcal{C}$ is an oval for any q.

Elementary calculation shows that the equation of the tangent line to $\mathcal{C}$ at the affine point (t_0, t_0^2) is

$$Y - t_0^2 = 2t_0(X - t_0).$$

If q is even then $2 = 0$, and the equation becomes $Y = t_0^2$. The tangent line to $\mathcal{C}$ at the point $(0:0:1)$ is the line at infinity. Hence each tangent contains the point $(0:1:0)$. Thus $\mathcal{C} \cup \{(0:1:0)\}$ is a hyperoval if q is even. □

Theorem 1.9. *Let Ω be an oval in the plane Π_n, n odd. Then the points of $\Pi_n \setminus \Omega$ are divided into two classes. There are $(n+1)n/2$ points which lie on two tangents to Ω (external points), and there are $(n-1)n/2$ points none of which lie on a tangent to Ω (internal points).*

Proof. Let ℓ be the tangent to Ω at P and $P_1, P_2, \ldots, P_n$ be the other points of ℓ. Let t_i be the number of tangents to Ω through P_i. Ω contains an even number of points, hence $t_i > 0$ must be an even number, too. There are n tangents to Ω distinct from ℓ, each of these meets ℓ in a unique point, hence $\sum(t_i - 1) = n$. Thus $t_i = 2$ because of the Pigeonhole Principle.

So the number of external points is $(n+1)n/2$, while the number of internal points is $n^2+n+1-(n+1)-(n+1)n/2 = (n-1)n/2$. □

Theorem 1.10. *Let Ω be a hyperoval in the plane Π_n, n even. Then the lines of $\Pi_n \setminus \Omega$ are divided into two classes. There are $(n+2)(n+1)/2$ secants to Ω, and there are $(n-1)n/2$ external lines to Ω.*

2 How can we organize a soccer championship?

Let $G = (V, E)$ be a simple graph. A *one-factor* of G is a set of pairwise disjoint edges of G such that every vertex of G is contained in exactly one of them. A *one-factorization* of G is a decomposition of E into edge-disjoint one-factors.

Not every graph has a one-factor. Obviously necessary condition that a graph with a one-factor must have an even number of vertices. However this is not sufficient.

A graph $G = (V, E)$ has a one-factor if and only if for each subset $W \subset V$ the number of the components of $G - W$ having an odd number of vertices is less than or equal to the number of the vertices of W. For more on graph factorizations we refer to [16].

In particular, the complete graph on $2n$ vertices, K_{2n}, has a one-factor, and it is easy to see that it has a lot of one-factorizations, too.

The one-factorizations of K_{2n} have an interesting and important application. Suppose that several soccer teams play against each other in a league (e.g. 18 teams in Serie A). The competition can be represented by a graph with the teams as vertices and games as edges (the edge uv corresponds to the game between the two teams u and v). If every pair of teams plays exactly once, then the graph is complete. Several matches are played simultaneously, every team must compete at once, the set of games held at the same time is called a round.

Thus a round of games corresponds to a one-factor of the underlying graph. The schedule of the championship is the same as a one-factorization of K_{2n}.

If n is small then it is easy to organize the championship. The bigger n the more difficult schedule. There are several methods of constructions of one-factorization of K_{2n}. Now we present two constructions, both of them are based on the geometric properties of complete arcs in finite planes.

2.1 Schedule from an oval

Suppose that the projective plane Π_{2n-1} contains an oval $\Omega = \{P_1, P_2, \dots, P_{2n}\}$. Take the points of Ω as the vertices of K_{2n}. Let E be an external point of Ω. Then E defines a one-factor $\mathcal{F}$ of K_{2n} in the following way: $\mathcal{F}$ consists of the edges P_jP_k if the points P_j, P_k and E are collinear, and the edge $P_\ell P_m$ if the lines EP_ℓ and EP_m are the two tangent lines to Ω through E.

Let ℓ be the tangent line to Ω at the point P_{2n}, let $L_1, L_2, \dots, L_{2n-1}$ be the points on ℓ distinct from P_{2n}, and let $\mathcal{F}_i$ be the one-factor belonging to the point L_i.

Lemma 2.1. The union of the one-factors $\mathcal{F}_i$ gives a one-factorization of K_{2n}.

The edge $P_{2n}P_k$ belongs to $\mathcal{F}_k$. If $i \neq 2n \neq j$ and $i \neq j$ then there is a unique point of intersection of the lines P_iP_j and ℓ, say L_k. Hence there is a unique one-factor $\mathcal{F}_k$ containing the edge P_iP_j.

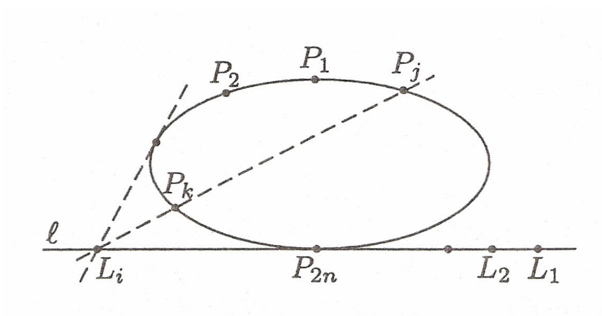


Figure 3

This construction works on arbitrary projective planes of odd order which contains an oval (although all the known planes have prime power order), in particular for $q = 17$ the plane $\mathrm{PG}(2, 17)$ contains an oval and our method gives a possible schedule for important championships, e.g. Serie A and Bundesliga. In the case $q = 19$ the plane $\mathrm{PG}(2, 19)$ also contains an oval and we get a schedule for Primera Division.

2.2 Schedule from a hyperoval

The following similar construction gives a one-factorization of K_{2n} if there exists a projective plane of order $2n - 2$ which contains a hyperoval $\mathcal{H} = \{P_1, P_2, \dots, P_{2n}\}$ (again, all the known examples have prime power order).

Take the points $P_1, P_2, \dots, P_{2n}$ as the vertices of K_{2n}. Let ℓ be an external line to $\mathcal{H}$ and let $L_1, L_2, \dots L_{2n-1}$ be the points of ℓ.

The one-factor $\mathcal{F}_i$ belonging to the point L_i is defined to consist of the edges P_jP_k if the points P_j, P_k and L_i are collinear. The union of the one-factors $\mathcal{F}_i$ is a one-factorization of K_{2n} because there is a unique point of intersection of the lines P_iP_j and ℓ.

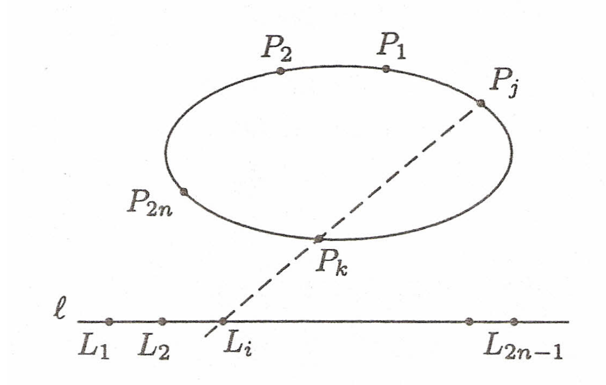


Figure 4

If $n = 16$, then the plane $\mathrm{PG}(2,16)$ contains hyperovals with 18 points, so we can make the schedule for Serie A and Bundesliga in this way, too.

3 How can we win on football pool games?

Football pool game (Totocalcio) is very popular in many countries. There is an *organizer* and there are *players*. The organizer assigns some matches (throughout this section n denotes the number of matches), and the players try to guess the outcomes of the competitions. The players do not have to guess the exact final score, the number of possible outcomes is only three: the home team wins, loses or plays a draw, usually these possibilities are denoted by 1,2 and x, respectively. The players make forecasts (a forecast is a sequence of length n, each of its element is 1,2 or x), and pay a fixed amount per forecast for the organizer. The more to pay, the bigger chance to win. After the matches have been played an entirely correct forecast wins the first prize, and in general a forecast with $(i-1)$ incorrect guesses wins the i^{th} prize. Furthermore, the amount of the i^{th} prize depends on the number of forecasts winning this prize. The organizer uses a fixed percentage of the stakes to pay the prizes (in Hungary, approximately 45 %), so it seems to be better to be the organizer than to be a player. Unfortunately to organize a football pool game is a monopoly of the state, so we cannot become rich in this way. There is really nothing one can do to get more money than one invested. Of course, there are some dirty possibilities. If the game between the teams A and B is assigned, and our guess is 1, then we can offer some money to the forwards of team A for playing extremely well, or we can offer some money to the goalkeeper of team B for playing not so well, or offer some money to the referee for making "good" decisions. But these are unfair things. We are interested in fair matches. We assume that each of the three possible outcomes has the same probability.

