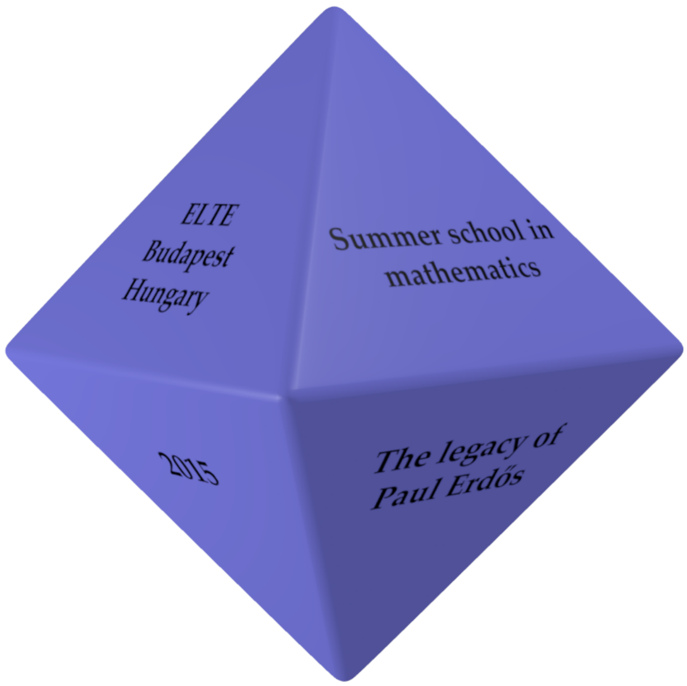
Summer School in Mathematics
The legacy of Paul Erdős

Eötvös Loránd University, Budapest

8-12 June, 2015



Special thanks go to Wiebke Höhn and Max Klimm for providing the original template of this guide book.

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Welcome to Summer School in Mathematics 2015!

In this short guide we would like to provide you with some basic information about practical issues as well as a rather incomplete list of sights, museums, restaurants, bars and pubs.

If you have any question concerning the summer school or your stay in Budapest, do not hesitate to ask us!

We wish you a pleasant stay in Budapest!

The Organizers

Venue

The summer school takes place in the Southern building ("Déli tömb") at the Lágymányosi Campus of the Eötvös Loránd University. The lectures are given in room 4-710, refreshments are provided in room 4-713.

Internet access

Wireless internet connection is available in the lecture room. If you need a scanner or a printer, please contact the organizers.

Lunch

There are several places around the university where you can have lunch. A cafeteria with a limited menu (including vegetarian choices) is in the northern building of the university. For a more complete list of restaurants, see the section "Around the campus".

Contact

If you have any questions or problems, please contact the organizers or the lecturers.

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Getting around by taxi

The taxi fares are uniformly calculated as follows: the base fee is 450 HUF with an additional 280 HUF/km or 70 HUF/min. Altogether, getting around by taxi is rather expensive and you might want to consider using public transport instead.

Getting around by bike

You can rent bikes for 2500 - 3500 HUF per day at many places. Here are some possibilities:

- **Budapestbike** - http://www.budapestbike.hu/
- **Yellow Zebra Bikes and Segways** - http://www.yellowzebrabikes.com/
- **Bikebase** - http://bikebase.hu/home
- **Bestbikerental** - http://www.bestbikerental.hu/

Since last year a large bicycle rental network (so called "Bubi") with many rental and return locations is run as part of the public transport system. You will find many automated rental places mostly in the downtown areas. If you want to rent a bicycle, you will have to pay a one week access fee of 2000 HUF plus the usage fee which is free for the first 30 minutes. A deposit is also required. Please, note that the fares are designed to encourage short term use. For the fares and further details see their website:

- **MOL Bubi** - http://molbubi.bkk.hu/dijszabas.php

Getting around by public transport

The public transportation system in Budapest is a favourite internal travel option for a number of Budapest visitors. The system is efficient, inexpensive and runs throughout all of the major tourist areas of Budapest. The system consists of a combination of the bus, trolley-bus, tram, metro, and train lines and is streamlined so that tickets for all of them can generally be purchased at the same locations.

All regular transportation services stop around midnight (varies by route). However, night buses (blue coloured buses, marked with black in the schedule, numbered 900-999 and tram line 6) replace the metro lines, major tram and bus routes and run through the night until normal service resumes in the morning. Separate schedules for night and day buses are posted at every stop. In the inner areas buses run very frequently (appr. 10-15 min.) Please note: there's front door boarding-only on most lines, except tram 6 and articulated buses.

Budapest currently has four metro lines - M1 (Yellow), M2 (Red), M3 (Blue) and M4 (Green). The Yellow line is the oldest underground transportation line in continental Europe and retains much of its old-fashioned charm. Lines M1, M2 and M3 meet at Deák Ferenc Tér in central Pest. Line M4 opened on 28 March 2014 and connects the Kelenföld railway station located in Buda, and the eastern Keleti station in Pest. Trains run frequently (3-5 minutes between rush hours and weekend, 1-3 minutes in rush-hours, 10 minutes late night).

Budapest has an extensive system of above-ground trams. The most useful lines for tourists are the famous 4 and 6, which follows the large ring road that encircles the Budapest city center and crosses Margaret bridge before terminating at Széll Kálmán Tér on the Buda side on the North - and Petőfi bridge before it terminates at Móricz Zsigmond Körtér, also on the Buda side; no 47 and 49, which runs through central Pest and across the river to Hotel Gellért; no. 2, which follows the Danube River on the Pest side; and no. 19, which follows the Danube River on the Buda Side. Please, note that the ongoing reconstruction work on Széll Kálmán tér makes commuting through this important interchange point somewhat difficult.

Bus lines of use to most tourists are the 7 and 107 which connect the busy Keleti railway station and the area around the Kelenföld railway station on the Buda side. Some other notable places that they stop along is Blaha Lujza tér (connection to the red M2 metro line, also trams 4 and 6), Ferenciek tere (very near Váci utca), and in front of the Rudas bath and the Gellért Hotel both on the Buda side. Bus 86 is also very useful as it has a stop near the Gellért Hotel and runs along the river bank on the Buda side. Bus 27 takes you to the top of Gellért Hill from Móricz Zsigmond körtér while bus 26 connects Nyugati station (Metro line 3) and Árpád bridge (on the same metro line) via Margaret Island.

All public transport in Budapest is run by the company BKK. Connections can be easily checked at http://www.bkk.hu/en/timetables/ or by using the convenient smart phone apps available for **Android** or **iPhone**. Online information - based on onboard GPS devices - about position of buses and timetable of bus, tram and trolley stops is available at http://futar.bkk.hu. The map also contains information about the number of available bicycles at the Bubi stations.

On the metro lines, **tickets need to be bought and validated before boarding** while on buses and trams you have to validate your ticket on the spot. For a complete list of tickets and conditions see http://www.bkk.hu/en/prices/.

Single ticket 350 HUF
Valid for one uninterrupted trip without change. On the metro lines the ticket must be validated before the start of the trip; on other vehicles immediately after boarding or after the vehicle has departed. Validity period is 60 minutes after stamping; it is 120 minutes on night services.

Block of 10 tickets 3000 HUF
You can buy 10 tickets in a block with some discount compared to buying 10 single tickets separately.

Transfer ticket 530 HUF
Valid for one trip with one change. Trip interruptions - other than changes - and return trips are not permitted. The ticket must be validated at the printed number grids at either end: first when starting a trip at one end and at the other end when changing, with the exception of changes between metro lines.

Short section metro ticket for up to 3 stops 300 HUF
Valid on the metro for one short trip of up to 3 stops for 30 minutes after validation. Trip interruptions and return trips

are not permitted.

Single ticket for public transport boat 750 HUF
24-hour, 72-hour, 7-day travelcards and Monthly Budapest pass is valid on weekdays.

Budapest 24-hour travelcard 1650 HUF
Valid for 24 hours from the indicated date and time (month, day, hour, minute) for an unlimited number of trips.

5/30 BKK 24-hour travelcard 4550 HUF
The 5/30 BKK travelcard consists of 5 slips, each with a validity period of 24 hours. The block can be purchased with any starting day with a validity period of 30 days from the starting day.

Budapest 72-hour travelcard 4150 HUF
Valid for 72 hours from the indicated date and time (month, day, hour, minute) for unlimited number of trips on the public transport services ordered by BKK on tram, trolleybus, underground, metro, cogwheel railway on the whole length of the lines on all days; on the whole length of boat services but only on working days.

Budapest 7-day travelcard 4950 HUF
Valid from 00:00 on the indicated starting day until 02:00 on the following 7th day for an unlimited number of trips. The travelcard is to be used only by one person; it is non-transferable as it is issued specifically for the holder.

Monthly Budapest pass for students 3450 HUF
Valid from 0:00 of the indicated optional starting day to 2:00 of the same day of the following month. Valid for students in higher education together with a Hungarian or EU or ISIC student ID.

Giraffe Hop On Hop Off

Giraffe Hop On Hop Off tours offer 2 bus, 1 boat and 1 walking tour in Budapest for tourists. They pass several sights on their way; the RED and YELLOW lines are audio guided in 20 languages and the BLUE boat line is audio guided in English and German. The ticket is valid on the day of the first departure while the next day is free.

Things to do in Budapest

- Walk along the Danube on the Pest side between Elisabeth Bridge and Chain Bridge. Then cross the Danube and continue towards Margareth Bridge to see the Parliament Building from Batthyány tér.
- Take Metro 1 from Vörösmarty tér to Hősök tere and see the monument there. You may take a walk in the City Park (Városliget) or go to the Zoo or the Museum of Fine Arts.
- After 7:00pm take a short walk along Ráday utca from Kálvin tér. You may want to enter one of the cafés or restaurants.
- Go to the Great Market Hall on Vámház körút, close to Liberty Bridge (Szabadság híd). After all, this is one of the few things Margareth Thatcher did when she visited Budapest in the 1980's (she bought garlic :)).
- Go to Gellért Hill to get a glimpse of the city from above.
- Go to a concert in one of the major or smaller concert halls, churches or open air locations. Some of them are free.
- Go to Margaret Island and see the fountain on the southern end or the music tower on the northern end of the island. In between you will find a garden of roses and a small zoo.
- Go at night to the Palace of Arts (across Eötvös University, close to Rákóczi Bridge on the Pest side) and enjoy the view of the National Theatre or of the glass walls of the Palace of Arts.
- See the bridges at night. You get a good view from Castle Hill.
- Go to some of the baths in Budapest (Széchenyi bath in the City park, Rudas bath at the Buda side of Elisabeth Bridge or Gellért bath in Hotel Gellért at the Buda side of Liberty Bridge).

Words of caution

- Don't go to a restaurant or café without checking the price list first. A reasonable dinner should not cost you more than 20 Euros (6000 HUF). (Of course you may be willing to pay more but you should know in advance.)
- Don't leave your valuables unattended, especially not in places frequently visited by tourists. Be aware of pickpockets on crowded buses or trams.
- Budapest is relatively safe even at night, nevertheless if possible, try to avoid being alone on empty streets at night. Some pubs should also be avoided.
- Don't carry too much cash with you: direct payment banking cards and most credit cards are widely accepted. If you withdraw money from a banking machine, be careful and try to do it in a public place.
- If you get on a bus, tram, trolley or metro, usually you have to have a pass or a prepaid ticket which you have to validate upon boarding (or when entering the metro station). Most of the tickets are valid for a single trip only (even if it is only for a short distance). If you get a pass for a week, you have to enter on the ticket the number of a photo id (passport, id card) which you have to carry with you when using the pass. - On some buses you may get a single ticket from the driver but be prepared to have change with you. Even tickets bought from the driver have to be validated.

Shopping centres

Mammut. 1024 Budapest, II. district, Lövőház utca 2, +36 1 345 8000 `www.mammut.hu`

Westend City Center. 1062 Budapest, VI. district, Váci út 1-3, +36 1 238 7777 `www.westend.hu`

Corvin Plaza. 1083 Budapest, VIII. district, Futó utca 37, +36 1 301 0160 `corvinplaza.hu`

Arena Plaza. 1087 Budapest, VIII. district, Kerepesi út 9, +36 1 880 7000 `www.arenaplaza.hu`

Allee. 1117 Budapest, XI. district, Október huszonharmadika utca 8-10, +36 1 372 7208 `www.allee.hu`

Market halls

Batthyány téri Vásárcsarnok. 1011 Budapest, I. district, Batthyány tér 5

Rákóczi téri Vásárcsarnok. 1084 Budapest, VIII. district, Rákóczi tér 7-9, Mon 6:00am–4:00 pm, Tue–Fri 6:00am–6:00pm, Sat 6:00am–1:00pm

Vámház körúti Vásárcsarnok. 1093 Budapest, IX. district, Vámház körút 1-3, Mon 6:30am–5:00 pm, Tue–Fri 6:30am–6:00pm, Sat 6:30am–3:00pm

Fehérvári úti Vásárcsarnok. 1117 Budapest, XI. district, Kőrösi J. utca 7-9, Mon 6:30am–5:00 pm, Tue–Fri 6:30am–6:00pm, Sat 6:30am–3:00pm, Sun 7:00am–2:00pm

Supermarkets

CBA	`www.cba.hu/uzletek`

CBA Déli ABC Nagyáruház. 1013 Budapest, I. district, Krisztina krt. 37, Mon–Sat 6:00am–10:00pm, Sun 6:00am–8:00pm

CBA Ferenciek Tere. 1053 Budapest, V. district, Ferenciek tere 2, Mon–Fri 6:00am–10:00pm, Sat 7:00am–10:00pm, Sun 8:00am–8:00pm

CBA Élelmiszer. 1051 Budapest, V. district, József Attila u. 16, Mon–Fri 6:30am–9:00pm, Sat 7:00am–8:00pm

CBA Millenium Príma élelmiszerüzlet. 1061 Budapest, VI. district, Andrássy út 30, Mon–Fri 7:00am–10:00pm, Sat 8:00am–8:00pm, Sun 9:00am–4:00pm

CBA Rákóczi út. 1074 Budapest, VII. district, Rákóczi u. 48-50, Mon–Fri 6:00–10:00pm, Sat 7:00am–10:00pm