We try to construct suitable systems to win some prize by using as little money as possible. This is the *classical football pool problem.*

From now on we will write 0 instead of x. We can formulate our problem in the following way:

Let n and k be positive integers. Let $\mathcal{E}_n$ denote the set of all $(0,1,2)$-sequences of length n. We consider $\mathcal{E}_n$ as the set of elements $\mathbf{Z}_3^n$. Find the smallest subset $\mathcal{T} \subset \mathcal{E}_n$ with the following property: for each $\mathbf{e} \in \mathcal{E}_n$ there exists $\mathbf{t} \in \mathcal{T}$ such that they differ from each other in at most k positions.

In another point of view this is a special case of the *covering radius problem* of coding theory, it has been widely studied in information theory. For more on this subject, we refer to [1]. Some notations come from this field. The elements of $\mathcal{E}_n$ will be called *words.* If

$$\mathbf{e} = (e_1, e_2, \ldots, e_n)$$

is an element of $\mathcal{E}_n$, then e_i will be called the i^{th} coordinate of $\mathbf{e}$.

3.1 The Hamming distance

The number of mistakes in a forecast is the number of coordinates in which the forecast and the result differ from each other. Thus the following definition is a natural measure of the correctness of forecasts.

Definition 3.1. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ be two elements of $\mathcal{E}_n$. Their Hamming distance, $d(\mathbf{x}, \mathbf{y})$, is the number of indices i for which $x_i \neq y_i$.

Proposition 3.2. *The pair $(\mathcal{E}_n, d)$ is a metric space. This means that for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{E}_n$:*

1. $d(\mathbf{x}, \mathbf{y}) \geq 0$, *and equality holds if and only if* $\mathbf{x} = \mathbf{y}$.
2. $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$.
3. $d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z}) \geq d(\mathbf{x}, \mathbf{z})$.

Proof. The first and second statements are trivially true. The proof of the third (the *triangle inequality*) as follows.

Let $d(\mathbf{x}, \mathbf{z}) = m$, then there are exactly m indices $i_1, i_2, \ldots, i_m$ for which $x_{i_j} \neq z_{i_j}$. Consider the corresponding coordinates of $\mathbf{y}$. For all j at least one of the two conditions $x_{i_j} \neq y_{i_j}$ and $z_{i_j} \neq y_{i_j}$ is satisfied. Hence the number of indices for which $\mathbf{y}$ differs from $\mathbf{x}$ or $\mathbf{z}$ (or both) is at least m. This proves the third statement. $\square$

If there is a distance, then we can define the spheres (balls) in the usual way.

Definition 3.3. Let $\mathbf{c} \in \mathcal{E}_n$ be a given word and $0 \leq r$ be a given integer. Then the sphere with centre $\mathbf{c}$ and radius r is the subset $B(\mathbf{c}, r) \subset \mathcal{E}_n$ which contains those elements of $\mathcal{E}_n$ whose Hamming distance from $\mathbf{c}$ is at most r. Thus:

$$B(\mathbf{c}, r) = \{\mathbf{x} \in \mathcal{E}_n : d(\mathbf{c}, \mathbf{x}) \leq r\}.$$

The number of words contained in a sphere is depend on only the radius of the sphere:

Proposition 3.4. *A sphere of radius $0 \leq r \leq n$ contains exactly*

$$\sum_{i=0}^{r} \binom{n}{i} \cdot 2^i$$

words from $\mathcal{E}_n$. This number does not depend on the centre of the sphere.

Proof. Let $\mathbf{c}$ be the centre of the sphere. Then the sphere contains those words, which differ from $\mathbf{c}$ not more than r coordinates. If a word differs from $\mathbf{c}$ in exactly i coordinates, then we have $\binom{n}{i}$ possibilities to choose these i positions, and we can write two possible numbers into each of these coordinates independently. $\square$

If we would like to guarantee the $(n-k+1)^{\text{st}}$ prize, then we have to construct a system of m forecasts $\{\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_m\}$ such that the union of the spheres of radius $(n-k)$ centered at the words $\mathbf{t}_i$ contains $\mathcal{E}_n$. Because if it holds, then for each word $\mathbf{e} \in \mathcal{E}_n$ there exists at least one forecast $\mathbf{t}_i$ such that $d(\mathbf{t}_i, \mathbf{e}) \leq n-k$, thus the forecast contains at least $n-(n-k) = k$ correct guesses, hence guarantees at least the $(n-k+1)^{\text{st}}$ prize.

From this observation we get the following theorem, the so-called *sphere covering bound.*

Theorem 3.5. *If the system of forecasts $\mathcal{T}$ guarantees the $(n-k+1)^{\text{st}}$ prize, and $|\mathcal{T}|$ denotes the cardinality of $\mathcal{T}$, then*

$$|\mathcal{T}| \geq \frac{3^n}{\sum_{i=0}^{n-k} \binom{n}{i} \cdot 2^i}.$$

Proof. Proposition 3.4 states that each sphere of radius $n-k$ contains $\sum_{i=0}^{n-k} \binom{n}{i} \cdot 2^i$ words. If $\mathcal{T}$ guarantees the $(n-k+1)^{\text{st}}$ prize, then the union of spheres of radius $(n-k)$ centered at the elements of $\mathcal{T}$ must contain $\mathcal{E}_n$, hence the number of the words in the union of these spheres is greater than or equal to the number of elements of $\mathcal{E}_n$, that is 3^n. □

3.2 Good forecasts

First consider two trivial cases.

If a set of forecasts $\mathcal{T}$ guarantees the first prize, then the spheres of radius 0 centered at the words of $\mathcal{T}$ must contain each elements of $\mathcal{E}_n$. But a sphere of radius 0 consists of only its centre, hence $\mathcal{T}$ must contain each element of $\mathcal{E}_n$. (Of course, we have already known it before we defined the Hamming distance and the spheres.)

If $n = 13$ (this is the most important case in practical point of view, because in lot of countries, e.g. Hungary and Italy, there are 13 matches on the official football pool game), and we would like to guarantee the 9^{th} prize, we can do it easily. For $i = 0, 1, 2$ let $\mathbf{t}_i$ be the word whose each coordinate is i. This set of three forecasts guarantees the 9^{th} prize, because each word of length 13 contains at least 5 coordinates which are equal. (Otherwise the length of the word would be at most $3 \times 4 = 12$.) Using the spheres, our statement is:

$$\bigcup_{i=0}^{2} B(\mathbf{t}_i, 8) \supset \mathcal{E}_{13}.$$

Unfortunately the 9^{th} prize does not pay any money. Usually we can win only with the first, second and third prizes. Hence the really interesting cases are those, when $n-k$ is small. From now on we introduce the notation $n-k = r$. If we would like to guarantee the $(r+1)^{\text{st}}$ prize, then the number $0 \leq r \leq n$ is an upper bound on the cardinality of bad guesses.

To find the smallest subset which guarantees the $(r+1)^{\text{st}}$ prize is open in general, also for $r = 1$. For the best currently known systems and estimates we refer to [6]

The constructions often arise from finite geometry. Now we construct an optimal system in the case $n = 4$ and $r = 1$. This construction is based on the geometric properties of the finite affine plane of order three, AG(2, 3). Let us denote this plane by $\mathcal{H}$. As we have already seen there are $3 \times 3 = 9$ points and $3 \times 3 + 3 = 12$ lines on the plane. Each line of $\mathcal{H}$ contains three points and there are four lines through each point of $\mathcal{H}$. The lines of $\mathcal{H}$ can be divided into four classes, each class contains three parallel lines.

The forecasts correspond to the points of $\mathcal{H}$. For each point P we associate a word of length four in the following way: Consider the lines passing through P. There are four lines, one from each parallel clases. If the equations of these lines are

$$X = c, \quad Y = d, \quad Y = X + e \quad \text{and} \quad Y = 2X + f,$$

respectively, then let the word associate to P be

$$P \quad \mapsto \quad \mathbf{p} = (c, d, e, f).$$

If P and Q are distinct points of $\mathcal{H}$ and the associated words are $\mathbf{p}$ and $\mathbf{q}$, then there is exactly one subscript $1 \leq i \leq 4$ for which $p_i = q_i$ holds, because there is a unique line joining P and Q. Hence their Hamming distance

$d(\mathbf{p},\mathbf{q}) = 3$. This implies that the spheres of radius 1 centered at the words corresponding to the points of $\mathcal{H}$ are pairwise distinct. Because if **k** would be a common point of the spheres centered at **p** and **q**, then applying the triangle inequality we would get

$$3 = d(\mathbf{p},\mathbf{q}) \leq d(\mathbf{p},\mathbf{k}) + d(\mathbf{q},\mathbf{k}) \leq 1 + 1 = 2,$$

contradiction.

Each sphere contains $1+4\cdot 2 = 9$ words, hence the union of the 9 spheres associated to the points of $\mathcal{H}$ contains $9 \cdot 9 = 81 = 3^4$ words. But $\mathcal{E}_4$ has 81 elements, thus each element of $\mathcal{E}_4$ belongs to the union of the spheres. So the following system of nine forecasts guarantees the second prize:

(0,0,0,0)	(0,1,1,1)	(0,2,2,2)
(1,0,2,1)	(1,1,0,2)	(1,2,1,0)
(2,0,1,2)	(2,1,2,0)	(2,2,0,1)

The words correspond to the points of $\mathcal{H}$, the notation comes from Figure 1.

[1] Baumert, L.D.: *Cyclic difference sets*, Lecture Notes in Mathematics 182, Springer, Berlin, 1971.

[2] Bichara, A.: *An elementary proof of the nonexistence of a projective plane of order six*, Mitt. Math. Sem. Giessen 192 (1989), 89–93.

[3] Cameron, P.J. and Van Lint J.H.: *Designs, graphs, codes and their links*, vol. 22 of London Mathematical Society Student Texts, Cambridge University Press, Cambridge, 1991.

[4] Conway, J.H. and Sloane, N. J. A.: *Sphere Packings, Lattices and Groups,* Springer-Verlag, Berlin-New York, 1988.

[5] Hall, M., Swift, J.D. and Walker, R.J.: *Uniqueness of the projective plane of order eight*, Math. Tables Aids Comput. 10 (1956), 186–194.

[6] Hämäläinen, H., Honkala, I., Lytsin, S. and Östergård, P.: *Football Pools - A Game for Mathematicians,* American Math. Monthly, August-Sept. 1995, 579-588.

[7] Hirschfeld, J.W.P.: *Finite Projective Spaces of Three Dimensions,* Clarendon Press, Oxford, 1985.

[8] Hirschfeld, J.W.P. and Thas, J.A.: *General Galois Geometries,* Clarendon Press, Oxford, 1989.

[9] Kárteszi, F.: *Introduction to finite geometries*, Akadémiai Kiadó, Budapest, 1976.

[10] Kiss, Gy.: *One-factorizations of complete multigraphs and quadrics in $PG(n,q)$,* J. Combin. Designs 19 (2002), 139–143.

[11] Kiss Gy.: *Hogyan nyerjünk a TOTÓ-n?,* (in Hungarian,) Középiskolai Mat. Fiz. Lapok 57 (2007), 267-273.

[12] Kiss, Gy. and Szőnyi, T.: *Véges geometriák (in Hungarian),* Polygon Kiadó, Szeged, 2001.

[13] Lam, C.W.H. and Thiel, L.H.: *The nonexistence of finite projective planes of order 10*, Canad. J. Math. 41 (1989), 1117–1123.

[14] Pickert, G.: *Projektive Ebenen*, Springer, Berlin, 1955.

[15] Segre, B.: *Ovals in a finite projective plane*, Canad. J. Math. 7 (1955), 414–416.

[16] W.D. Wallis, *One-Factorizations,* Mathematics and its Applications, vol. 390, Kluwer Academic Publisher Group, Dordrecht, 1997.

When probability meets geometry, seemingly unlikely things happen. As a warm-up, we will discuss Bertrand's paradox, and show that 1/2=1/3=1/4 – or not?

Then, we will move on to consider classical problems in geometric probability. What is the probability that the convex hull of a number of points randomly selected from a circle will contain the center? Will a grid catch needles thrown at it at random? These are problems where the goal is to understand the result of a random algorithm that generates a geometric configuration. Through these problems, we will introduce the basics of probabilistic constructions in discrete geometry.

Finally, we will discuss a construction of Erdős and Füredi that shows that one can find surprisingly many vectors in Euclidean d-space that are pairwise almost orthogonal.

The main goal in the selection of the topics is to present a diverse set of methods and thus invite you to a field where combinatorics, probability and analysis all come together.

All proofs are accessible at the BSc. level and yet, lead to current research, and thus, are aimed at graduate students as well.

1 Bertrand's Paradox

Joseph Bertrand posed the following problem in 1889 in his book *Calcul des probabilités* [3].

What is the probability that a randomly chosen chord of the unit circle is longer than the side of the equilateral triangle inscribed in that circle?

He computed this probability by three equally legitimate methods. The problem is not the problem itself, but the fact that the three answers are different.

Method 1. We pick the two endpoints of the chord on the circle independently according to the uniform distribution. It is not hard to see that the probability in question is 1/3. *(Check it yourself.)*

Method 2. We pick a point P on the circle according to the uniform distribution. Then, we consider the radius OP, and randomly select a point on this segment, according to the uniform distribution on the segment. It is not difficult to see that now, the desired probability is 1/2. *(Check it yourself.)*

Method 3. We pick a point P in the unit disk according to the uniform distribution, and draw the chord whose mid-point is P. (In the case when P is the center –which case is of zero probability– we take any fixed diameter). The chord will be longer than the side of the triangle if, and only if, P falls inside the inscribed disk of the triangle, which is of radius 1/2. Its area is one quarter of the area of the unit disk. Thus now, the probability is 1/4.

The solution of the paradox is, as you may already expect, that the question itself is not well posed. "Choosing a chord at random" does not determine a probability distribution on the set of all chords. The three methods above are three interpretations of the phrase "at random", and yield three distinct distributions on the set of chords.

The message is simple: when dealing with probability, one has to make clear what distribution is used or assumed in the process.

2 Buffons's Needle Problem

A century before Bertrand, in 1777, Georges-Louis Leclerc, Comte de Buffon asked the following simple sounding question.

Assume we have a floor made of parallel strips of wood, each of the same width. We drop a needle onto the *floor. What is the probability that the needle will lie across a line between two strips?*

Having learnt the lesson from Bertrand (thanks to living yet another century later), we will first specify the random process by which the needle is dropped. On the coordinate plane, we are given the lines $x = k$, where $k \in \mathbb{Z}$. First, we will specify where the left end of the needle lies. Since only the x-coordinate matters, and it only matters modulo one, we will choose a number t in the interval $[0,1]$ with the uniform distribution. Then independently, we choose the angle of the needle with the horizontal direction (ie. the x-axis). That is a number α chosen in the interval $[-\pi/2, \pi/2]$ with the uniform distribution.

Suppose that the length ℓ of the needle is $\ell < 1$. Let an angle $\Theta \in [-\pi/2, \pi/2]$ be given. Then the probability that the needle hits a line under the condition $\alpha = \Theta$ is

$$\mathbb{P}(\text{the needle hits a line} \mid \alpha = \Theta) = \ell \cos\Theta.$$

Thus,

$$\mathbb{P}(\text{the needle hits a line}) = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \ell \cos\Theta \, \mathrm{d}\Theta = \frac{2\ell}{\pi}. \tag{19}$$

A nice interpretation of (19) is that π can can be measured: Do the experiment a large number of times, and you obtain an approximation of π.

We leave the case of the long needle as an exercise.

Exercise 2.1. *Prove that if $\ell \geq 1$ then*

$$\mathbb{P}(\text{the needle hits a line}) = \frac{2\ell}{\pi} - \frac{2}{\pi}\left(\sqrt{\ell^2 - 1} + \arcsin(1/\ell)\right) + 1.$$

A variation of the question is *Buffon's noodle problem*. Now we are throwing a rectifiable curve of length ℓ on the ruled plane. Clearly, the probability that this curve hits a line depends on its shape. However, as we will see, the expectation of the number of intersection points of the curve and the lines does not. The computation is very simple but demonstrates the use of a fundamental property of the expected value: its additivity which holds for non-independent random variables, too.