Körúti Élelmiszer. 1075 Budapest, VII. district, Károly krt. 9, Mon–Fri 6:30am–9:30pm, Sat 7:00am–8:00pm, Sun 7:00am–7:00pm

Görög Csemege. 1085 Budapest, VIII. district, József krt. 31, Mon–Fri 6:00am–10:00pm, Sat 8:00am–10:00pm, Sun 9.00am–9.00pm

CBA Prima. 1085 Budapest, VIII. district, József krt 84, Mon–Sat 6:00am–9:30pm

Corvin Átrium CBA. 1085 Budapest, VIII. district, Futó utca 37, Mon–Fri 7:00am-10:00pm, Sat 7:00am–8:00pm

Bakáts csemege. 1092 Budapest, IX. district, Bakáts tér 3, Mon–Fri 6:00am–7:30pm, Sat 7:00am–1:00pm

Szatócs delikátesz. 1111 Budapest, XI. district, Bartók Béla út 32, Mon–Fri 6:30am-8:00pm, Sat 7:00am–6:00pm

CBA Élelmiszer. 1111 Budapest, XI. district, Karinthy Frigyes u. 30-32, Mon–Fri 6:30am–9:00pm, Sat 7:00am–2:00pm

CBA Élelmiszer. 1117 Budapest, Kőrösi József u. 7-9., Mon 6:30am–5:00pm, Tue–Fri 6.30am–6.00pm, Sat 6:30am–3:00pm, Sun 7:00am–2:00pm

Spar/Interspar	www.spar.hu

SPAR. 1011 Budapest, I. district, Batthyány tér 5-6, Mon–Fri 7:00am–9:00pm, Sat 8:00am–8:00pm

SPAR. 1024 Budapest, II. district Lövőház utca 2-6, Mon–Sat 6:30am–10:00pm

City SPAR. 1052 Budapest, V. district, Károly körút 22-24, Mon–Sat 7:00am–10:00pm

SPAR. 1066 Budapest, VI. district, Teréz körút 28, Mon–Sat 6:30am-10:00pm

SPAR. 1066 Budapest, VI. district, Nyugati tér 1-2, Mon–Sat 6:30am-10:00pm

SPAR. 1073 Budapest, VII. district, Erzsébet körút 24, Mon–Sat 7:00am–10:00pm

City SPAR. 1076 Budapest, VII. district, Thököly út 8, Mon–Sat 6:30am–10:00pm

SPAR. 1085 Budapest, VIII. district, Blaha Lujza tér 1, Mon–Sat 6:30am–10:00pm

City SPAR. 1092 Budapest, IX. district, Ráday utca 32, Mon–Sat 7:00am–10:00pm

City SPAR. 1095 BUDAPEST, IX. district, Mester utca 1, Mon–Sat 6:30am–10:00pm

SPAR. 1095 Budapest, IX. district, Soroksári út 1, Mon–Sat 6:30am–9:00pm

SPAR Market. 1111 Budapest, XI. district, Bartók Béla út 14, Mon–Sat 6:00am–10:00am

SPAR. 1117 Budapest, XI. district, Irinyi József utca 34, Mon–Sat 7:00am–8:00pm

INTERSPAR. 1117 Budapest, XI. district, Október 23-a utca 8-10 (in the basement of the Allee shopping center), Mon–Sat 7:00am–10:00pm, Sun 8:30am–8:00pm

SPAR. 1123 Budapest, XII. district, Alkotás utca 53, Mon–Sat 7:30am–10:00pm

Penny Market	www.penny.hu

Penny Market. 1085 Budapest, VIII. district, József Körút 45, Mon–Wed 6:00am–9:00pm, Thu–Sat 6:00am–10:00pm

ALDI	www.aldi.hu

ALDI. 1053 Budapest, V. district, KOssuth Lajos utca 13, Mon–Sun 7:00am–10:00pm

ALDI. 1054 Budapest, V. district, Báthory utca 8, Mon–Sun 7:00am–9:00pm

ALDI. 1081 Budapest, VIII. district, Rákóczi út 65, Mon–Sat 7:00am–10:00pm

ALDI. 1093 Budapest, IX. district, Vámház körút 1-3, Mon–Sat 6:00am–9:00pm, Sun 8:00am–9:00pm

ALDI. 1094 Budapest, IX. district, Tűzoltó utca 10-16, Mon–Sat 7:00am–10:00pm

Lidl	www.lidl.hu

Lidl. 1061 Budapest, VI. district, Bajcsy-Zsilinszky út 61, Mon–Sat 7:00am–10:00pm

Lidl. 1061 Budapest, VI. district, Király utca utca 112, Mon–Sun 7:00am–10:00pm

Lidl. 1082 Budapest, VIII. district, Leonardo Da Vinci utca 23, Mon–Sat 7:00am–10:00pm

Lidl. 1114 Budapest, XI. district, Bartók Béla út 47, Mon–Sat 7:00am–10:00pm

Tesco	www.tesco.hu

Tesco Astoria Szupermaket. 1088 Budapest, VIII. district, Rákóczi út 1-3, Mon–Sat 6:00am–10:00pm

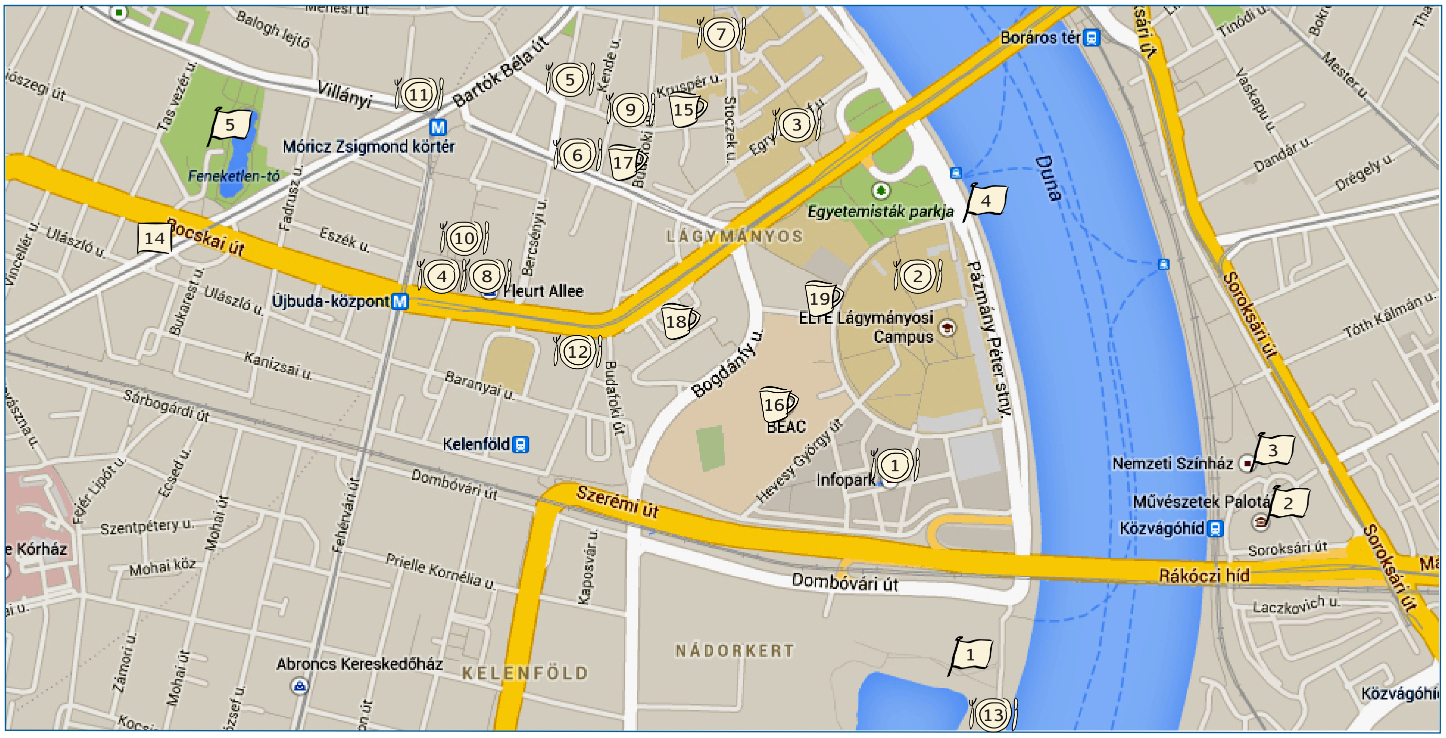
Tesco Expressz. 1088 Budapest, VIII. district, Rákóczi út 20, Mon–Sat 6:00am–10:00pm

Tesco Expressz. 1088 Budapest, IX. district, Kálvin tér 7, Mon–Sat 6:00am–10:00pm

Tesco Arena Plaza Hipermarket. 1087 Budapest, VIII. district, Kerepesi út 9-11, Mon–Sat 6:00am–10:00pm

Tesco Soroksári úti Hipermaket. 1097 Budapest, IX. district, Koppány utca 2-4, Mon–Sat 6:00am–10:00pm

Tesco Új Buda Center Hipermarket. 1117 Budapest, XI. district, Hengermalom út 19-21, Mon–Sun 6:00am-10:00pm



Sights

1 **Kopaszi-gát.** *1117 Budapest, Kopaszi gát 5, Bus 103, 6:00am–10:00pm.*

Kopaszi-gát is a beautifully landscaped narrow peninsula in south Buda, next to Rákóczi Bridge. Nested in between the Danube on one side and a protected bay, it has a lovely beach feel. Kopaszi-gát is also a favourite picnic spot and the park offers lots of outdoor activities from biking to ball games. The sign in the park says it all: "Fűre lépni szabad!", which means "Stepping on the grass is permitted!"

2 **Palace of Arts.** *1095 Budapest, Komor Marcell utca 1, Suburban railway 7.*

The Palace of Arts in Budapest, also known as MÜPA for short (Művészetek Palotája), is located within the Millennium Quarter of the city, between Petőfi and Lágymányosi bridges. It is one of the most buzzing cultural and musical centres in Budapest, and as such one of the liveliest Budapest attractions. Think of the Palace of Arts as a cultural complex, which includes the Festival Theatre, the Béla Bartók National Concert Hall and the Ludwig Museum.

3 **National Theatre.** *1095 Budapest, Bajor Gizi park 1, Suburban railway 7.*

The building lies on the bank of the Danube, in the Ferencváros district, between the Soroksári road, the Grand Boulevard and the Lágymányosi bridge, and is a five-minute walk from the Csepel HÉV (Suburban railway 7). The area of the theatre can be functionally separated into three parts. The central part is the nearly round building of the auditorium and stage, surrounded by corridors and public areas. The second is the U-shaped industrial section around the main stage. The third section is the park that surrounds the area, containing numerous memorials commemorating the Hungarian drama and film industry.

4 **A38 Ship.** *1117 Budapest, a little South from Petőfi bridge, Buda side, Trams 4 and 6 (Petőfi híd, budai hídfő), Mon–Sun 11:00am–4:00am.*

The world's most famous repurposed Ukrainian cargo ship, A38 is a concert hall, cultural center and restaurant floating on the Danube near the abutment of Petőfi Bridge on the Buda-side with a beautiful panorama. Since its opening it has become one of Budapest's most important venues, and according to artists' feedback, one of Europe's coolest clubs.

5 **Feneketlen-tó.** *1114 Budapest, Bus 86, Tram 19, 49.*

Feneketlen-tó, which means bottomless lake, is surrounded by a beautiful park filled with paths, statues and children's playgrounds. The lake is not as deep as its name suggests. In the 19th century there was a brickyard in its place and the large hole dug by the workers filled with water when they accidentally hit a spring. Ever since, locals cherish the park and they come to feed the ducks, relax on the benches or take a stroll around the lake. The lake's water quality in the 1980s began to deteriorate, until a water circulation device was built. The lake today is a popular urban place for fishing.

Restaurants & Eateries

1 **Infopark.** *Next to the university campus, Mon–Fri 8:00am–6:00pm.*

Infopark is the first innovation and technology park of Central and Eastern Europe. There are several cafeterias and smaller sandwich bars hidden in the buildings, most of them are really crowded between 12:00am-2:00pm.

2 **University Cafeteria.** *University campus, Northern building, Mon–Fri 8:00am–4:00pm.*

The university has a cafeteria on the ground floor of the Northern building. You can also buy sandwiches, bakeries, etc here.

3 **Goldmann restaurant.** *1111 Budapest, Goldmann György tér 1, Mon–Fri 11:00am–3:00pm.*

Goldmann is a cafeteria of the Technical University, popular among students for its reasonable offers. Soups are usually quite good.

4 **Fehérvári úti vásárcsarnok.** *1117 Budapest, Kőrösi József utca 7–9, Mon 6:30am–5:00pm, Tue–Fri 6:30am–6:00pm, Sat 6:30am–3:00pm.*
A farmers market with lots of cheap and fairly good native canteens (e.g. Marika Étkezde) on the upper floors. You can also find cheese, cakes, fruits, vegetables etc.

5 **Anyu.** *1111 Budapest, Bercsényi utca 8, Mon–Fri 8:00am–8:00pm.*
Tiny bistro selling home-made soup, sandwiches and cakes.

6 **Turkish restaurant.** *1111 Budapest, Karinthy Frigyes út 26, Mon–Sun 10:00am–0:00am.*
This tiny Turkish restaurant offers gyros, baklava and salads at a reasonable price.

7 **Stoczek.** *1118 Budapest, Stoczek utca 1-3, Mon–Fri 11:00am–3:00pm.*
Stoczek is a cafeteria of the Technical University. It offers decent portions for good price. There are two floors, a café can be found downstairs.

8 **Allee.** *1117 Budapest, Október huszonharmadika utca 8-10, Mon–Sun 10:00am–10:00pm.*
A nearby mall with several restaurants on its 2nd floor.

9 **Íz-lelő étkezde.** *1111 Budapest, Lágymányosi utca 17, Mon–Fri 11:00am–5:00pm.*
Decent lunch for low price, and student friendly atmosphere. Only open from Monday to Friday!

10 **Cserpes Milk Bar.** *1117 Budapest, Október Huszonharmadika utca 8-10, Mon–Sat 7:30am–10:00pm, Sun 9:00am–8:00pm.*
A milk bar just next to the shopping center Allee. Great place for having a breakfast or a quick lunch.