Let us approximate the given curve γ by a polygonal curve Γ of n short line segments. Denote by X_i the indicator of the event that the i^{th} segment intersects a line, that is X_i is 1 if the i^{th} segment intersects a line, and X_i is 0 otherwise. Then for the expectation of the number of intersection points of Γ with the lines we have

$$\mathbb{E}\left(\#\left(\gamma \cap \left(\bigcup_{k\in\mathbb{Z}} \{x = k\}\right)\right)\right) = \mathbb{E}\left(\sum_{i=1}^{n} X_i\right) =$$

$$\sum_{i=1}^{n} \mathbb{P}(\text{the } i^{\text{th}} \text{ segment intersects a line}) = \sum_{i=1}^{n} \frac{2\ell_i}{\pi}.$$

By taking the limit for a series of polygons that approximates γ, we obtain

Theorem 2.2. *In the above setting, the expectation of the number of intersection points of γ with the lines is*

$$\frac{2 \cdot \text{length}(\gamma)}{\pi}$$

independently of the shape of γ.

3 Wendel's Problem

Our next random geometric construction is even simpler than the previous ones, and at the same time, it is one of the first questions in a field of strong current interest, the study of random polytopes.

Theorem 3.1 (J.G. Wendel [12]). *We choose n points of the unit circle uniformly and independently. Then the probability that their convex hull contains the center is $1-\frac{2n}{2^n}$.*

Proof. Let $Q_1,\dots,Q_n$ be random points of the unit circle chosen uniformly and independently. We will apply the following simple, but very powerful symmetrization technique: take your vectors, add some coin tosses, and the computation will be nicer. Slightly more precisely, let $\varepsilon_1,\dots,\varepsilon_n$ be random Bernoulli variables with values 1 and -1, each of probability $1/2$. Let $P_i=\varepsilon_i Q_i$ $(i=1,\dots,n)$.

Now, the P_is are independent and are chosen uniformly on the circle. Furthermore, with probability one, $Q_1,\dots,Q_n,Q_{n+1}:=-Q_1,\dots,Q_{2n}:=-Q_n$ are $2n$ distinct points.

We fix n points $q_1,\dots,q_n$ on the unit circle, no two of which forming a opposite pair. Let x denote the number of those n-element subsets of $\{q_1,\dots,q_n,-q_1,\dots,-q_n\}$ that are contained in an open half-plane whose bounding line passes through the origin. Clearly, $x=2n$.

The probability that the origin is not in the convex hull of the P_is under the condition that $Q_1=q_1,\dots,Q_n=q_n$ is easy to compute:

$$\mathbb{P}\left(o\notin \operatorname{conv}\{P_1,\dots,P_n\}\mid Q_1=q_1,\dots,Q_n=q_n\right)=\frac{x}{2^n}=\frac{2n}{2^n}.$$

The main thing to notice here is that it is independent of the choice of the q_is. Thus,

$$\mathbb{P}\left(o\notin \operatorname{conv}\{P_1,\dots,P_n\}\right)=\frac{2n}{2^n},$$

as needed. □

Exercise 3.2. *Solve the same problem with the following modification: We now pick the points uniformly on the disk and not on the unit circle.*

Exercise 3.3. *Alice and Bob each go to the grocery store independently at a random time between noon and one in the afternoon, and spend 5 minutes there. What is the probability that they will meet?*

4 Antipodal Sets

The following question was asked by Erdős [5] in 1957. How many points can we find in Euclidean d-space, $\mathbb{R}^d$ such that no three determine an obtuse (ie. larger than a right angle) angle? A stricter version of the question prohibits right angles as well.

Definition 4.1. *A hyperplane is said to* support *a set X in $\mathbb{R}^d$ if the set lies in one of the closed half-spaces bounded by the hyperplane. A set X in $\mathbb{R}^d$ is* antipodal, *if for any two points $x_1,x_2\in X$ there is a pair of distinct parallel hyperplanes through x_1 and x_2 supporting X. Furthermore, X is* strictly antipodal, *if for any x_1 and x_2, the supporting hyperplanes H_1 and H_2 of X can be chosen such that $X\cap H_1=\{x_1\}$ and $X\cap H_2=\{x_2\}$.*

Antipodal sets were defined by Klee in [10], where he posed the problem of finding the maximum cardinality of such a set.

Exercise 4.2. *Show that any set that satisfies Erdős' original condition (resp., the strict version of his condition) is antipodal (resp., strictly antipodal).*

Danzer and Grünbaum [4] gave a complete answer to Klee's question.

Theorem 4.3 (Danzer–Grünbaum, [4]). *The maximum cardinality of an antipodal set in $\mathbb{R}^d$ is 2^d, which is only attained by the set of vertices of an affine image of a cube.*

Proof. Let $X = \{x_1,\ldots,x_n\} \subset \mathbb{R}^d$ be an antipodal set. First, observe that X is in convex position, that is, no point of X is contained in the convex hull of the others. *(Observe it.)*

Consider the convex set $K = \mathrm{conv}X$. (It is the convex hull of finitely many points in $\mathbb{R}^d$, and thus, is called a polytope.) I will assume that the affine hull of X is $\mathbb{R}^d$ (ie., X does not lie in the translate of a proper linear subspace of $\mathbb{R}^d$), and leave the verification of the other case to you.

Now, contract K by factor one half with center x_i to obtain K_i for $i = 1,\ldots,n$. From the assumption that X is antipodal, it follows that the K_is are pairwise non-overlapping, that is, the interiors of any two are disjoint. *(Check it.)* On the other hand, since K is convex, $K_i \subset K$ for each i.

A simple volume argument finishes the proof of the upper bound:

$$\mathrm{vol}(K) \geq \sum_{i=1}^{n} \mathrm{vol}K_i = n\frac{\mathrm{vol}K}{2^d}. \tag{20}$$

The case of equality is treated by the following result of Groemer.

Lemma 4.4 (Groemer [8]). *Let K be the convex hull of $x_1,\ldots,x_{2^d}$ in $\mathbb{R}^d$. Suppose that $K = \bigcup_{i=1}^{2^d} \frac{1}{2}(K+x_i)$. Then K is a parallelotope with vertices $x_1,\ldots,x_{2^d}$.*

□

Exercise 4.5. *Prove Lemma 4.4.*

The question of the maximal cardinality of a strictly antipodal set remains. Clearly, it is at most $2^d - 1$ by Theorem 4.3. In fact, that is all we know.

Danzer and Grünbaum conjectured that the cardinality of a strictly antipodal set in $\mathbb{R}^d$ is at most $2d-1$, and Grünbaum [8] gave a proof of this conjecture for $d = 3$. The conjecture, however, turned out to be wrong, in fact, by a large margin.

Theorem 4.6 (Erdős–Füredi [6]). *There is a $d_0 > 0$ such that for every $d > d_0$, there is a set in $\mathbb{R}^d$ of cardinality 1.15^d with the property that no three points of the set determine a non-acute angle (ie. all angles are smaller than a right angle).*

Open problem 4.7. Is there a $c < 2$ such that the cardinality of a strict antipodal set in $\mathbb{R}^d$ is at most c^d?

Proof of Theorem 4.6 by David Bevan [1]. Let $m := \left\lfloor \frac{\sqrt{6}}{9}\left(\frac{2}{\sqrt{3}}\right)^d \right\rfloor$. We will show that one can find a strictly antipodal subset of the vertices of the $[0,1]^d$ cube, which is of cardinality $2m$.

Exercise 4.8. *Prove that no three points of $\{0,1\}^d$ determine an obtuse triangle.*

Let $x_1,\ldots,x_{3m}$ be $3m$ randomly chosen elements of $\{0,1\}^d$. Consider a triple, x_i, x_j, x_k. When is $\sphericalangle(x_i,x_j,x_k) = \perp$? $\sphericalangle(x_i,x_j,x_k) = \perp$ if, and only if, $0 = (x_i - x_j)(x_k - x_j)$. The latter is $\sum_{\ell=1}^{d}(x_i^\ell - x_j^\ell)(x_k^\ell - x_j^\ell)$, where upper index ℓ denotes the ℓ^{th} coordinate. However, each summand is non-negative, thus the sum will be zero only if, for each $\ell = 1,\ldots,d$, we have that $x_i^\ell = x_j^\ell$ or $x_k^\ell = x_j^\ell$.

Thus, the probability that i,j,k is a "bad triple" is

$$\mathbb{P}(\sphericalangle(x_i,x_j,x_k) = \perp) = \left(\frac{3}{4}\right)^d.$$

There are $N := 3\binom{3m}{3}$ triples in total. Let $\xi_1, \dots, \xi_N$ be indicators for the event that the first, second,...,N^{th} triple is bad, ie. $\xi_t = 1$ if the t^{th} triple is bad, and $\xi_t = 0$ otherwise. Now, the expected number of bad triples is

$$\mathbb{E}(\xi_1 + \ldots + \xi_N) = \sum_{t=1}^{N} \mathbb{E}(\xi_t) = \sum_{t=1}^{N} \mathbb{P}(\text{the } t^{\text{th}} \text{ triple is bad}) =$$

$$N\left(\frac{3}{4}\right)^d < 3\frac{(3m)^3}{6}\left(\frac{3}{4}\right)^d \le m.$$

It follows that with non-zero probability the number of bad triples in the chosen $3m$ points is less than m. Consider such a choice of $3m$ points and for each bad triple leave out one of the points in the triple. Thus, at least $2m$ points remain, and they are such that no three form a bad triple, ie. no three are vertices of a right triangle. That proves the Theorem. □

5 Almost Orthogonal Vectors

Exercise 5.1. *Prove that any set of pairwise orthogonal vectors in $\mathbb{R}^d$ is of cardinality at most d.*

Exercise 5.2. *Show that the maximum cardinality of a set of vectors in $\mathbb{R}^d$ whose pairwise angles are all obtuse is $d+1$.*

What happens if we relax the condition in Exercise 5.1, and require only that the vectors be pairwise close to orthogonal? A surprise again, one which can be obtained by a probabilistic argument: there are exponentially many such vectors.

Theorem 5.3 (Füredi-Lagarias-Morgan [7])**.** *For any $\varepsilon > 0$ there is a $c = c(\varepsilon) > 1$ such that, for any sufficiently large d, there is a set of $N \ge c^d$ unit vectors $x_1, \dots, x_N$ in $\mathbb{R}^d$ with $\frac{\pi}{2} - \varepsilon < \sphericalangle(x_i, x_j) < \frac{\pi}{2} + \varepsilon$.*

We will give a probabilistic construction, ie. choose vectors in $\mathbb{S}^{d-1}$ randomly, and show that with non-zero probability, we obtain a desired set of vectors. But what probability distribution can we use on $\mathbb{S}^{d-1}$? I will give two equivalent answers, pick the one dearest to you (or, just pick one randomly).

5.1 A Probability Distribution on the Sphere

First, we define the *angular distance* on the sphere: for two points $x, y \in \mathbb{S}^{d-1}$, their distance is the angle $\sphericalangle(x, y)$ of their position vectors. This turns $\mathbb{S}^{d-1}$ into a metric space. Balls in this metric space are often referred to as *spherical caps*, and we denote them by $C(x, \phi) := \{y \in \mathbb{S}^{d-1} : \sphericalangle(x, y) < \phi\}$.

One way to measure sets on the sphere is to introduce the surface area, or, using its more elegant name, the $(d-1)$-dimensional Lebesgue measure on $\mathbb{S}^{d-1}$. We will denote it by λ_{d-1}. We will not build λ_{d-1} here, as it would be a long analytical argument, instead, we will assume that you have at least an intuitive understanding of measuring the surface area of a the sphere itself, and of its (Lebesgue measurable) subsets. Then, for any Lebesgue measurable set $A \subseteq \mathbb{S}^{d-1}$ let

$$\sigma(A) = \lambda_{d-1}(A) / \lambda_{d-1}\left(\mathbb{S}^{d-1}\right). \tag{21}$$

In this manner, we obtain a probability measure σ on $\mathbb{S}^{d-1}$.

Another approach is slightly more abstract but a good deal more general. If you find it a bit too difficult to follow at this point, then skip the rest of this subsection freely and proceed to the proof of Theorem 5.3.

Let (M, ρ) be a compact metric space. We call a Borel measure on M *regular*, if $\mu(M) < \infty$, $\mu(U) > 0$ for all open subsets $U \subseteq M$, $\mu(E) = \inf\{\mu(U) : E \subseteq U, U \text{ open}\}$ and $\mu(E) = \sup\{\mu(K) : K \subseteq E, K \text{ compact}\}$.

Let G be a subgroup of the isometry group of (M,ρ), that is, $\rho(g(a),g(b)) = \rho(a,b)$ for every $a,b \in M$ and every $g \in G$.

We call a measure μ on M *left-invariant with respect to* G, if for any measurable subset $A \subseteq M$ and any $g \in G$ we have $\mu(gA) = \mu(A)$, where $gA = \{ga \ : \ a \in A\}$.

Theorem 5.4 (Haar, 1933, Weil, 1940)**.** *Let (M,ρ) be a compact metric space, and G a subgroup of its isometry group. Then there is a left-invariant regular Borel measure on M with respect to G. This measure is unique up to a constant factor.*

A measure whose existence is guaranteed by this theorem is called a *Haar measure on M with respect to G.*

We will not provide a proof here, it can be found in several books, cf. [11].

Now, we apply this result to the metric space $M = \mathbb{S}^{d-1}$ and its full isometry group, $G = O(d)$ (the group of real $d\times d$ orthogonal matrices), and we normalize the Haar measure so that the measure of the whole sphere is one. This way we obtain a probability measure σ on $\mathbb{S}^{d-1}$.

Exercise 5.5. *Show that the Haar probability measure on $\mathbb{S}^{d-1}$ is identical to measure σ defined in* (21).

5.2 Proof of Theorem 5.3

The main tool in the proof is an estimate on the measure of caps in the unit sphere $\mathbb{S}^{d-1}$ of $\mathbb{R}^d$.

Lemma 5.6. *Let $0 < \phi < \arccos\frac{1}{\sqrt{d}}$, and $x \in \mathbb{S}^{d-1}$ an arbitrary point. Then*

$$\sigma(C(x,\phi)) \quad < \quad \frac{\sin^{d-1}\phi}{\sqrt{2\pi(d-1)}\cos\phi}.$$

Now, the proof is simple. We pick N random vectors $x_1,\dots,x_N$ independently according to σ on $\mathbb{S}^{d-1}$. For any two x_i,x_j, the probability that they are a "bad pair" is

$$\mathbb{P}\left(\sphericalangle(x_i,x_j) > \frac{\pi}{2}+\varepsilon \text{ or } \sphericalangle(x_i,x_j) < \frac{\pi}{2}-\varepsilon\right) = 2\sigma\left(C\left(x,\frac{\pi}{2}-\varepsilon\right)\right).$$

Denote by $T := \sin\left(\frac{\pi}{2}-\varepsilon\right)$. Thus, the probability that no two of the N chosen points is a bad pair is

$$\mathbb{P}\left(\frac{\pi}{2}-\varepsilon < \sphericalangle(x_i,x_j) < \frac{\pi}{2}+\varepsilon \text{ for all } i,j\right) > 1 - N^2\frac{T^{d-1}}{\sqrt{2\pi(d-1)}\cos\left(\frac{\pi}{2}-\varepsilon\right)}$$

Let c be a number strictly between one and $\frac{1}{\sqrt{T}}$, and let $N = \lfloor c^d \rfloor$. We obtain that if d is large enough, then the right hand side is positive, and thus, with non-zero probability, the set $x_1,\dots,x_N$ is as promised by the Theorem.

Exercise 5.7. *Prove that for any $\varepsilon > 0$ there is a $c = c(\varepsilon) > 1$ such that, for any sufficiently large d, there is a set of $N \geq c^d$ points $x_1,\dots,x_N$ in $\mathbb{R}^d$ such that the angle determined by any three of them is at most $\pi/3+\varepsilon$.*

6 Looking Further

In the examples that we have discussed so far, the question was the (likely) result of a probabilistic algorithm in some geometric setting. These types of questions, however, go beyond the goal of understanding the probabilistic question itself. In the theory of convex bodies, often it is the bodies, ie., the geometric setting that we need to understand, and probability is a tool, or a language to do that. We give a simple example here.

Let K be an origin-symmetric convex body, ie. $K = -K$, in $\mathbb{R}^d$. Then K defines a norm on $\mathbb{R}^d$ as $\|x\|_K = \inf\{\lambda \geq 0 \ : \ x \in \lambda K\}$. We now pick two points uniformly and independently from K. What is the likely distance of the two points with respect to the norm $\|.\|_K$? See more on this problem in [2], and in references given therein.

Exercise 6.1. *Show that if K is the Euclidean ball than the distance of the two points is very likely to be around $\sqrt{2}$. More precisely, for every $\varepsilon > 0$ there is a $c = c(\varepsilon) < 1$ such that, for any sufficiently large d, we have*

$$\mathbb{P}(|\|x - y\|_K - \sqrt{2}| > \varepsilon) < c^d. \tag{22}$$

Exercise 6.2. *Show that if K is the cube $[-1,1]^d$ than $\sqrt{2}$ is replaced by 2 in (22), ie., the following holds.*

$$\mathbb{P}(|\|x - y\|_K - 2| > \varepsilon) < c^d. \tag{23}$$

Inequalities (22) and (23) show a true geometric difference between the Euclidean ball and the cube.