11 **Wikinger Bistro.** *1114 Budapest, Móricz Zsigmond körtér 4, Mon–Sun 10:00am–21:00pm.*
If you are up for hamburgers, Wikinger Bistro offers a huge selection of different burgers.

12 **Hai Nam Bistro.** *1117 Budapest, Október huszonharmadika utca 27, Mon–Sun 10:00am–9:00pm.*
If you like Vietnamese cuisine and Pho, this may be the best place in the city. It is a small place, so be careful, it is totally full around 1:00pm.

13 **Vakvarjú.** *1117 Budapest, Kopaszi gát 2, Mon–Sun 11:30am–11:30pm.*
Vakvarjú can be found on the Kopaszi gát. It is a nice open-air restaurant where you can have lunch and relax next to the Danube for a reasonable price.

Others

14 **Gondola.** *1115 Budapest, Bartók Béla út 69-71, Mon–Sun 10:00am–8:00pm.*
This is a nice little ice cream shop right next to the Feneketlen-tó (Bottomless Lake).

Pubs

15 **Bölcső.** *1111 Budapest, Lágymányosi utca 19, 11:30am–11:00pm.*
Bölcső has a nice selection of Hungarian and Czech craft beers and one of the best all-organic homemade burger of the city. Other than burgers, the menu contains homemade beer snacks such as pickled cheese, hermelin (a typical Czech bar snack), and breadsticks. Bölcső also boasts a weekly menu that makes a perfect lunch or dinner.

16 **Szertár.** *1117 Budapest, Bogdánfy utca 10.*
Szertár is a small pub close to the university campus. It is located at the BEAC Sports Center and offers sandwiches and hamburgers as well. A perfect place to relax after a long day at the university where you can also play kicker.

17 **Pinyó.** *1117 Budapest, Karinty Frigyes út 26, Mon 10:00am–0:00am, Tue–Sat 10:00am–1:00am, Sun 4:00pm–0:00am.*
Squeezed to a basement, Pinyó looks like being after a tornado: old armchairs, kicker table, tennis racket on the wall, ugly chairs and tables. It does not promise a lot, but from the bright side, it is completely foolproof. Popular meeting place among students.

18 **Lusta Macska.** *1117 Budapest, Irinyi József utca 38, Mon–Sat 2:00pm–0:00am.*
Lusta Macska is a cheap pub for students close to the Schönherz dormitory of the Technical University. It is a tiny place with very simple furniture.

19 **Kocka.** *1111 Budapest, Warga László út 1, Mon–Fri 6:30am–7:50pm.*
Nearby the campus, the Kocka Pub is a rather cheap place mainly for students.

SOUTHERN PART OF THE CITY CENTER

Sights

1 **Great Market Hall.** *1093 Budapest, Vámház körút 1-3, Metro 4, Mon 6:00am–5:00pm, Tue–Fri 6:00am–6:00pm, Sat 6:00am–3:00pm.*

Central Market Hall is the largest and oldest indoor market in Budapest. Though the building is a sight in itself with its huge interior and its colourful Zsolnay tiling, it is also a perfect place for shopping. Most of the stalls sell fruits and vegetables but you can also find bakery products, meat, dairy products and souvenir shops. In the basement there is a supermarket.

2 **Károlyi Garden.** *1053 Budapest, Károlyi Mihály utca 16, Metro 2, Tram 47, 49.*

Károlyi Garden is maybe the most beautiful park in the center of Budapest. It was once the garden of the castle next to it (Károlyi Castle, now houses the Petőfi Literature Museum). In 1932 it was opened as a public garden. In the nearby Ferenczy utca you can see a fragment of Budapest's old town wall (if you walk in the direction of Múzeum körút).

3 **Gellért Hill and the Citadel.** *1118 Budapest, Metro 4, Bus 7, 86, 173.*

The Gellért Hill is a 235 m high hill overlooking the Danube. It received its name after St. Gellért who came to Hungary as a missionary bishop upon the invitation of King St. Stephen I. around 1000 a.d. If you approach the hill from Gellért square, you can visit the Gellért Hill Cave, which is a little chapel. The fortress of the Citadel was built by the Habsburgs in 1851 to demonstrate their control over the Hungarians. Though it was equipped with 60 cannons, it was used as threat rather than a working fortification.
From the panorama terraces one can have a stunning view of the city, especially at night. By a short walk one can reach the Liberation Monument.

4 **National Museum.** *1088 Budapest, Múzeum körút 14-16, Metro 3, 4, Tram 47, 49, Tue–Sun 10:00am–6:00pm, 1100 HUF.*

The Hungarian National Museum (Hungarian: Magyar Nemzeti Múzeum) was founded in 1802 and is the national museum for the history, art and archaeology of Hungary, including areas not within Hungary's modern borders such as Transylvania; it is not to be confused with the collection of international art of the Hungarian National Gallery. The museum is in Budapest VIII in a purpose-built Neoclassical building from 1837-47 by the architect Mihály Pollack.

5 **Rudas Gyógyfürdő és Uszoda.** *1013 Budapest, Döbrentei tér 9 (a little South from Erzsébet bridge, Buda side), Buses 5, 7, 8, Trams 18, 19, Mon–Sun 6:00am–8:00pm; Night bath: Fri–Sat 10:00pm–4:00am, 1500–4200 HUF.*

Centered around the famous Turkish bath built in the 16th century, Rudas Spa offers you several thermal baths and swimming pools with water temperatures varying from 16 to 42 Celsius degrees.

6 **Gellért Gyógyfürdő és Uszoda.** *1118 Budapest, Kelenhegyi út 4 (at Gellért tér), Metro 4, Buses 7, 86, Trams 18, 19, 47, 49, Mon–Sun 6:00am–8:00pm, 4900–5500 HUF.*

Gellért Thermal Bath and Swimming Pool is a nice spa in the center of the city.

7 **Sziklatemplom (Cave Church).** *1111 Budapest, Szent Gellért tér, Metro 4, Tram 18, 19, 41, 47, 49, Bus 7, 86, Mon–Sat 9:30am–7:30pm, 500 HUF.*

The Cave Church, located inside Gellért Hill, isn't your typical church with high ceilings and gilded interior. It has a unique setting inside a natural cave system formed by thermal springs.

Restaurants & Eateries

1 **Pagony.** *1114 Budapest, Kemenes utca 10, Mon–Sun 10:00am–1:00am.*

If you are looking for a cool spot in the blazing summer heat of Budapest, look no further. This joint was created by its resourceful proprietor by converting an unused toddler's pool section of the Gellért bath into a trendy pub. While there is no water (yet) in the pools, you can definitely find a table with comfy chairs which are actually in a wading pool.

2 **Hummus Bar.** *1225 Budapest, Bartók Béla út 6, Mon–Fri 10:00am–10:00pm, Sat–Sun 12:00am–10:00pm.*
The famous homemade Hummus can be enjoyed in variety of different dishes. The menu offers everything from a wide variety of quality salads, soups, desserts, meats and vegetarian dishes. The food is prepared with great care using only high quality products, and focusing on the simplicity of preparation - thus allowing affordable pricing.

3 **Főzelékfaló Ételbár.** *1114 Budapest, Bartók Béla út 43–47, Mon–Fri 10:00am–9:30pm, Sat 12:00–8:00pm.*
Főzelékfaló Ételbár boasts a selection centered on főzelék, a Hungarian vegetable dish that is the transition between a soup and a stew, but you can get fried meats, several side dishes, and desserts as well.

4 **Főzelékfaló Ételbár.** *1053 Budapest, Petőfi Sándor utca 1 (Ferenciek tere), Mon–Fri 10:00am–8:00pm, Sat 12:00–8:00pm.*
Főzelékfaló Ételbár boasts a selection centered on főzelék, a Hungarian vegetable dish that is the transition between a soup and a stew, but you can get fried meats, several side dishes, and desserts as well.

5 **Púder.** *1091 Budapest, Ráday utca 8, Sun–Thu 12:00am–1:00am, Fri–Sat 12:00am–2:00am.*
Restaurant and bar with a progressive, eclectic interior that was created by Hungarian wizards of visual arts. Its back room gives home to a studio theatre. Many indoor and outdoor cafés, bars, restaurants and galleries are located in the same street, the bustling neighborhood of Ráday Street is often referred to as "Budapest Soho".

Cafés

6 **CD-fű.** *1053 Budapest, Fejér György utca 1, Mon–Sat 4:00pm–12:00pm.*
As the third teahouse of Budapest, CD-fű is located in a slightly labyrinth-like basement. With its five rooms it is a bit larger than usual, and also gives place for several cultural events.

7 **Hadik kávéház.** *1111 Budapest, Bartók Béla út 36, Mon–Sat 9:00am–11:00pm.*
A lovely place to relax and soak up the atmosphere of prewar years in Budapest. Hadik is a pleasant, old-fashioned café serving excellent food.

8 **Sirius Teaház.** *1088 Budapest, Bródy Sándor utca 13, Mon–Sun 12:00am–10:00pm.*
Sirius teahouse has the perfect atmosphere to have a cup of tea with your friends, but it is better to pay attention to the street numbers, this teahouse is very hard to find, there is no banner above the entrance. Customers can choose from 80 different types of tea.

Pubs

9 **Mélypont Pub.** *1053 Budapest, Magyar utca 23, Mon–Tue 6:00pm–1:00am, Wed–Sat 6:00pm–2:00am.*
Basement pub in the old city center. Homey atmosphere with old furniture.

10 **Trapéz.** *1093 Budapest, Imre utca 2, Mon–Tue 10:00am–0:00am, Wed–Fri 10:00am–2:00am, Sat 12:00am–2:00am.*
Nice ruin pub in an old house behind the Great Market Hall which also has an open-air area. You can watch sports events and play kicker on the upper floor.

11 **Élesztő.** *1094 Budapest, Tűzoltó utca 22, Mon–Sun 3:00pm–3:00am.*
Élesztő is the Gettysburg battlefield of the Hungarian craft beer revolution; it's a like a mixture of a pilgrimage site for beer lovers, and a ruin-pub with 17 beer taps, a home brew bar, a theater, a hostel, a craft pálinka bar, a restaurant and a café.

12 **Mr. & Mrs. Columbo.** *1013 Budapest, Szarvas tér 1, Mon–Sat 4:00pm–11:00pm.*
A nice pub with excellent food and czech beers. Their hermelin is really good.

13 **Aréna Corner.** *1114, Budapest, Bartók Béla út 76, Sun–Thu 12:00am–0:00am, Fri–Sat 12:00am-2:00am.*
A nice place to watch World Cup matches while drinking Czech beer.

Others

14 **Mikszáth square.** *1088 Budapest, Mikszáth Kálmán tér.*
Mikszáth tér and the surrounding streets are home to many cafés, pubs and restaurants usually with nice outdoor terraces. Many places there provide big screens to watch World Cup matches.

Sights

1 **St. Stephen's Basilica.** *1051 Budapest, Szent István tér 1, Tram 2, Guided tours Mon–Fri 10:00am–3:00pm, 1600 HUF, student 1200 HUF.*
This Roman Chatolic Basilica is the most important church building in Hungary, one of the most significant tourist attractions and the third highest building in Hungary. Equal with the Hungarian Parliament Building, it is one of the two tallest buildings in Budapest at 96 metres (315 ft) - this equation symbolises that worldly and spiritual thinking have the same importance. According to current regulations there cannot be taller building in Budapest than 96 metres (315 ft). Visitors may access the dome by elevators or by climbing 364 stairs for a 360° view overlooking Budapest.

2 **Opera.** *1061 Budapest, Andrássy út 22, Metro 1, Tours start at 3:00pm and 4:00pm, 2900 HUF, Students: 1900 HUF.*
The Opera House was opened in 1884 among great splendour in the presence of King Franz Joseph. The building was planned and constructed by Miklós Ybl, who won the tender among other famous contemporary architects.

3 **Kossuth Lajos Square.** *1055 Budapest, Metro 2, Tram 2.*
The history of Kossuth Lajos Square goes back to the first half of the 19th century. Besides the Parliament, other attractions in the square refer to the Museum of Ethnography (which borders the square on the side facing the Parliament) and to several monuments and statues. The square is easily accessible, since the namesake metro station is located on the south side of the square.

4 **Parliament.** *1055 Budapest, Kossuth Lajos tér 1-3, Metro 2, Tram 2, Mon–Fri 8:00am–6:00pm, Sat–Sun 8:00am–4:00pm, 3500 HUF, EU citizens and students 1750 HUF, EU students 875 HUF.*
The commanding building of Budapest Parliament stretches between Chain Bridge and Margaret Bridge on the Pest bank of the Danube. It draws your attention from almost every riverside point. The Gellért Hill and the Castle Hill on the opposite bank offer the best panorama of this huge edifice. The Hungarian Parliament building is splendid from the inside too. You can visit it on organised tours. Same-day tickets can be purchased in limited numbers at our ticket office in the Museum of Ethnography. Advance tickets are available online at `www.jegymester.hu/parlament`.

5 **Buda Castle and the National Gallery.** *1014 Budapest, Szent György tér 2, Bus 16, Funicular, Tue–Sun 10:00am–6:00pm, 900 HUF.*
Buda Castle is the old royal castle of Hungary, which was damaged and rebuilt several times, last time after World War II. Now it houses the Széchényi Library and the National Gallery, which exhibits Hungarian paintings from the middle ages up to now. The entrance to the castle court is free (except if there is some festival event inside). One of the highlights of the court is the Matthias fountain which shows a group of hunters, and the monument of Prince Eugene Savoy. From the terrace of the monument you have a very nice view of the city.

6 **Fishermen's Bastion.** *1014 Budapest, Hess Andras Square 1-3, Bus 16, 16A, 116, all day, tower: daily 9:00am–11:00pm, free, tower: 700 HUF, students: 350 HUF.*
On the top of the old fortress walls, the Fishermen's Bastion was only constructed between 1895-1902. It is named after the fishermen's guild because according to customs in the middle ages this guild was in charge of defending this part of the castle wall. As a matter of fact it has never had a defending function. The architect was Frigyes Schulek, who planned the building in neo-gothic style.