[1] Martin Aigner and Günter M. Ziegler. *Proofs from The Book*. Springer-Verlag, Berlin, fourth edition, 2010.

[2] J. Arias-de-Reyna, K. Ball, R. Villa, *Concentration of the distance in finite-dimensional normed spaces*, Mathematika 45 (1998), no. 2, 245–252.

[3] Joseph Bertrand. *Calcul des probabilités*. Bronx, N.Y. Chelsea Pub. Co., 1972. Reprint of the 1907 ed.

[4] L. Danzer, B. Grünbaum, *Über zwei Probleme bezüglich konvexer Körper von P. Erdős und von V.L. Klee*, Math. Z., vol. **79** (1962), pp. 95-99.

[5] Erdős, P., *Some unsolved problems*, Michigan Math J., vol. **4** (1957), pp. 291-300

[6] P. Erdős and Z. Füredi. The greatest angle among n points in the d-dimensional Euclidean space. In *Combinatorial mathematics (Marseille-Luminy, 1981)*, volume 75 of *North-Holland Math. Stud.*, pages 275–283. North-Holland, Amsterdam, 1983.

[7] Z. Füredi, J.C. Lagarias, and F. Morgan. Singularities of minimal surfaces and networks and related extremal problems in Minkowski space. In *Discrete and computational geometry: Papers from the DIMACS special year. Proceedings of workshops, held at DIMACS from September 1989 through August 1990*, pages 95–109. Providence, RI: American Mathematical Society; New York: Association for Computing Machinery, 1991.

[8] Helmut Groemer. Abschätzungen für die anzahl der konvexen körper, die einen konvexen körper berühren. *Monatshefte für Mathematik*, 65:74–81, 1961.

[9] Branko Grünbaum. Strictly antipodal sets. *Israel J. Math.*, 1:5–10, 1963.

[10] Klee, V., *Unsolved problems in intuitive geometry*, Hectographical lecture notes, Seatle, 1960.

[11] V.D. Milman and G. Schechtman. *Asymptotic Theory of Finite Dimensional Normed Spaces: Isoperimetric Inequalities in Riemannian Manifolds*. Lecture Notes in Mathematics. Springer, 1986.

[12] James G. Wendel. A problem in geometric probability. *Mathematica Scandinavica*, 11(0), 1962.

The lectures focus on two interesting results and their corollaries from the field of algorithms in geometry. They do not build on each other, so missing a class is no problem.

First, I present a simple algorithm that draws a planar graph with n vertices onto a $(2n-4)\times(n-2)$ grid.

Second, I will talk about online competitive algorithms focusing on the k-server problem. Here we have a graph on n vertices, with given edge lengths and some information in each vertex. We can move k servers that can read out and transfer the information. At every step one of the data is requested and our goal is to move the servers such that the sum of the distances they travel is minimized. How much does it help if we know the queries in advance? It turns out that it helps at most a constant factor.

1 Embedding planar graphs on a small grid

1.1 Planar graphs

A graph is *planar* if it can be embedded in the plane such that its vertices are points and its edges are non-crossing simple curves. We call an already embedded graph a *plane graph*. Planar graphs have several useful properties, maybe the most well-known is Euler's formula: $v(G)-e(G)+f(G)\geq 2$ where equality holds if and only if G is connected. Here $v(G)$ is the number of vertices, $e(G)$ the number of edges and $f(G)$ the number of faces, so we can already conclude that the number of faces does not depend on the embedding, only on the graph. From the formula we can also reduce that if G has at least 3 vertices, then $e(G)\leq 3v(G)-6$ and $f(G)\leq 2v(G)-4$ where equality holds for *triangulated* graphs, i.e. if all faces of the graph have 3 sides. Another simple consequence is that the *chromatic number* of any planar graph G, $\chi(G)$ is at most 5. The strengthening of this, $\chi(G)\leq 4$, is the famous Four color theorem that was a conjecture for a long time.

Here we will focus on straight-line drawings, i.e. when every edge is embedded as a straight-line segment. The existence of such an embedding was discovered independently by Fáry, Tutte and Wagner. We will give a different proof of this result. In our case not only the edges will be straight-line segments, but even the vertices will have small, integer coordinates. This has applications in computer science to draw a graph on a screen or can be used in theoretical computer science to give a polynomial *witness* of the planarity of a graph.

1.2 Canonical ordering

We need the following observation.

Lemma 1.1. *Let G be a plane graph, whose exterior face is bounded by a cycle $u_1,u_2,\dots,u_k$. Then there is a vertex u_i $(i\neq 1,k)$ not adjacent to any u_j other than u_{i-1} and u_{i+1}.*

Proof. If there are no two non-consecutive vertices along the boundary of the exterior face that are adjacent, then there is nothing to prove. Otherwise, pick an edge $u_iu_j\in E(G)$, for which $j>i+1$ and $j-i$ is minimal. Then u_{i+1} cannot be adjacent to any element of $\left\{u_1,\dots,u_{i-1},u_{j+1},\dots,u_k\right\}$ by planarity, nor can it be adjacent to any other vertex of the exterior face different from u_i and u_{i+2}, by minimality of $j-i$. □

Theorem 1.2 (Canonical Ordering)**.** *Let G be a triangulation of n vertices, with exterior face uvw. Then there is an ordering of the vertices $v_1=u,v_2=v,v_3,\dots,v_n=w$ satisfying the following conditions for every k $(4\leq k\leq n)$:*

- (i) *the boundary of the exterior face of the subgraph G_{k-1} of G induced by $\{v_1,v_2,\dots,v_{k-1}\}$ is a cycle C_{k-1} containing the edge uv;*
- (ii) *v_k is in the exterior face of G_{k-1}, and its neighbors in $V(G_{k-1})$ are some (at least two) consecutive elements along the path obtained from C_{k-1} by removal of the edge uv. (See Figure 8)*

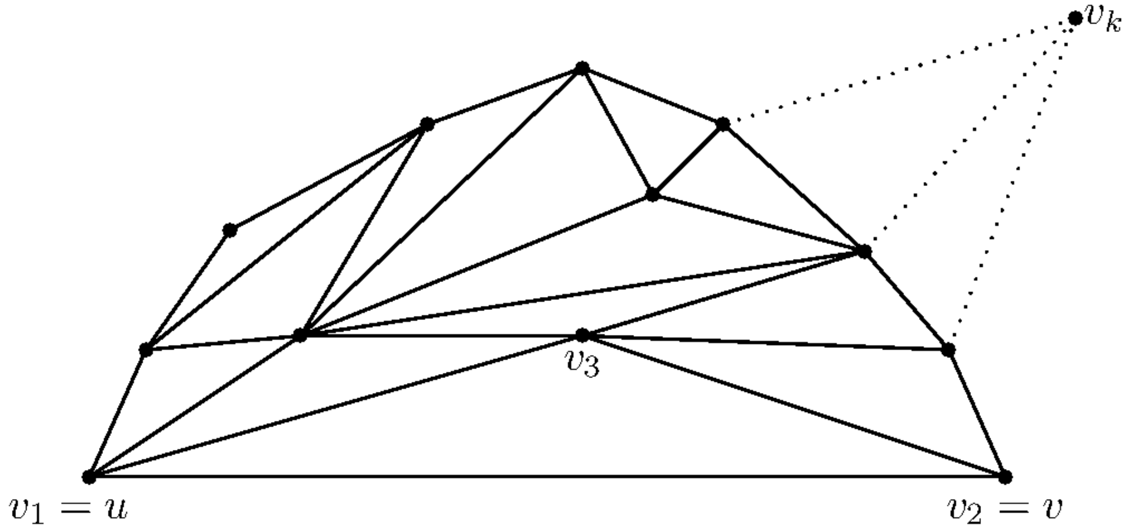


Figure 8: G_{k-1} and v_k in the exterior

Proof. The vertices $v_n, v_{n-1}, \ldots, v_3$ will be defined by reverse induction. Set $v_n = w$, and let G_{n-1} be the graph obtained from G by the deletion of v_n. Since G is a triangulation, the neighbors of w form a cycle C_{n-1} containing uv, and this cycle is the boundary of the exterior face of G_{n-1}.

Let $4 \le k \le n$ be fixed and assume that $v_n, v_{n-1}, \ldots, v_k$ have already been determined so that the subgraph G_{k-1} induced by $V(G) \setminus \{v_k, v_{k+1}, \ldots, v_n\}$ satisfies condition (i) and (ii). Let C_{k-1} denote the boundary of the exterior face of G_{k-1}. Applying Lemma 1.1 to G_{k-1}, we obtain that there is a vertex u' on C_{k-1}, different from u and v, which is adjacent only to two other points of C_{k-1} (i.e., to its immediate neighbors). Letting $v_{k-1} = u'$, the subgraph $G_{k-2} \subseteq G$ induced by $V(G) \setminus \{v_{k-1}, v_k, \ldots, v_n\}$ obviously meets the requirements. □

Using this theorem, we can easily prove the main result of this section.

Corollary 1.3. *Every planar graph has a straight-line embedding in the plane.*

Proof. It is sufficient to show that the statement is true for triangular planar graphs.

Let G be any triangulation with the canonical ordering $v_1 = u$, $v_2 = v$, $v_3, \ldots, v_n = w$, described above. We will determine the positions $f(v_k) = (x(v_k), y(v_k))$ of the vertices by induction on k.

Set $f(v_1) = (0,0)$, $f(v_2) = (2,0)$, $f(v_3) = (1,1)$. Assume that $f(v_1), f(v_2), \ldots, f(v_{k-1})$ have already been defined for some $k \ge 4$ such that, connecting the images of the adjacent vertex pairs by segments, we obtain a straight-line embedding of G_{k-1}, whose exterior face is bounded by the segments corresponding to the edges of C_{k-1}. Suppose further that

$$\begin{aligned} &x(u_1) < x(u_2) < \ldots < x(u_m), \\ &y(u_i) > 0 \qquad \text{for } 1 < i < m, \end{aligned} \tag{24}$$

where $u_1 = u$, u_2, u_3, $\ldots$, $u_m = v$ denote the vertices of C_{k-1} listed in cyclic order. By condition (ii) of Theorem 1.2, v_k is connected to u_p, $u_{p+1}, \ldots, u_q$ for some $1 \le p \le q \le m$. Let $x(v_k)$ be any number strictly between $x(u_p)$ and $x(u_q)$. If we choose $y(v_k) > 0$ to be sufficiently large and connect $f(v_k) = (x(v_k), y(v_k))$ to $f(u_p), f(u_{p+1}), \ldots, f(u_q)$ by segments, then we obtain a straight-line embedding of G_k meeting all the requirements (including the auxiliary Hypothesis (24) for the vertices of C_k). □

1.3 Embedding on the grid

Now we shall restrict our attention to straight-line drawings, where each point is mapped into a *grid point*, i.e. a point with integer coordinates. Our goal is to minimize the size of the grid needed for the embedding of any planar graph of n vertices. The set of all grid points (x,y) with $0 \le x \le m$, $0 \le y \le n$ is said to be an *$m \times n$ grid*.

Theorem 1.4. *Any planar graph with n vertices has a straight-line embedding on the $2n-4$ by $n-2$ grid.*

Proof. It suffices to prove the theorem for triangulations. Let G be a triangulation with exterior face uvw, and let $v_1 = u, v_2 = v, v_3, \ldots, v_n = w$ be a canonical labelling of the vertices (see Theorem 1.2).

We are going to show by induction on k that G_k, the subgraph of G induced by $\{v_1, v_2, \ldots, v_k\}$, can be straight-line embedded on the $2k-4$ by $k-2$ grid, for every $k \ge 3$. Let f_3 be the following embedding of G_3:

$$f_3(v_1) = (0,0), f_3(v_2) = (2,0), f_3(v_3) = (1,1).$$

Suppose now that for some $k \ge 4$ we have already found an embedding $f_{k-1}(v_i) = (x_{k-1}(v_i), y_{k-1}(v_i))$, $1 \le i \le k-1$, with the following properties:

(a) $f_{k-1}(v_1) = (0,0)$, $f_{k-1}(v_2) = (2k-6, 0)$;

(b) If $u_1 = u, u_2, \ldots, u_m = w$ denote the vertices of the exterior face of G_{k-1} in cyclic order, then

$$x_{k-1}(u_1) < x_{k-1}(u_2) < \ldots < x_{k-1}(u_m);$$

(c) The segments $f_{k-1}(u_i)f_{k-1}(u_{i+1})$, $1 \le i < m$, all have slope $+1$ or -1.

Note that (c) implies that the Manhattan distance $|x_{k-1}(u_j) - x_{k-1}(u_i)| + |y_{k-1}(u_j) - y_{k-1}(u_i)|$ between the image of any two vertices u_i and u_j on the exterior face of G_{k-1} is even. Consequently, if we take a line with slope $+1$ through u_i and a line with slope -1 through u_j, then they always intersect at a grid point $P(u_i, u_j)$.

Let $u_p, u_{p+1}, \ldots, u_q$ be the neighbours of v_k in G_k $(1 \le p < q \le m)$. Clearly, $P(u_p, u_q)$ is a good candidate for $f_k(v_k)$, except that we may not be able to connect it to e.g. $f_{k-1}(u_p)$ by a segment avoiding $f_{k-1}(u_{p+1})$. To resolve this problem, we have to modify f_{k-1} before embedding v_k. We shall move the image of $u_{p+1}, u_{p+2}, \ldots, u_m$ one unit to the right, and then move the images of $u_q, u_{q+1}, \ldots, u_m$ to the right by an additional unit. That is, let

$$\tilde{x}_k(u_i) = \begin{cases} x_{k-1}(u_i), & \text{for } 1 \le i \le p, \\ x_{k-1}(u_i) + 1, & \text{for } p < i < q, \\ x_{k-1}(u_i) + 2, & \text{for } q \le i \le m, \end{cases}$$

$$y_k(u_i) = y_{k-1}(u_i), \quad \text{for } 1 \le i \le m,$$

and let $f_k(v_k)$ be the point of intersection of the lines of slope $+1$ and -1 through $f_k(u_p)$ and $f_k(u_q)$, respectively. Of course, $f_k(v_k)$ is a grid point that can be connected by disjoint segments to the points $f_k(u_i) = (x_k(u_i), y_k(u_i))$, $p \le i \le q$, without intersecting the polygon $f_k(u_1)f_k(u_2)\ldots f_k(u_m)$. However, as we move the image of some u_i, it may be necessary to move some other points (not on the exterior face) as well, otherwise we may create crossing edges.

In order to tell exactly which set of points has to move together with the image of a given exterior vertex u_i, we define recursively a total order '$\prec$' on $\{v_1, v_2, \ldots, v_n\}$. Originally, let $v_1 \prec v_3 \prec v_2$. If the order has already been defined on $\{v_1, v_2, \ldots, v_{k-1}\}$, then insert v_k just before u_{p+1}. According to this rule, obviously

$$u_1 \prec u_2 \prec \cdots \prec u_m.$$

Now we can extend the definition of f_k to the interior vertices of G_{k-1}, as follows. For any $1 \le i \le k-1$, let

$$\tilde{x}_k(v_i) = \begin{cases} x_{k-1}(v_i), & \text{if } v_i \prec u_{p+1}, \\ x_{k-1}(v_i) + 1, & \text{if } u_{p+1} \preceq v_i \prec u_q, \\ x_{k-1}(v_i) + 2, & \text{if } u_q \preceq v_i, \end{cases}$$

$$y_k(v_i) = y_{k-1}(v_i).$$

Evidently, f_k satisfies conditions (a),(b) and (c).

To complete the proof, it remains to verify that f_k is a straight-line embedding, i.e., no two segments cross each other. A slightly stronger statement follows by straightforward induction.

Lemma 1.5. *Let $f_{k-1} = (x_{k-1}, y_{k-1})$ be the straight-line embedding of $G_k - 1$, defined above, and let $\alpha_1, \alpha_2, \ldots, \alpha_m \geq 0$. For any $1 \leq i \leq k-1$, $1 \leq j \leq m$, let*

$$x(v_i) = x_{k-1}(v_i) + \alpha_1 + \alpha_2 + \cdots + \alpha_j \text{ if } u_j \preceq v_i \prec u_{j+1},$$

$$y(v_i) = y_{k-1}(v_i).$$

Then $f'_{k-1} = (x, y)$ is also a straight-line embedding of G_{k-1} .