7 **Matthias Church.** *1014 Budapest, Szentháromság tér 2, Bus 16, 16A, 116, Mon–Fri 9:00am–5:00pm, Sat 9:00am–1:00pm, Sun 1:00pm–5:00pm, 1200 HUF, students: 800 HUF.*

Matthias Church (Mátyás-templom) is a Roman Catholic church located in front of the Fisherman's Bastion at the heart of Buda's Castle District. According to church tradition, it was originally built in Romanesque style in 1015. The current building was constructed in the florid late Gothic style in the second half of the 14th century and was extensively restored in the late 19th century. It was the second largest church of medieval Buda and the seventh largest church of medieval Hungarian Kingdom.

8 **Heroes Square.** *1146 Budapest, Hősök tere, Metro 1.*

The monumental square at the end of Andrássy Avenue sums up the history of Hungary. The millennium memorial commemorates the 1000th anniversary of the arrival of the Hungarians in the Carpathian Basin.

9 **Városliget.** *1146 Budapest, Városliget, Metro 1.*

Városliget (City Park) is a public park close to the centre of Budapest. It is the largest park in the city, the first trees and walkways were established here in 1751. Its main entrance is at Heroes Square, one of Hungary's World Heritage sites.

10 **Vajdahunyad vára.** *1146 Budapest, Városliget, Metro 1, Courtyard always open, Castle Tue–Sun 10:00am-5:00pm, Courtyard free, Castle 1100 HUF.*

Vajdahunyad Castle is one of the romantic castles in Budapest, Hungary, located in the City Park by the boating lake / skating rink. The castle, despite all appearances, was built in 1896, and is in fact a fantasy pastiche showcasing the architectural evolution through centuries and styles in Hungary. The castle is the home of several festivals, concerts and the exhibitions of the Hungarian Agricultural Museum.

11 **Museum of Fine Arts (Szépművészeti Múzeum).** *1146 Budapest, Dózsa György út 41, Metro 1, Tue–Sun 10:00am–6:00pm, 1800 HUF.*

The Museum of Fine Arts is a museum in Heroes' Square, Budapest, Hungary. The museum's collection is made up of international art (other than Hungarian), including all periods of European art, and comprises more than 100,000 pieces. The Museum's collection is made up of six departments: Egyptian, Antique, Old sculpture gallery, Old painter gallery, Modern collection, Graphics collection.

12 **Zoo Budapest (Fővárosi Állat és Növénykert).** *1146 Budapest, Állatkerti körút 6-12, +36 1 273 4900, Metro 1, Mon–Thu 9:00am–6:00pm, Fri–Sat 9:00am–7:00pm, 1900 HUF.*

The Budapest Zoo and Botanical Garden is one of the oldest in the world with its almost 150 years of history. Some of its old animal houses were designed by famous Hungarian architects. Nowadays it houses more than 1000 different species. Currently the greatest attraction is Asha, the child elephant.

13 **Holnemvolt Park.** *1146 Budapest, Állatkerti körút 6-12, Metro 1, Mon–Thu 10:00am–6:00pm, Fri–Sat 10:00am–7:00pm, 500 HUF.*

Holnemvolt Park is situated next to the Zoo. It opened recently in the place of the old amusement park. It can either be visited with a Zoo ticket, or separately. Besides some local and some exotic species, it offers entertainment rides, some of which are nearly a hundred years old, and have been inherited from the oldest amusement parks of the city (wooden roller coaster, traditional carousel, enchanted castle).

14 **Great Synagogue.** *, Metro 2, Bus 7,9, Tram 47,49, Sun–Thu 10:00am–4:00pm, 3000 HUF.*

The Great Synagogue in Dohány Street is the largest Synagogue in Europe and the second largest in the world. It can accommodate close to 3,000 worshipers. It was built between 1854 and 1859 in Neo-Moorish style. During World War II, the Great Synagogue was used as a stable and as a radio communication center by the Germans. Today, it's the main center for the Jewish community.

15 **Millenáris.** *1024 Budapest, Kis Rókus utca 16–20, Tram 4, 6 (Széna tér), Fri–Sun 6:00am–11:00pm.*

Located next to the Mammut mall, at the site of the onetime Ganz Electric Works, Millenáris is a nice park and venue for exhibitions, concerts, performances. You can also see a huge hyperbolic quadric and its two reguli.

16 **Batthyány tér.** *1011 Budapest, Metro M2, Tram 19, 41, Bus 86, Suburban railway 5.*

Batthyány square has a great view on the beautiful Hungarian Parliament, one of Europe's oldest legislative buildings, a notable landmark of Hungary.

17 **Erzsébet tér.** *1051 Budapest, Tram 47, 49, Metro 1, 2, 3.*

Erzsébet Square is the largest green area in Budapest's inner city. The square was named after Elisabeth, 'Sisi', wife of Habsburg Emperor Franz Joseph, in 1858. The square's main attraction is the Danubius Fountain, located in the middle of the square, symbolizing Hungary's rivers. One of the world's largest mobile Ferris wheels can be also found on the square. The giant wheel offers fantastic views over Budapest day and night. Standing 65 meters tall, the wheel with its 42 cars is Europe's largest mobile Ferris wheel.

18 **Hungarian Academy of Sciences.** *1051 Budapest, Széchenyi István tér 9, Tram 2.*

The Hungarian Academy of Sciences is the most important and prestigious learned society of Hungary. Its seat is at the bank of the Danube in Budapest.

19 **Playground for adults.** *1124 Budapest, Vérmező.*

A playground for adults? Yes, this indeed exists and can be found in a nice park on the Buda side, close to the castle.

Restaurants & Eateries

1 **Onyx restaurant.** *1051 Budapest, Vörösmarty tér 7, Tue–Fri 12:00am–2:30pm, 6:30pm–11pm; Sat 6:30pm–11:00pm.*

Exclusive atmosphere, excellent and expensive food – Onyx is a highly elegant restaurant with one Michlein Star. Do not forget to reserve a table.

2 **Pizza King.** *1072 Budapest, Akácfa utca 9, Mon–Sun 10:00am–9:00pm.*
During lunchtime on weekdays offers nice menus for 900 HUF, and you can buy cheap pizza there at any time of the day. Also runs pizza takeaways at many locations in the city.

Pubs

3 **Snaps Galéria.** *1077 Budapest, Király utca 95, Mon–Fri 2:00pm–0:00am, Sat 6:00pm–0:00am.*
Snaps is a tiny two-floor pub located in the sixth district. From the outside it is nothing special, but entering it has a calm atmosphere. The beers selection - Belgian and Czech beers- is quite extraordinary compared to other same level pubs.

4 **Noiret Pool and Darts Hall, Cocktail Bar and Pub.** *1066 Budapest, Dessewffy utca 8-10, Mon–Sun 10:00am–0:40am.*
A good place to have a drink and play pool, darts, snooker, or watch soccer.

5 **Szimpla Kert.** *1075 Budapest, Kazinczy utca 14, Mon–Sun 12:00am–3:00am.*
Szimpla Kert (Simple Garden) is the pioneer of Hungarian ruin pubs. It is really a cult place giving new trends. Undoubtedly the best known ruin pub among the locals and the tourists, as well.

Others

6 **Mammut Shopping and Entertainment Centre.** *1024 Budapest, Lövőház utca 2, Fri–Sat 10:00am–9:00pm, Sun 10:00am–6:00pm.*
A twin mall in the heart of Buda.

7 **WestEnd City Center.** *1062 Budapest, Váci út 1-3, Mon–Sun 8:00am–11:00pm.*
A big mall with stores, restaurants etc. and a roof garden.

8 **Corvintető.** *1085 Budapest, Blaha Lujza tér 1, Mon–Sun 6:00pm–6:00am.*
Situated on the rooftop of once-glorious Corvin Department Store, Corvintető offers world-class DJs and concerts every day of the week. Recommended by The New York Times. Do not mess it up with Corvin Negyed, another stop of trams 4 and 6.

MARGARET ISLAND

Margaret Island (Margitsziget) is the green heart of Budapest. It lies in the middle of the Danube between Margaret Bridge and Árpád Bridge. Apart from a couple of hotels and sport facilities, there are no buildings on the Island, it is a huge green park with promenades and benches, great for a date or a picnic. Everyone can find their own cup of tea here: there is the Hajós Alfréd National Sports Swimming Pool, the Palatinus and the running track for the sporty, the petting zoo, the music fountain and the Water Tower for families, and we recommend the Japanese Garden or a ride on a 4-wheel bike car for couples. If you're hungry for culture, check out the open-air stages and the medieval ruins of the Island.

Sights

1 **Entrance of Margit-sziget.** *Budapest, Margit híd, Trams 4 and 6 (Margit híd, Margit-sziget).*

Here you can enter the beautiful Margit-sziget (Margaret Island) on foot. However, you may also take bus 26 to get to the Island.

2 **Kiscelli Múzeum.** *1037 Budapest, Kiscelli utca 108, Bus 17, 160, 260, Tue–Sun 10:00am–6:00pm, 500 HUF.*

Kiscelli Múzeum is located in a beautiful baroque monastery in Old-Buda. It offers exhibitions on the history of Budapest between the 18–21. centuries.

3 **Görzenál.** *1036 Árpád fejedelem útja 125, Bus 29, Suburban railway 5, Mon–Fri 2:00pm–8:00pm, Sat–Sun 9:30am–8:00pm, Mon–Fri 500 HUF, Sat–Sun 900 HUF.*

Görzenál currently is the biggest outdoor roller skating rink in Europe. The skating surface of the Gorzenal Roller Skate and Recreational Park is 14,000 square meters. This rink, which is located in picturesque surroundings along the Danube and Margaret Island, has a skating track as well as park structures for aggressive roller sports and BMX.

4 **Pál-völgyi Cave.** *1025 Budapest, Szépvölgyi út 162, Bus 65, Tue–Sun 10:00am–4:00pm, 1300 HUF.*

An 500-metre long route in a cave with narrow, canyon-like corridors, large level differences, astonishing stone formations, drip stones, glittering calcium-crystals and prints of primeval shells. Even with the 120 steps and the ladder that have to be mounted, the whole tour can easily be fulfilled in normal clothes and comfortable shoes.

5 **Gül baba's türbe.** *1023 Budapest, Mecset utca 14 (entrance: Türbe tér 1), Tram 4, 6, Tue–Sun 10:00am–6:00pm, free.*

The tomb of Gül Baba, "the father of roses", who was a Turkish poet and companion of Sultan Suleiman the Magnificent. He died shortly after the Turkish occupation of Buda in 1541 and his tomb is said to be the northernmost pilgrimage site of the muslims in the world. It is located on a hilltop, surrounded by a beautiful garden which offers a nice view of the city.

Bars

1 **Holdudvar Courtyard.** *1138 Budapest, Margitsziget, Mon–Tue 11:00am–0:00am, Wed 11:00am–2:00am, Thu 11:00am–4:00am, Fri–Sat 11:00–5:00, Sun 11:00–0:00am.*

A great entertainment spot in Budapest where everybody finds something to do: an open-air cinema, café, bar. The gallery exhibits works of contemporary fine art. Holdudvar hosts fashion shows and various cultural events.

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<tr><th></th><th>Monday</th><th>Tuesday</th><th>Wednesday</th><th>Thursday</th><th>Friday</th></tr>
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Erdős – The man and the mathematician</td><td rowspan="6">László Lovász</td><td></td></tr>
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1 Introduction

Modelling large networks by random graphs has been a fastly developing area in the last fifteen years [10, 16]. The main motivation was finding models that can be adjusted to fit real-world networks appropriately (e.g. the internet [2], social networks, biological networks describing the interactions of cells, proteins or other organic compounds). A possible application can be predicting whether two vertices are connected, based on their connections that are already explored. These applications gave rise to many theoretical or mathematical questions as well (see e.g. [4], which will be briefly discussed in Section 5).

Among these random graph models, the one defined by Erdős and Rényi [12] and Gilbert [14], which was defined much earlier, is rather simple and it has a homogeneous structure. Thus, from several points of view, it is not flexible enough for modelling real world networks. However, as we will see later, some interesting phenomena appear even in this model. In addition, if we want to understand stochastic processes (e.g. epidemic spread) on more complicated networks, the starting point can be the Erdős–Rényi model; the answers are often not easy, and the methods may help further research concerning more complex models.

Historically, these random graphs were first mentioned in the middle of the last century to solve some problems in combinatorics [12, 13, Erdős–Rényi, 1959, 1960]. The basic idea, which is now known as the probabilistic method, is the following. We want to prove that a graph (or other combinatorial object) with a given property exists. To do this, we choose a graph at random (with an appropriately chosen distribution), and show that it satisfies the given property with positive probability. This immediately implies that this object must exist. Furthermore, the probabilistic method often goes beyond counting arguments. The original problem was giving a lower bound for so-called Ramsey numbers. More precisely, the goal was showing that the edges of the complete graph on $\sqrt{2}^s$ vertices can be colored red and blue such that neither blue, nor red complete graph of size s appears (in fact, a more precise exponential lower bound was proved). For a large variety of applications of the probabilistic method, see e.g. the book of Alon and Spencer [1].

In this lecture note, first we define Erdős–Rényi random graphs and present some results of Erdős and Rényi about its global properties, like connectivity and sizes of connected components. Then we proceed to subgraph densities, which leads to the notion of graph limits and convergence of dense graphs and random graphs [7, 8, 19]. Finally, we survey some recent achievements about epidemic spread on random graphs.

2 Erdős–Rényi random graphs

The Erdős–Rényi random graph model is a simple graph, i.e. it does not contain neither loops nor multiple edges. It has a fixed number of vertices. The vertices are labelled, but the random structure added is homogeneous. In the original formulation of Erdős and Rényi [13], the number of edges was also fixed, and the graph was defined as follows. Choose positive integers n and m such that $m \le \binom{n}{2}$. Consider all simple graphs on n vertices with m edges, and choose one uniformly at random. Notice that the probability that two given vertices are connected with an edge is $2m/(n(n-1))$. The expected degree of a fixed vertex is $2m/n$.

In our setting and for many applications, it is much more convenient to use a modified version, which includes independence – we will use this one in the sequel. This is also usually called Erdős–Rényi graph, despite the fact that it probably appeared first in the paper of Gilbert [14], also around 1960. In this model, the number of vertices and the probability that a given edge appears are fixed. This probability is the same for all pairs of vertices. This makes the model homogeneous: easier to handle, but not flexible enough for modelling various real-world large networks. The asymptotic behavior of this version is often very similar to the original model of Erdős and Rényi, when the expected number of edges in the latter is equal to m and the number of vertices goes to infinity.

Definition 2.1 (Erdős–Rényi graph)**.** Fix a positive integer n and $0 \le p \le 1$. Let $\mathcal{G}(n,p)$ be (the distribution of) the random graph defined as follows. The vertex set is $\{v_1, v_2, \ldots, v_n\}$. For $1 \le i < j \le n$, vertices v_i and v_j are connected with probability p, independently for all pairs of vertices.

We start with some basic observations about this model.

- We denote by D_i the degree of v_i $(i = 1, \dots, n)$ in a $\mathcal{G}(n,p)$ random graph. Then D_i has distribution Bin$(n-1, p)$, because v_i is connected to all other vertices independently with probability p.
- It follows that the expected degree of a vertex v_i is $(n-1)p$.
- Let M be the total number of edges in a $\mathcal{G}(n,p)$ Erdős–Rényi graph. Then M has binomial distribution with parameters $\binom{n}{2}$ and p:

$$\mathbb{P}(M = m) = \binom{\binom{n}{2}}{m} p^m (1-p)^{\binom{n}{2}-m} \quad \text{for } m = 0, 1, \dots, \binom{n}{2}.$$

- The expected number of edges is as follows.

$$\mathbb{E}(M) = \binom{n}{2} p.$$

- The variance of the number is edges is given by

$$\mathrm{Var}(M) = \binom{n}{2} p(1-p).$$

- The number of edges divided by the number of pairs of vertices is called the *edge density* of the graph. Notice that the expected edge density is equal to p for every n:

$$\frac{\mathbb{E}(M)}{\binom{n}{2}} = p.$$

3 Connected components

Some questions about the global properties of an Erdős–Rényi random graph $\mathbb{G} \sim \mathcal{G}(n,p)$ are the following. What are the sizes of the connected components of $\mathbb{G}$? What is the size of the largest (or the second largest) component? What is the probability that $\mathbb{G}$ is connected, i.e. there is a consecutive path between any two vertices? How this probability depends on p and n?

For the last one, we can say something about how it depends on p. The following is a reasonable guess: given n, the larger p is, the larger the probability that $\mathbb{G}$ is connected is. In fact, this holds in a more general sense. Consider a monotone graph property: if a graph satisfies the property, and we add some edges to it, then it still satisfies the property (connectivity is an easy example). Then, the larger p is, the larger the probability that $\mathbb{G}$ satisfies the property.

The questions about connected components become much more interesting if we let the probability p depend on n. More precisely, fix a sequence of probabilities (p_n), and let $\mathbb{G}_n$ be a $\mathcal{G}(n, p_n)$ Erdős–Rényi random graph. The key step is looking at the expected degree of a given vertex, which is $p_n(n-1)$. The question is whether this quantity is smaller or larger than 1. To put it in another way, if we start exploring the graph from a given vertex, the expected number of descendants of the actual vertex is smaller or greater than 1. We present some of the results of Erdős and Rényi [13] based on [10]. See also [16] to see how this is related to branching processes.

Theorem 3.1 (Component sizes for Erdős–Rényi graphs)**.** *Fix $\lambda > 0$. Let $p_n = \lambda/n$ and $\mathbb{G}_n$ a $\mathcal{G}(n, p_n)$ Erdős–Rényi random graph. We denote by C_n the size of the largest connected component in $\mathbb{G}_n$.*

1. *If $\lambda < 1$, then*

$$\mathbb{P}\left(C_n > \frac{1}{\lambda - 1 - \log \lambda} \log n\right) \to 0 \quad \text{as } n \to \infty.$$

2. *If $\lambda = 1$, then $C_n n^{-2/3}$ is convergent in distribution.*

3. *If $\lambda > 1$, then there exists $\beta > 0$ such that the probability that exactly one component of $\mathbb{G}_n$ is larger than $\beta \log n$ converges to 1 as $n \to \infty$. In addition, C_n/n converges in probability to some $0 < h(\lambda) < 1$.*

That is, if the expected degree is smaller than 1, then all components have at most logarithmic size with high probability. If it is larger than 1, then exactly one giant component appears, which contains a positive proportion of the vertices (but its diameter is logarithmic), and all the other components have at most logarithmic size.

A more refined analysis leads to the following result of Erdős and Rényi [13].

Theorem 3.2 (Connectivity threshold for Erdős–Rényi graphs)**.** *Fix $a > 0$. Let $p_n = a \log n/n$ and $\mathbb{G}_n$ a $\mathcal{G}(n, p_n)$ Erdős–Rényi random graph.*

1. *If $a < 1$, then*
$$\mathbb{P}(\mathbb{G}_n \textit{ is connected}) \to 0 \quad \textit{as } n \to \infty.$$

2. *If $a > 1$, then*
$$\mathbb{P}(\mathbb{G}_n \textit{ is connected}) \to 1 \quad \textit{as } n \to \infty.$$

Bollobás, Janson and Riordan [6] gave similar and even more deeper results for an inhomogeneous random graph model, which includes the Erdős–Rényi model as a particular case.

4 Dense graph limits

In this section, we look at quantities closely related to subgraph densities, which will lead to the notion of graph limits. We will also see how the Erdős–Rényi model and other random graphs are related to this concept. This section is based on the work of Borgs, Chayes, Lovász, Szegedy, T. Sós and Vesztergombi [7, 8, 9, 19]. The examination of dense graph limits started around 2004. We refer to the monograph of Lovász [18, 2012], which provides deep results related to this topic, and presents the development of this area in the last decade. For some more recent achievements on graph limits, see e.g. [11, 15, 17].

In this lecture note, we follow the definitions and formulations of [19]; we will focus on the connection to Erdős–Rényi graphs and the results related to random graphs.

4.1 Homomorphism densities

Definition 4.1 (Homomorphism density)**.** Let F and G be finite simple graphs. We say that a map $\varphi : V(F) \to V(G)$ is a homomorphism if $(\varphi(u), \varphi(v)) \in E(G)$ holds for all $(u, v) \in E(F)$. We denote by $\hom(F, G)$ the number of homomorphisms from F into G. The homomorphism density of F with respect to G is defined by

$$t(F, G) = \frac{\hom(F, G)}{|V(G)|^{|V(F)|}}.$$

To put it in another way, a map φ is a homomorphism if it preserves edges (and it is not required that it maps non-edges to non-edges). Then the homomorphism density is the probability that a random map from $V(F)$ to $V(G)$ (where the vertices are chosen independently, uniformly at random) is a homomorphism. Notice that if F is a single edge, the homomorphism density is slightly different from the edge density defined in Section 2; there we normalized with $\binom{n}{2}$ instead of n^2, where n is the number of vertices of G.

4.2 Homomorphism densities in Erdős–Rényi graphs

Fix $0 < p < 1$, and a finite simple graph F. Let $\mathbb{G}_n$ be a $\mathcal{G}(n,p)$ Erdős–Rényi random graph. We calculate the expectation of the homomorphism density $t(F,\mathbb{G}_n)$, and look at its asymptotic behavior as $n \to \infty$. Given $\mathbb{G}_n$, take a random map $\varphi : V(F) \to V(\mathbb{G}_n)$ (again, the images of the vertices are chosen independently, uniformly at random). Since there are not any loops in $\mathbb{G}_n$, φ can not be a homomorphism if it maps the endpoint of any edge in F to the same vertices. In order to avoid this case, let A_n be the event that the images of the vertices of F are all different at φ. Conditionally on A_n, any pair is connected independently with probability p in $\mathbb{G}_n$. The random map φ is a homomorphism if the endpoints of all edges are connected; this has probability $p^{|E(F)|}$. This is expectation of the probability that a given φ is a homomorphism.

We conclude that

$$\mathbb{E}t(F,\mathbb{G}_n) = p^{|E(F)|}\mathbb{P}(A_n) + \mathbb{P}(\varphi \text{ is a homomorphism}|\overline{A}_n)\mathbb{P}(\overline{A}_n).$$

Notice that

$$\mathbb{P}(A_n) = \frac{n(n-1)\dots(n-|V(F)|+1)}{n^{|V(F)|}} \to 1 \qquad (n \to \infty).$$

Therefore we obtain

$$\lim_{n\to\infty} \mathbb{E}t(F,\mathbb{G}_n) = p^{|E(F)|}.$$

This means that the sequence of expected homomorphism densities with respect to Erdős–Rényi graphs is convergent for all finite simple test graphs F, when p is fixed. We will see later that more is true, and $(\mathbb{G}_n)$ is convergent almost surely in the following sense.

Definition 4.2 (Convergent graph sequence)**.** Let $(G_n)_{n=1}^{\infty}$ be a sequence of finite simple graphs. We say that $(G_n)_{n=1}^{\infty}$ is convergent if, for every finite simple graph F, the sequence $(t(F,G_n))_{n=1}^{\infty}$ is convergent as $n \to \infty$.

A convergent sequence is also called left-convergent. On the other hand, notice that if the edge density tends to 0 as $n \to \infty$, then each limit is equal to zero. Hence this definition provides an interesting notion only for dense graph sequences. Erdős–Rényi random graphs are dense in this sense if p is fixed, at least in expectation. (On the other hand, there are several notions for limits of sequences of sparse graphs.)

4.3 Graphons as limit objects

A theorem of Lovász and Szegedy guarantees that one can find appropriate limit objects to convergent graph sequences, as follows.

Definition 4.3 (Graphon)**.** A symmetric measurable function

$$W : [0,1]^2 \to [0,1]$$

is called a graphon.

Given a finite simple graph F with vertices $1,2,\dots,k$ and a graphon W, the homomorphism density of F with respect to W is defined by

$$t(F,W) = \int_{[0,1]^k} \prod_{(i,j)\in E(F)} W(x_i,x_j)dx_1\dots dx_k. \tag{1}$$

We say that a graph sequence $(G_n)_{n=1}^{\infty}$ converges to the graphon W if

$$t(F,G_n) \to t(F,W) \qquad (n \to \infty)$$

holds for all finite simple graphs F.

Let us see how equation (1) gives back the original definition of homomorphism densities. Given a finite simple graph G with n vertices, we think of $[0,1]^2$ as the union of n^2 congruent squares of side length $1/n$. If v_i and v_j are connected in G, then we define W to be equal to 1 on the square in the ith row and jth column. Otherwise we set W to be equal to 0 on that square. (One can associate graphons to weighted graphs in a similar way.) It is an exercise to check that $t(F,W) = t(F,G)$ holds.

The following theorem gives the connection of graph convergence and graphons.

Theorem 4.4 (Lovász–Szegedy, 2006)**.** *The following are equivalent.*

1. *The sequence $(G_n)_{n=1}^{\infty}$ is convergent according to Definition 4.2.*
2. *There exists a graphon $W : [0,1]^2 \to [0,1]$ that is the limit of $(G_n)_{n=1}^{\infty}$; that is,*

$$t(F,G_n) \to t(F,W) \qquad (n \to \infty)$$

   holds for all finite simple graphs F (as in Definition 4.3).

An important element of the proof is using Szemerédi's regularity lemma and understanding its connection to the notion of cut metric for graphs. In the final step, martingale convergence theorem is applied to construct the appropriate graphon.

4.4 Graphons and random graphs

Lovász and Szegedy [19] proposed the following way to get random graphs from graphons.

Definition 4.5 (Random graph from a graphon)**.** Fix a graphon W and a positive integer n. We define a random graph $\mathcal{G}(n,W)$ on the vertex set $\{1,\ldots,n\}$. Let $X_1,\ldots,X_n$ be independent random variables with uniform distribution on $[0,1]$. For each $1 \le i < j \le n$, we connect vertices i and j with probability $W(X_i,X_j)$, independently.

Lovász and Szegedy proved the following (by using martingale arguments and Azuma's inequality).

Theorem 4.6 (Lovász–Szegedy, 2006)**.** *Fix an arbitrary graphon W. Let $\mathbb{G}_n$ be a $\mathcal{G}(n,W)$ random graph ($n \in \mathbb{N}$). Then the sequence $(\mathbb{G}_n)$ is convergent with probability 1, and its limit is the graphon W.*

Moreover, they could characterize the families of random graph sequences (G_n) arising from some graphon W. The following three conditions together are necessary and sufficient. (a) The distribution of G_n is invariant under relabelling the vertices. (b) If we delete vertex n from G_n, the rest has the same distribution as G_{n-1}. (c) For every $1 < k < n$, the subgraphs induced by $\{1,\ldots,k\}$ and $\{k+1,\ldots,n\}$ are independent.

4.5 Limits of Erdős–Rényi graphs

Consider Definition 4.5 with $W \equiv p$ for fixed $p \in [0,1]$. Then $\mathcal{G}(n,W)$ will have the same distribution as the Erdős–Rényi graph $\mathcal{G}(n,p)$. This leads to the following statement.

Corollary 4.7. *Fix $p \in [0,1]$ and let $\mathbb{G}_n$ be a $\mathcal{G}(n,p)$ Erdős–Rényi random graph for all $n \in \mathbb{N}$. Then $(\mathbb{G}_n)_{n=1}^{\infty}$ converges almost surely to the constant p graphon. That is, for every finite simple graph F the following holds:*

$$\lim_{n\to\infty} t(F,\mathbb{G}_n) = p^{|E(F)|}.$$

This immediately implies the convergence of expected homomorphism densities, which we could verify by a simple argument.

4.6 Convergence of random graphs

Based on the work of Borgs, Chayes, Lovász, T. Sós and Vesztergombi [8], we present some examples of random graph sequences, which converge almost surely to some graphons, although they are not generated from this limit object.

Growing uniform attachment graph

Let (U_n) be a sequence of simple random graphs defined as follows. U_1 is a single vertex. Given U_n, we add a new vertex and we connect every pair of nonadjacent vertices independently with probability $1/n$ to get U_{n+1}. The steps are also performed independently. Then the following holds.