The claim is trivial for $k = 4$. Assume that it has already been confirmed for some $k \geq 4$, and we want to prove the same statement for G_k. The vertices of the exterior face of G_k are $u_1, \ldots, u_p, v_k, u_q, \ldots, u_m$. Fix now any nonnegative numbers $\alpha(u_1), \ldots, \alpha(u_p), \alpha(v_k), \alpha(u_q), \ldots, \alpha(u_m)$. Applying the induction hypothesis to G_{k-1} with $\alpha_1 = \alpha(u_1), \ldots, \alpha_p = \alpha(u_p), \alpha_{p+1} = \alpha(v_k) + 1, \alpha_{p+2} = \cdots = \alpha_{k-1} = 0, \alpha_q = \alpha(u_q) + 1, \alpha_{q+1} = \alpha(u_{q+1}), \ldots, \alpha_m = \alpha(u_m)$, we obtain that the restriction of f'_k to G_{k-1} is a straight-line embedding. To see that the edges of G_k incident to v_k do not create any crossing, it is enough to notice that f_k and f'_k map $\{u_{p+1}, \ldots, u_{q-1}\}$ into congruent sets. □

2 Online competitive algorithms

2.1 Baby example

As an example, consider the following problem. When a baby is born, the parents need a *baby scale* to measure how much she eats. To get a baby scale, they have three options.
1) Buy one for 30€.
2) Rent one for 5€/month.
3) Borrow one from a friend.

Let us rule out the mathematically less fascinating third option and suppose they only have the first two options. It is not hard to decide which to choose *if* they know for how long they need the scale; for less than five months rent and for more months buy. (Here we suppose that the scale will have no value for them later - we could easily modify this condition by subtracting the price for which they can sell it later from the initial price.) But what if they have no clue at all? One option would be to guess and calculate some expected values from the probabilities. However, there can be too many factors (how well the baby is gaining weight, number of future children) to make any reasonable estimates. Another option is to try to minimize their later regrets, to make sure they could not have done much better.

For example if they decide to buy one (30€) and need it for only one month (5€), then their *competitive ratio* is 6:1 (compared to the best possible choice they could have made). However, if they decide to rent and are blessed with many children and, say, 18 months of scale usage (90€), their ratio becomes 3:1 and could be even worse. So what should they do to minimize the competitive ratio?

The answer first might seem counterintuitive but the best is to mix the above strategies - first rent for a while, then buy. After a little thinking, we can realize that in fact this is the only thing that makes sense and turns out to be not that a crazy idea after all. Now the only thing left to decide is for how long to rent before buying.

Suppose we rent for R month and then buy if we still need the scale. This way we spend $5i$ if we need it for $i \leq R$ months and $5R + 30$ if we need it for at least $R + 1$. The best option would be either to rent the whole time (for $5i$ if we need it for $i \leq 6$ months) or to buy immediately (for 30 if we need it for at least 6 months). Our ratio against the renting option is worst if we need the scale for exactly $R + 1$ months, in this case we get $\frac{5R+30}{5R+5}$. Our ratio against the buying option is of course also worst if we need the scale for at least $R+1$ months, in this case we get $\frac{5R+30}{30}$. So our goal is to minimize $max(\frac{5R+30}{5R+5}, \frac{5R+30}{30})$ by suitably choosing R. For this we solve $\frac{5R+30}{5R+5} = \frac{5R+30}{30}$ which gives $R = 5$, so we have to buy in the sixth month, which is exactly what we would have done with my wife, but we needed the scale for only five month. So with the next child, we buy from the start...

2.2 k-server problem

In the *k-server problem* we control k servers each of which occupies one point in a given finite metric space from which it can move to another one for the cost of the distance between them. There is a series of requests, each of which is a point where we have to move a server (if there is no server present there at the moment). Our goal is to keep our total cost as small as possible. Since we do not know anything against the requests, the best we can try is to minimize the competitive ratio of our algorithm against the cost of what would have been the best sequence of moves, known as the *offline optimum*. It is conjectured that there is an algorithm that is k-competitive[1] but the best known algorithm is only $2k-1$-competitive. An interesting special case is when all distances are the same is called the *k-paging problem*. First we show that already in this case we cannot hope to have a $< k$-competitive algorithm.

Claim 2.1. *No online algorithm can achieve a better competitive ratio than k for the k-paging problem if the metric space has at least $k+1$ points.*

Proof. Suppose that the space has exactly $k+1$ points (if it has more, we never request them). Every time we request the point that has no server on it (no optimal algorithm would put two servers to the same point, so we can suppose that there is exactly one such point). This way after R requests, the cost of the algorithm is R. However, the best choice would be at each step to move the server whose location would be requested the latest, so after at least $k-1$ further requests. Thus the offline optimum is at most $\lceil \frac{R}{k} \rceil$. □

Next we present an algorithm that for a space with $k+1$ points achieves a competitive ratio of k. Denote by $D(i)$ the distance traveled by server i before the request and by $d(i)$ the distance of server i from the requested location. The algorithm called $BALANCE$ has the following simple rule:

Always move the server for which $D(i)+d(i)$ is minimized.

So if e.g. we have three servers, the first has traveled 3, the second 4 and the third 6 units until the query, which is at distance 4 from the first, at distance 2 from the second and at distance 1 from the third server. In this case $BALANCE$ moves the second server, as that gives a minimum distance of 6 after the move, while the other two would give 7.

Proof. We can suppose that the request is always the only unoccupied location. First we need to make some definitions. Define $d(i,j)$ as the distance between location i and j. Let R^t be the t-th request. Let opt_i^t be the offline optimum of the first t requests that has no server on location i (if $R^t = i$, i.e. the last request was i, then an extra move must be made after it to move away the server from it). Finally, let D_i^t be the distance traveled by the server at location i after t requests (if $i \neq R^{t+1}$, since that place is unoccupied).

Observation 2.2. *$opt_i^{t+1} = opt_i^t$ if $i \neq R^{t+1}$ and $opt_{R^{t+1}}^{t+1} = min_{i \neq R^{t+1}} opt_i^t + d(i, R^{t+1})$.*

Lemma 2.3. *For every $i \neq R^{t+1}$ we have $D_i^t \leq opt_i^t$.*

Proof. We prove this by induction on t. Let $h = R^{t+1}$ and m denote the location for which $D_m^t + d(m,h)$ is minimal (in fact $m = R^{t+2}$). If $i \neq m,h$, then $D_i^{t+1} = D_i^t$ and since $opt_i^{t+1} \geq opt_i^t$, we are done. Otherwise, we have $D_h^{t+1} \leq D_i^t + d(i,h)$ for all $i \neq h$, by the choice of the server we moved to the empty position. But using induction we have

[1] Here in the definition of the ratio we are interested in the asymptotic behavior and ignore additive constants.

$D_i^t + d(i,h) \leq opt_i^t + d(i,h)$ and using the previous observation there is an $i \neq h$ for which $opt_i^t + d(i,h) = opt_h^{t+1}$. Putting the inequalities together, we get exactly what we wanted, $D_h^{t+1} \leq opt_i^t + d(i,h)$. $\square$

From here the proof of the theorem follows from $\sum_i D_i^t \leq \sum_i opt_i^t \leq k \cdot (\textit{offline optimum} + \textit{largest distance})$. $\square$

2.3 Randomization

Another, very interesting problem emerges if we allow *randomized* online algorithms. Here we can measure the competitiveness depending on what kind of offline optimum we take. We imagine that the requests are given by some adversary and we distinguish the three following types.

Oblivious: The requests are generated in advance.

Adaptive Online: The requests are generated depending on the moves so far but the adversary must also make its moves online.

Adaptive Offline: The requests are generated depending on the moves so far and the adversary can decide its moves after all the requests are made.

By definition, we have the following relations among the respective competitive ratios: $\mathcal{C}_{OB} \leq \mathcal{C}_{ADON} \leq \mathcal{C}_{ADOFF} \leq \mathcal{C}_{DET}$.

While $\mathcal{C}_{OB}$ can be much smaller than $\mathcal{C}_{DET}$ (e.g. about $\log k$ for the k-paging problem with $k+1$ locations), the other two quantities are not that far. We can prove this through a few simple statements.

Claim 2.4. *If for G and H algorithms we have $\mathcal{C}_{ADON}(G) \leq \alpha$ and $\mathcal{C}_{OB}(H) \leq \beta$, then $\mathcal{C}_{ADOFF}(G) \leq \alpha\beta$.*

Claim 2.5. *If there are finitely many options at each request, then $\mathcal{C}_{ADOFF} = \mathcal{C}_{DET}$.*

Corollary 2.6. *If there are finitely many options at each request, then $\mathcal{C}_{DET} \leq (\mathcal{C}_{ADON})^2$.*