Theorem 4.8. *[8, 2011] The sequence of random graphs (U_n) converges almost surely to the graphon $W(x,y) = 1-\max(x,y)$.*

Notice that the probability that the ith and jth vertex $(i < j)$ are not connected after n steps is equal to $\frac{j}{j+1} \cdot \frac{j+1}{j+2} \cdots \frac{n-1}{n} = \frac{j}{n}$. In addition, these events are independent.

Growing ranked attachment graph

Let (R_n) be a sequence of simple random graphs, as follows. R_1 is a single vertex with label 1. At the nth step, a new vertex with label n is added to get the next graph. For all $1 \le i < n$, the new vertex is connected to vertex i independently, with probability $1-i/n$. Moreover, in the same step, each pair of nonadjacent vertices is connected with probability $2/n$. All these choices are independent of each other.

Theorem 4.9. *[8, 2011] The sequence of random graphs (R_n) converges almost surely to the graphon $W(x,y) = 1-xy$.*

Preferential attachment graph with a fixed vertex set

Following [8], we define the $\mathrm{PAG}(n,m)$ random graph on vertex set $\{v_1,\dots,v_n\}$. First we produce a list of vertices: given $v_1,\dots,v_n,v_{n+1},\dots,v_{n+k}$, we choose one element uniformly from this list of length $n+k$, and add a new copy of this to the end of the list as v_{n+k+1}. We repeat this for $k = 0,\dots,2m-1$. Finally, the list consists of copies of $v_1,\dots,v_n$ in some order. So far, this process is a Pólya urn model with colors $1,\dots,n$. In addition, vertices that occured more times have larger chance to be chosen again; this is the preferential attachment property. To get the graph, we connect vertices v_{n+2k-1} and v_{n+2k} for $k = 1,\dots,m$. Finally, we delete loops, and replace parallel edges by single edges. The following result is due to Ráth and Szakács.

Theorem 4.10. *[20, 2012] Given $c > 0$, the sequence of random graphs $\mathrm{PAG}(n,\lceil cn^2\rceil)$ converges with probability 1 to the graphon $W(x,y) = 1-\exp(-c\ln x \ln y)$.*

5 Epidemic spread on Erdős–Rényi graphs and related models

In this section we briefly summarize some questions and results about epidemic spread on random graphs. There are many possibilities to model the outbreak of a contagious illness on networks; here we focus on the so-called contact process, which may be also called SIS model. (In SIR models, vertices are removed after they get recovered.)

5.1 The contact process

The general setup is the following. First we fix the set of vertices V_n of size n, and choose a random graph $G_n = (V_n, E_n)$ (e.g. according to the Erdős–Rényi model). At the beginning, a single vertex is infected. An infected vertex v recovers and gets susceptible again; the length of the infective period is exponentially distributed and has mean 1. These lengths are independent for different vertices and also for the different infected periods of the same vertex. Let us say that w is a susceptible neighbor of v in G_n. During its infected period, v transfers the disease to w at random moments that occur according to a Poisson process of rate λ. If w is susceptible, it gets infected (and stays infected for a period of exponential length, while it may transfer the disease to its neighbors). Notice that a susceptible vertex w gets infected at rate λk, where k is the actual number of its infected neighbors.

5.2 Erdős–Rényi graphs

In [6], Bollobás, Janson and Riordan considered an inhomogeneous random graph model and answered many questions, for example about connectivity. They formulated the following questions for the case when $\mathbb{G}_n$ is a $\mathcal{G}(n, c/n)$ Erdős–Rényi random graph for some $c > 0$. Recall the concept of large component from Section 1.

- Is there a critical value λ_c such that for $\lambda < \lambda_c$, the expected number of vertices ever infected is bounded (i.e. the bound does not depend on n)?
- Is there a critical value λ_c such that for $\lambda > \lambda_c$, with probability $q > 0$, almost all vertices of the large component become infected, and there is at least one infected vertex for exponentially long time (as a function of n)?

As for the second question, Sivakoff [21] showed that in the case $c\lambda > 1$, the epidemic stays alive for exponentially long time conditionally on the event that it initially infects a positive proportion of the vertices. In fact, he also considered an inhomogeneous network, consisting of components of Erdős–Rényi graphs of a given density which are connected by edges chosen independently at random, but with a smaller probability. The random time when the epidemic is transferred from one community to the other one was investigated.

We highlight the isoperimetric inequality from [21], which is a crucial point of the proof there.

Definition 5.1. Let $G = (V, E)$ be a finite simple graph. For $\varepsilon > 0$, its ε-isoperimetric number is defined by

$$i_\varepsilon(G) = \min\left\{\frac{|\partial U|}{|U|} : U \subset V, |U| \le \varepsilon |V|\right\},$$

where ∂U is the set of egdes that have one endpoint in U and the other one in $V \setminus U$.

Proposition 5.2 (Isoperimetric inequality for Erdős–Rényi graphs, [21])**.** *Let $\mathbb{G}_n \sim \mathcal{G}(n, p)$ be an Erdős–Rényi random graph. Suppose that $np \le 28(\log n)^3$ holds. Then for any fixed $\varepsilon > 0$ we have*

$$\lim_{n\to\infty} \mathbb{P}(i_\varepsilon(\mathbb{G}_n) \le (1-\varepsilon)np - (np)^{2/3}) = 1.$$

5.3 General graph models with given degree sequence

In [3], Barbour and Reinert proved general results in the direction of the questions above for bounded degree graphs; their results also show that if the epidemic spread does not die out soon (which happens with positive probability if λ is large enough), then it follows the next pattern. In the early stages, it is similar to a branching process: each individual has a random number of offsprings (which are different from each other) with a given rate. After that, the epidemic spread can be well approximated by a constant speed deterministic process. The change point between the two stages is also random. Bhamidi, van der Hofstad and Komjáthy [5] achieved similar results for a model where (expected) degrees are not necessarily bounded, but they have finite third moments. Their underlying random graph model is the configuration model; they choose the degrees of the vertices at random, given by independent and identically distributed random variables. Then a graph is chosen uniformly among all simple graphs with this degree distribution. From many points of view, Erdős–Rényi random graphs and configuration models with Poisson degrees show similar behavior.

5.4 Preferential attachment graphs

It might be interesting to compare these results with the ones arising in the family of so-called preferential attachment models. According to the results of Berger, Borgs, Chayes and Saberi [4], the critical value λ_c mentioned above is 0. That is, for arbitrary infection rate $\lambda > 0$, with positive probability, a positive proportion of the vertices will get infected. However, these models are not homogeneous any more, so one has to pay attention to the starting vertex of the epidemic.

More precisely, in the simplest case, the random graph is built up as follows. We start with some initial configuration. At each step, a new vertex is added to the graph. We choose an already existing vertex at random, such that the probability that a given vertex is chosen is proportional to its actual degree. Then this old vertex is connected to the new one with an edge. (It is possible to define the model such that $m > 1$ edges are added at each step; see e.g. [4] for the details.) This is often called the Barabási model [2]. Notice that, typically, old vertices have larger degree, and have higher chance to get new edges in the future. This is why the starting point of the epidemic matters. Loosely speaking, the results of [4] state the following. For an $1 - O(\lambda^2)$ proportion of the vertices, the epidemic started from that vertex has probability $\lambda^{\Theta\left(\frac{\log 1/\lambda}{\log\log 1/\lambda}\right)}$ to survive. But, the average probability of survival for all starting points is $\lambda^{\Theta(1)}$. In any case, this shows that epidemic spread has a positive probability to infect a positive proportion of vertices with arbitrary infection rate, which does not seem to be true for Erdős–Rényi graphs.

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1 Introduction

Given a graph, finding a subgraph whose degrees meet certain prescriptions is a central topic of graph theory. Such problems include the existence of perfect matchings in bipartite or in general graphs, the existence of b-factors, etc. Some of these problems are well-known and can be solved in polynomial time while others proved to be difficult. However, these questions become significantly easier when restricted to complete graphs. In this case, solutions for factorization problems give characterizations of the existence of graphs with given degree sequences.

A finite sequence $\mathbf{d} = (d_1, \ldots, d_n)$ of non-negative integers is called **graphical** or **realizable** if it is the degree sequence of an undirected graph $G = (V, E)$. In this case we say that G **realizes** $\mathbf{d}$. A natural question is the following: when is a given sequence realizable?

Similar questions can be asked about directed graphs. Two sequences $d_1^+, \ldots, d_n^+$ and $d_1^-, \ldots, d_n^-$ of non-negative integers are **realizable** if they are the in- and out-degree sequences of a directed graph $D = (V, A)$, respectively.

The first part of the minicourse discusses some basic result including the Erdős-Gallai theorem solving the graph realization problem, the Havel-Hakimi algorithm, or the Kundu-Lovász theorem.

In the second part we prove an abstract result on covering supermodular functions by simple bipartite graphs. Based on this result, we give a new proof for Ryser's maximum term rank formula and give a characterization of the existence of a wooded hypergraph with given degree sequence and hyperedge sizes.

2 Realizable degree sequences

Given an undirected graph $G = (V, E)$ and a subset $F \subseteq E$ of edges, $F(v)$ denotes the set of edges in F incident to a node $v \in V$. The **degree** of v in F is the cardinality of $F(v)$ plus the number of self-loops in F on v, and is denoted by $d_F(v)$ (in other words, $d_F(v)$ counts the loops in F on v twice). We say that F **covers** a subset of nodes $X \subseteq V$ if $d_F(v) \geq 1$ for every $v \in X$. Let $b : V \to \mathbb{Z}_+$ be an upper bound function. A subset $N \subseteq E$ of edges is called a b-**factor** if $d_N(v) = b(v)$ for every node $v \in V$. For some integer $t \geq 2$, by a t-**factor** we mean a b-factor where $b(v) = t$ for every $v \in V$. The b-factor problem asks for the existence of a b-factor

It is easy to see that graph realization problems are special cases of the b-factor problem in a special graph. For example, $d_1, \ldots, d_n$ is the degree sequence of a simple graph if and only if the complete graph K_n has a b-factor where $v_1, \ldots, v_n$ is the set of nodes and $b(v_i) = d_i$ for $i = 1, \ldots, n$. Although results on b-factors could be applied to obtain necessary and sufficient conditions for $d_1, \ldots, d_n$ being graphical, more transparent conditions can be formulated by other approaches.

2.1 Trees and forests

We start by an easy result characterizing the degree sequences of trees.

Theorem 2.1. *Integers $d_1 \geq \cdots \geq d_n > 0$ are the degrees of a tree if and only if $d_1 + \cdots + d_n = 2n - 2$.*

Proof. Necessity is trivial as a tree on n nodes has $n - 1$ edges and the sum of the degrees of the nodes in a graph equals twice the number of the edges.

Assume now that $d_1 + \cdots + d_n = 2n - 2$. We prove by induction on n. The cases when $n \leq 2$ are trivial, hence assume that $n \geq 3$. Then $d_n = 1$ as otherwise $d_1 + \cdots + d_n \geq 2n > 2n - 2$, a contradiction. Meanwhile, $d_1 > 1$ as $d_1 = 1$ implies $d_1 + \cdots + d_n = n < 2n - 2$, a contradiction. As

$$(d_1 - 1) + d_2 + \cdots + d_{n-1} = 2n - 4 = 2(n - 1) - 2,$$

there exists a tree T with degree sequence $d_1 - 1, \ldots, d_{n-1}$. Add an extra node to T and an edge connecting the new node to the node of T having degree $d_1 - 1$. The tree thus obtained realizes $d_1, \ldots, d_n$. □

Degree sequences of forests can be characterized similarly.

Theorem 2.2. *Integers $d_1 \geq \cdots \geq d_n > 0$ are the degrees of a forest if and only if $d_1 + \cdots + d_n$ is even and $d_1 + \cdots + d_n \leq 2n - 2$.*

Proof. Necessity follows from the facts that a forest on n nodes has at most $n-1$ edges and the sum of the degrees of the nodes in a graph equals twice the number of the edges.

Assume now that $d_1 + \cdots + d_n$ is even and $d_1 + \cdots + d_n \leq 2n-2$. We prove by induction on n. The cases when $n \leq 2$ are trivial, hence assume that $n \geq 3$. Then $d_n = 1$ as otherwise $d_1 + \cdots + d_n \geq 2n > 2n-2$, a contradiction. If $d_1 = 1$ then $d_1 = \cdots = d_n = 1$, and a matching on n nodes satisfies the requirements of the theorem. Assume that $d_1 > 1$. As

$$(d_1 - 1) + d_2 + \cdots + d_{n-1} \leq 2n - 4 = 2(n-1) - 2,$$

there exists a forest F with degree sequence $d_1 - 1, \ldots, d_{n-1}$. Add an extra node to F and an edge connecting the new node to the node of F having degree $d_1 - 1$. The forest thus obtained realizes $d_1, \ldots, d_n$. □

2.2 Graphs

Without further restrictions on the structure of the graph in question, it is rather easy to characterize the existence of a graph realizing a given sequence $d_1, \ldots, d_n$.

Theorem 2.3. *Non-negative integers $d_1 \geq \cdots \geq d_n$ are the degrees of a graph if and only if $d_1 + \cdots + d_n$ is even.*

Proof. Necessity is easy. We prove sufficiency by induction on $\sum_{i=1}^n d_n$. If $\sum_{i=1}^n d_n = 0$ then $d_i = 0$ for $i = 1, \ldots, n$ and the empty graph on n nodes satisfies the conditions of the theorem.

Assume that $\sum_{i=1}^n d_n \geq 2$. If $d_1 \geq 2$, then, by induction, there exists a graph with degree sequence $d_1 - 2, \ldots, d_{n-1}, d_n$, and by adding a loop on the node with degree $d_1 - 2$ we get a realization of the original sequence.

If $d_1 = 1$ then $d_2 = 1$. By induction, there exists a graph with degree sequence $d_1 - 1, d_2 - 1, d_3, \ldots, d_n$, and by adding an edge between nodes with degrees $d_1 - 1$ and $d_2 - 1$ we get a realization of the original sequence. □

2.3 Loopless graphs

Theorem 2.4. *Non-negative integers $d_1 \geq \cdots \geq d_n$ are the degrees of a loopless graph if and only if $d_1 + \cdots + d_n$ is even and $d_1 \leq d_2 + \cdots + d_n$.*

Proof. Necessity is easy. We prove sufficiency by induction on $\sum_{i=1}^n d_n$. We distinguish two cases.

Case 1. $d_1 > d_3$.

In this case $d_1 - 1$ is maximal among $d_1 - 1, d_2 - 1, d_3, \ldots, d_n$. Moreover, $(d_1 - 1) + (d_2 - 1) + d_2 + \cdots + d_n$ is even and $d_1 - 1 \leq d_2 - 1 + d_3 + \cdots + d_n$.

Case 2. $d_1 = d_3$.

In this case $d_1 = d_2 = d_3$ and $(d_1 - 1) + (d_2 - 1) + d_2 + \cdots + d_n$ is clearly even. We claim that $d_3 \leq d_1 - 1 + d_2 - 1 + d_4 + \cdots + d_n$ holds. Indeed, if $d_3 \geq 2$ this follows from $d_1 = d_2 = d_3$, while in case of $d_3 = 1$ the inequality follows as the right hand side is odd.

By the above, $d_1 - 1, d_2 - 1, d_3, \dots, d_n$ satisfies the conditions of the theorem, hence there exists a loopless graph realizing it. By adding a new edge between nodes with degrees $d_1 - 1$ and $d_2 - 1$ we get a loopless graph realizing the original sequence. □

2.4 Simple directed graphs

Theorem 2.5. *Non-negative integers $d_1^+, \dots, d_n^+, d_1^-, \dots, d_n^-$ are in- and out-degrees of a simple directed graph if and only if $\sum_{i=1}^n d_i^+ = \sum_{i=1}^n d_i^-$ and*

$$\sum_{i \in I} d_i^+ \leq \sum_{j \in J} d_j^- + |I|(n - |J|) - |I - J|, \tag{1}$$

where $I, J \subseteq \{1, \dots, n\}$.

Sketch of the proof. Define a network flow problem as follows. Let $A = \{u_1, \dots, u_n\}$ and $B = \{v_1, \dots, v_n\}$ be two sets of nodes. For each $i \neq j$, add a directed edge from u_i to v_j. Let s and t be two extra nodes and extend the digraph by adding edges su_i, v_it $(i = 1, \dots, n)$. The capacity of su_i is set to d_i^- $(i = 1, \dots, n)$, the capacity of v_jt is set to d_j^+ $(j = 1, \dots, n)$, while the capacity of any edge going from A to B is set to 1.

It is not difficult to see that there exists a directed graph satisfying the requirements of the theorem if and only if there exists a feasible flow of value $\sum_{i=1}^n d_i^+ = \sum_{i=1}^n d_i^-$ in the above network. Thus the theorem follows from the Max-Flow-Min-Cut theorem of Ford and Fulkerson. □

2.5 Simple graphs

The fundamental theorem of Erdős and Gallai gives a necessary and sufficient condition for $d_1, \dots, d_n$ to be the degree sequence of a simple graph (a graph without loops and parallel edges). The theorem was published in 1960 by Paul Erdős and Tibor Gallai [1], after whom it is named.

Theorem 2.6 (Erdős and Gallai)**.** *Non-negative integers $d_1 \geq \dots \geq d_n$ are degrees of a simple graph if and only if $d_1 + \dots + d_n$ is even and*

$$\sum_{i=1}^{k} d_i \leq \sum_{i=k+1}^{n} \min\{d_i, k\} + k(k-1). \tag{2}$$

Proof. The left hand side of (2) sums up the degrees of the k highest degree nodes. This value can be bounded as follows: the set of these nodes spans at most $k(k-1)$ edges, while the number of edges having exactly one endpoint among the k highest degree nodes is at most $\sum_{i=k+1}^n \min\{d_i, k\}$, showing necessity of (2).

Now we prove sufficiency.

Lemma 2.7. *Assume that $d_1, \dots, d_n$ are non-negative integers such that $d_1 + \dots + d_n$ is even and there exists a loopless simple directed graph $H = (V, A)$ for what $\varrho_H(v_i) = \delta_H(v_i) = d_i$ $(i = 1, \dots, n)$. Then there exists an undirected simple graph $G = (V, E)$ such that $d_E(v_i) = d_i$ $(i = 1, \dots, n)$.*

Proof. Choose a directed graph $H = (V, A)$ satisfying the conditions of the lemma having a maximum number of directed cycles of length two. Let $\widetilde{H} = (V, \widetilde{A})$ denote the subgraph of H consisting of directed edges $uv \in A$ for what $vu \notin A$. We claim that $\widetilde{H}$ does not contain a directed cycle of even length. Suppose indirectly that $(x_1, \dots, x_{2k})$ is a directed cycle of even length in $\widetilde{H}$. Delete directed edges $x_{2i}x_{2i+1}$ $(i = 1, \dots, k,\ x_{2k+1} = x_1)$ and add edges $x_{2i}x_{2i-1}$ $(i = 1, \dots, k)$. The digraph thus obtained has the same in- and out-degree sequences as H but contains more directed cycles of length two, a contradiction.

The same reasoning shows that $\widetilde{H}$ does not contain a closed trail of even length. Since $\widetilde{H}$ is an Eulerian graph, it is the disjoint union of directed cycles. By the above, these cycles have odd lengths and any two of them are node-disjoint as otherwise their union would form a closed trail of even length.

Assume that C_1 and C_2 are two cycles in the above decomposition of $\widetilde{H}$, and let $C_1 = \{x_0,\dots,x_{2k}\}$ and $C_2 = \{y_0,\dots,y_{2l}\}$. If $x_0y_0 \in A$ then $x_0y_0 \notin \widetilde{A}$ by the above. This implies $y_0x_0 \in A$, so $C_1 \cup C_2 \cup \{x_0y_0, y_0x_0\}$ is a closed trail of even length, a contradiction. Hence $x_0y_0, y_0x_0 \notin A$.

Delete edges $x_0x_1, x_2x_3, \dots, x_{2k}x_0, y_0y_1, y_2y_3, \dots, y_{2l}y_0$ from H and add edges $x_2x_1, \dots, x_{2k}x_{2k-1}, y_2y_1, \dots, y_{2l}y_{2l-1}$, x_0y_0, y_0x_0 to H. The digraph thus obtained has the same in- and out-degree sequences as H but contains more directed cycles of length two, a contradiction.

Thus $\widetilde{H}$ contains at most one cycle. If it contains exactly one, then this is odd and $\sum_{i=1}^n d_i = |A|$ is odd, a contradiction of the assumption. This $\widetilde{H}$ has no edge, that is, H consists of pairs of opposite edges. Replacing each such edge by a single undirected edge we get a simple graph satisfying the requirements of the theorem. □

By Lemma 2.7, it suffices to show that there exists a simple digraph such that $\varrho_H(v_i) = \delta_H(v_i) = d_i$ $(i = 1,\dots,n)$. By Theorem 2.5, this is equivalent to

$$\sum_{i=1}^{k} d_i \le \sum_{j=n-l+1}^{n} d_j + k(n-l) - \min(k, n-l). \tag{3}$$

If $k + l > n$ then (3) is equivalent to

$$\sum_{i=1}^{n-l} d_i \le \sum_{j=k+1}^{n} d_j + k(n-l) - \min(k, n-l),$$

which follows from (3) by replacing k and l by $n-l$ and $n-k$, respectively.

If $k + l \le n$, then

$$\sum_{j=n-l+1}^{n} d_j + k(n-l) \ge \sum_{j=k+1}^{n} \min(d_j, k) + k^2.$$

Hence if

$$\sum_{i=1}^{k} d_i \le \sum_{j=k+1}^{n} \min(d_j, k) + k^2 - k, \tag{4}$$

then (3) is satisfied. On the other hand, (4) is a special case of (3) with choice $m = \max\{j : d_j \le k\}$ and $l = \min\{n-k, m\}$. Hence (4) is necessary and sufficient, concluding the proof. □

2.6 Simple connected graphs

Theorem 2.8. *Non-negative integers $d_1 \ge \dots \ge d_n$ are degrees of a simple connected graph if and only if they satisfy the conditions of the Erdős-Gallai theorem (Theorem 2.6), $d_n > 0$ and $\sum_{i=1}^n d_i \ge 2(n-1)$.*

Proof. Necessity is easy, we prove sufficiency. By Theorem 2.6, there exists a simple undirected graph with degree sequence $d_1,\dots,d_n$. Choose such a graph G with a minimum number of components. We claim that G is connected. Suppose indirectly that G is not connected. By $\sum_{i=1}^n d_i \ge 2(n-1)$, G has a component that contains a cycle. Let G_1 denote this component and let xy be an edge on the cycle. Choose another component G_2 arbitrarily and let uv be one of its edges (G_2 has an edge as there is no isolated node in G by $d_n \ge 1$). Then $G - \{xy, uv\} + \{xu, yv\}$ is a graph with degree sequence $d_1,\dots,d_n$ but having less components than G, a contradiction. □

2.7 Havel-Hakimi algorithm

The following simple procedure is due to Havel [6] and Hakimi [5] and checks whether or not a sequence is graphical.

Theorem 2.9 (Havel and Hakimi). *Non-negative integers $d_1 \geq \cdots \geq d_n$ are realizable as the degrees of a simple graph if and only if the numbers $d_1 - 1, \ldots, d_{d_n} - 1, d_{d_n+1}, \ldots, d_{n-1}$ are realizable.*

Proof. Sufficiency is easy. To prove necessity, it suffices to show that if $d_1, \ldots, d_n$ is graphical then it has a realization in which the node of degree d_n is adjacent to the d_n highest degree nodes. Let G be a simple graph with degrees $d_1, \ldots, d_n$ and assume that there are nodes v_i and v_j such that $1 \leq 1 < j \leq n-1$ and v_n is adjacent to v_j but not to v_i. Since $d_j \leq d_i$, there must be a node $v_k \neq v_i, v_j, v_n$ which is adjacent to v_i but not to v_j. Replace $v_k v_i$ and $v_n v_j$ by $v_n v_i$ and $v_k v_j$, we obtain another simple graph with the same degrees. Repeating this procedure, we get a graph with the required property. This completes the proof. □

2.8 Kundu-Lovász theorem

The following theorem was conjectured in a slightly weaker form by Grünbaum and proved independently by Kundu [7] and Lovász [9].

Theorem 2.10. *A sequence $d_1 \geq d_2 \geq \cdots \geq d_n$ of non-negative integers is the degree sequence of a simple graph containing a matching of size at least ν if and only if both $d_1, \ldots, d_n$ and $d_1 - 1, \ldots, d_{2\nu} - 1, d_{2\nu+1}, \ldots, d_n$ are realizable by a simple graph.*

Proof. The next lemma shows the necessity of the conditions.

Lemma 2.11. *Assume that $d_1 \geq d_2 \geq \cdots \geq d_n$ is realizable by a simple graph containing a matching of size ν. Then it is also realizable by a simple graph containing a matching of size ν covering $d_1, \ldots, d_{2\nu}$.*

Proof. Take a realization $G = (V, E)$ of $d_1, \ldots, d_n$ that has a matching $M \subseteq E$ of size ν covering the maximum number of nodes from $v_1, \ldots, v_{2\nu}$ where node v_i corresponds to degree d_i.

If M covers $v_1, \ldots, v_{2\nu}$ then we are done.

Otherwise there are indices $i \in \{1, \ldots, 2\nu\}$ and $j \in \{2\nu+1, \ldots, n\}$ such that v_i is not covered by M, v_j is covered by M and $d_i \geq d_j$. Let u denote the neighbour of v_j in M. If $v_i u \in E$, then $M - v_j u + v_i u$ is a better matching, a contradiction. If $v_i u \notin E$ then v_i has a neighbour w not connected to v_j. Then $G' = G - \{v_i w, v_j u\} + \{v_i u, v_j w\}$ is a graph realizing $d_1, \ldots, d_n$ with a better matching, a contradiction again. □

Now we prove sufficiency. Let $G = (V, E)$ and $G' = (V, E')$ be realizations of $d_1, \ldots, d_n$ and $d_1 - 1, \ldots, d_{2\nu} - 1$, $d_{2\nu+1}, \ldots, d_n$, respectively, such that $|E \cap E'|$ is as large as possible. It suffices to show that $E' \subseteq E$.

Assume to the contrary that this is not the case. Take a node $x \in V$ for what $(d_{E'-E}(v), d_{E-E'}(v))$ is lexicographically maximal. Let w be a neighbour of x in $E' - E$ and let $y_1, \ldots, y_k$ be the neighbours of x in $E - E'$. Moreover, let z be a neighbour of w in $E - E'$.

Assume first that $z \notin \{y_1, \ldots, y_k\}$. Observe that $zy_i \notin E'$ for $i = 1, \ldots, k$ as otherwise $G' - \{xw, zy_i\} + \{xy_i, wz\}$ would be a simple graph realizing $d_1 - 1, \ldots, d_{2\nu} - 1, d_{2\nu+1}, \ldots, d_n$, contradicting the choice of G and G'. On the other hand, $zy_i \in E$ for each $i = 1, \ldots, k$ as otherwise $G - \{xy_i, wz\} + \{xw, zy_i\}$ would be a simple graph realizing $d_1, \ldots, d_n$, contradicting the choice of G and G' again. We know that $d_{E'}(v) \geq d_E(v) - 1$ for each $v \in V$. But then z is lexicographically larger than x, a contradiction.

Assume now that $z = y_1$. Again, $zy_i \notin E'$ and $zy_i \in E$ for $i = 2, \ldots, k$ for the same reasons as in the previous case. That means that z is lexicographically larger than x, a contradiction. □

3 Covering supermodular functions

3.1 Preliminaries

Let $D = (S, T; A^*)$ be a complete bipartite digraph in which each edge is oriented from S to T. We denote $S \cup T$ by V. Subsets $X, Y \subseteq V$ are **comparable** if $X \subseteq Y$ or $Y \subseteq X$. A family of pairwise comparable sets is called a **chain**. The sets are ST**-intersecting** if $X \cap Y \cap T \neq \emptyset$ and **properly** ST**-intersecting** if, in addition, they are not comparable. The sets are ST**-crossing** if $X \cap Y \cap T \neq \emptyset$ and $S \not\subseteq X \cup Y$ and **properly** ST**-crossing** if they are not comparable. A set is **trivial** if $X \cap T = \emptyset$ or $S \subseteq X$. A function $p : 2^V \to \mathbb{R}$ is called **positively** ST**-crossing supermodular** if $p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y)$ for all ST-crossing pair $X, Y \subseteq V$ with $p(X) > 0, p(Y) > 0$. X and Y are ST**-independent** if they are not ST-crossing, that is, $X \cap Y \cap T = \emptyset$ or $S \subseteq X \cup Y$. A family of sets is ST**-independent** if its members are pairwise ST-independent.

Let $p : 2^V \to \mathbb{Z}_+$ be a positively ST-crossing supermodular function which is 0 on trivial sets. We say that a vector $z : A^* \to \mathbb{Z}_+$ **covers** p if $\varrho_z(X) \geq p(X)$ for each $X \subseteq V$. As p is 0 on trivial sets, there exists z covering p. If z corresponds to the incidence vector of a subset of edges $A \subseteq A^*$, then we say that A **covers** p.

For a set $\mathcal{F} \subseteq 2^V$ we call $\sum_{X \in \mathcal{F}} p(X)$ the p**-value of** $\mathcal{F}$ and denote it by $\widetilde{p}(\mathcal{F})$. Let τ_p be the minimum value of a cover of p, that is,

$$\tau_p = \min\{\widetilde{z}(A^*) : \ z : A^* \to \mathbb{Z}_+ \text{ covers } p\}.$$

Meanwhile, the maximum p-value of an ST-independent family is denoted by ν_p, so

$$\nu_p = \max\{\widetilde{p}(\mathcal{I}) : \ \mathcal{I} \subseteq 2^V \ ST\text{-independent}\}.$$

The following min-max theorem appeared in [3].

Theorem 3.1 (Frank, Jordán)**.** *Let $D = (S, T; A^*)$ be a complete bipartite digraph in which each edge is oriented from S to T. Let $V = S \cup T$ and assume that $p : 2^V \to \mathbb{R}$ is a positively ST-crossing supermodular function that is non-positive on trivial sets. Then $\tau_p = \nu_p$.*

3.2 Coverings by simple bipartite graphs

Theorem 3.2. *Let S and T be non-empty disjoint sets, p a positively intersecting supermodular set-function on T and $m_S : S \to \mathbb{Z}_+$ a degree specification on S for which $\widetilde{m}_S(S) = \gamma$. There exists a simple bipartite graph $G = (S, T; E)$ for which*

$$|\Gamma_G(Y)| \geq p(Y) \text{ holds for every } Y \subseteq T \tag{5}$$

and

$$d_G(s) = m_S(s) \text{ for every } s \in S \tag{6}$$

if and only if

$$m_S(s) \leq |T| \text{ for every } s \in S \tag{7}$$

and

$$\widetilde{m}_S(X) + \sum_{i=1}^{q} [p(T_i) - |X|] \leq \gamma \tag{8}$$

holds for every sub-partition $\mathcal{T} = \{T_1, \dots, T_q\}$ of T and subset X of S. Inequality (8) is equivalent to

$$\sum_{i=1}^{q} p(T_i) \leq \widetilde{m}(S - X) + q|X|. \tag{9}$$

Proof. First we prove necessity. Suppose that G is is a graph with the requested properties. By the simplicity of G, $m_S(s) = d_G(s) \leq |T|$ holds for every $s \in S$, that is, (6) is indeed necessary.

G has $\widetilde{m}_S(X)$ edges ending in X. Furthermore, each T_i has at least $p(T_i) - |X|$ neighbours in $S - X$ implying that there are at least $p(T_i) - |X|$ edges between T_i and $S - X$. Therefore the total number γ of edges is at least $\widetilde{m}_S(X) + \sum_{i=1}^{q}[p(T_i) - |X|]$, that is, (8) is also necessary.

Now we prove sufficiency. Recall that $V = S \cup T$. A set $Z \subseteq V$ is called **large** if $Z = V - s$ for some $s \in S$, otherwise it is **small**. Consider the following set function on V.

$$p'(Z) = \begin{cases} m_S(s), & \text{if } Z = V - s \text{ is a large set for } s \in S, \\ p(Z \cap T) - |Z \cap S|, & \text{if } Z \text{ is small.} \end{cases}$$

Condition (8), when applied to $\mathcal{T} = \{T\}$ and $X = S - s$, implies that $m_S(s) \geq p(T) - |S - s|$ for each $s \in S$, hence $p'(Z) = m_S(s) \geq p(Z \cap T) - |Z \cap S|$ for every large set $Z = V - s$. By choosing subpartition $\{T'\}$ together with $X = S$ in (8), we get $p(T') \leq |S|$ for every $T' \subseteq T$. If Z is a trivial set, then either $Z \cap T = \emptyset$ or $S \subseteq Z$. In both cases, $p'(Z) \leq 0$ follows from the observation and $p(\emptyset) = 0$.

Let $D = (S, T; A^*)$ be the complete bipartite digraph in which each arc is oriented from S to T. Note that $\tau_{p'} \geq \gamma$ as the set of every large sets is ST-independent with total p'-value γ.

Claim 3.3. *If $\tau_{p'} = \gamma$, then there exists a – not-necessarily simple – bipartite graph G satisfying (5) and (6).*

Proof. Let $z : A^* \to \mathbb{Z}_+$ be a vector covering p' such that $\widetilde{z}(A^*) = \tau_{p'} = \gamma$ and let $G = (S, T; E)$ be a bipartite graph in which there are $z(uv)$ parallel edges between $u \in S$ and $v \in T$. Note that $\widetilde{z}(A^*) = \gamma$ implies $d_G(s) = \delta_z(s) = m_S(s)$ for each $s \in S$, hence (6) is satisfied.

Suppose indirectly that $|\Gamma_G(Y)| < p(Y)$ for some set $Y \subseteq T$. Then $0 = \varrho_z(Y \cup \Gamma_G(Y)) \geq p(Y) - |\Gamma_G(Y)| > 0$, a contradiction, so (5) is also satisfied and the claim follows. □

By Theorem 3.1 and Claim 3.3, it suffices to show that (8) implies $\nu_{p'} = \gamma$.

Lemma 3.4. $\nu_{p'} = \gamma$.

Proof. Theorem 3.1 and $\tau_{p'} \geq \gamma$ implies $\nu_{p'} \geq \gamma$. Suppose indirectly that $\nu_{p'} > \gamma$. Let $\mathcal{I} \subseteq 2^V$ be an ST-independent family with $\widetilde{p'}(\mathcal{I}) = \nu_{p'}$, modulo this let $|\mathcal{I}|$ be minimal. Define $\mathcal{I}_1 = \{Z \in \mathcal{I} : Z \text{ is large}\}$ and $\mathcal{I}_2 = \{Z \in \mathcal{I} : Z \text{ is small}\}$.

We claim that $\{Z \cap T : Z \in \mathcal{I}_2\}$ is a subpartition of T. Indeed, if $Z_1 \cap Z_2 \cap T \neq \emptyset$ for some $Z_1, Z_2 \in \mathcal{I}_2$, then

$$\begin{aligned} p'(Z_1) + p'(Z_2) &= p(Z_1 \cap T) - |Z_1 \cap S| + p(Z_2 \cap T) - |Z_2 \cap S| \\ &\leq p((Z_1 \cap Z_2) \cap T) - |(Z_1 \cap Z_2) \cap S| + p((Z_1 \cup Z_2) \cap T) - |(Z_1 \cup Z_2) \cap S| \\ &\leq p'(Z_1 \cap Z_2), \end{aligned}$$

where the last inequality follows from $p'(Z_1 \cup Z_2) \leq 0$ as $Z_1 \cup Z_2$ is trivial. Hence replacing Z_1 and Z_2 with their intersection results in an independent family $\mathcal{I}'$ with $\widetilde{p'}(\mathcal{I}') \geq \widetilde{p'}(\mathcal{I})$ and $|\mathcal{I}'| < |\mathcal{I}|$, a contradiction.

Recall that $|S - Z| = 1$ for each $Z \in \mathcal{I}_1$. Let $X = \bigcup_{Z \in \mathcal{I}_1}(S - Z)$. As $\mathcal{I}$ is ST-independent, $X \subseteq Z$ for each $Z \in \mathcal{I}_2$.

By the above observations, we have

$$\begin{aligned} \gamma < \widetilde{p'}(\mathcal{I}) &= \sum_{Z \in \mathcal{I}} p'(Z) \\ &= \sum_{Z \in \mathcal{I}_1} p'(Z) + \sum_{Z \in \mathcal{I}_2} p'(Z) \\ &\leq \widetilde{m}(X) + \sum_{Z \in \mathcal{I}_2} [p(Z \cap T) - |X|], \end{aligned}$$

contradicting (8), thus proving the lemma. □

Take an arbitrary bipartite graph $G = (S, T; E)$ satisfying (5) and (6). If there exist parallel edges between nodes $u \in S$ and $v \in T$, then, by $m_S(u) \le |T|$ and $d_G(u) = m_S(u)$, there exists a node $w \in T$ such that $uw \notin E$. Delete an edge between u and v and add an edge uw to the graph. By repeating the above step we get a simple graph $G' = (S, T; E')$ still satisfying (5) and (6). This completes the proof of the theorem. □

3.3 Covering p by degree-specified bipartite graphs

Let S and T be two disjoint sets and $V := S \cup T$. We are given a non-negative integer-valued degree specification $m : V \to \mathbb{Z}_+$ whose restrictions to S and to T are denoted by m_S and m_T, respectively. We assume throughout that $\widetilde{m}_S(V) = \widetilde{m}_T(V)$ and this common value will be denoted by γ. We say that the pair m or (m_S, m_T) is a **degree specification** and that a bipartite graph $G = (S, T; E)$ **fits** this degree specification if $d_G(v) = m(v)$ holds for every node $v \in V$. Let $\mathcal{G}(m_S, m_T)$ denote the set of simple bipartite graphs fitting (m_S, m_T).

Gale [4] and Ryser [13] found in an equivalent form the following characterization.

Theorem 3.5 (Gale and Ryser)**.** *There is a bipartite graph G fitting the degree specification, that is, $\mathcal{G}(m_S, m_T)$ is non-empty if and only if*

$$\widetilde{m}_S(X) + \widetilde{m}_T(Z) - |X||Z| \le \gamma \text{ for every } X \subseteq S,\ Z \subseteq T. \tag{10}$$

Moreover, (10) follows from its special case when X consists of the i largest values of m_S and Z consists of the j largest values of m_T $(i = 1, \dots, |S|,\ j = 1, \dots, |T|)$.

Theorem 3.6. *Let p be a positively intersecting supermodular function p on T and let (m_S, m_T) be a degree specification. There is a simple bipartite graph $G = (S, T; E)$ fitting (m_S, m_T) (that is, a member of $\mathcal{G}(m_S, m_T)$) wich covers p if and only if*

$$\widetilde{m}_S(X) + \widetilde{m}_T(Z) - |Z||X| + \sum_{i=1}^{q} [p(T_i) - |X|] \le \gamma \text{ for every } X \subseteq S,\ Z \subseteq T \tag{11}$$

where $\{Z, T_1, \dots, T_q\}$ is a subpartition of T in which Z may be empty and q may be zero. The inequality in (11) is equivalent to

$$\widetilde{m}_T(Z) + \sum_{i=1}^{q} p(T_i) \le \widetilde{m}_S(S - X) + (q + |Z|)|X|. \tag{12}$$

Proof. Necessity. Suppose that G is a member of $\mathcal{G}(m_S, m_T)$ covering p. Furthermore each T_i has at least $p(T_i) - |X|$ neighbours in $S - X$ from which there are at least $p(T_i) - |X|$ edges between T_i and $S - X$. Therefore the total number γ of edges is at least $\widetilde{m}_S(X) + \widetilde{m}_T(Z) - |X||Z| + \sum_{i=1}^{q} [p(T_i) - |X|]$ that is, (11) is indeed necessary.

Sufficiency. Observe first that (11) in the special case when $q = 0$ gives back (10), and hence $\mathcal{G}(m_S, m_T)$ is non-empty by Theorem 3.5. Also, for any element $t \in T$, by applying (11) to $q = 1,\ T_1 = T - t,\ X = \emptyset$, we obtain that $\widetilde{m}_T(T - t) + p(t) \le \gamma$, that is, $p(t) \le m_T(t)$ for every $t \in T$.

For each $t \in T$, increase the value $p(t)$ to $m_T(t)$ and let p_1 denote the resulting function. Then p_1 is also intersecting supermodular, and (11) is equivalent to the inequality obtained from (8) by applying it to p_1.

Observe that (11), when applied to $X = \{s\}$, $Z = T$ and $q = 0$, requires that $m_S(s) + \widetilde{m}_T(T) - |T| \le \gamma$, that is, $m_S(s) \le |T|$. By applying Theorem 3.2 to p_1 in place of p, we obtain that there is a simple bipartite graph G covering p for which $d_G(s) = m_S(s)$ for every $s \in S$. Furthermore, $|E| = \sum[d_G(s) : s \in S] = \sum[m_S(s) : s \in S] = \gamma = \sum[m_T(t) : t \in T] \le \sum[d_G(t) : t \in t] = |E|$ from which we must have $d_G(t) = m_T(t)$ for every $t \in T$. □

4 Applications

Let $G=(S,T;E)$ be a bipartite graph. The **deficiency** $h(Y)$ of a subset $Y\subseteq T$ is defined by $h(Y):=|Y|-|\Gamma(Y)|$ where $\Gamma(Y)=\Gamma_G(Y)$ denotes the set of neighbours of Y. Let $\mu=\mu(G,T)$ denote the maximum deficiency of subsets of T while $\nu=\nu(G)$ is the maximum cardinality of a matching of G. The defect form of Hall's theorem (which is equivalent to Kőnig's theorem) is as follows.

Theorem 4.1. *In a bipartite graph $G=(S,T;E)$, there is a matching of ℓ edges if and only if the deficiency of every subset of T is at most $|T|-\ell$. Equivalently, $\nu(G)=|T|-\mu(G,T)$.*

4.1 Term rank of matrices

Graphs in $\mathcal{G}(m_S,m_T)$ can be identified with $(0,1)$-matrices of size $|S|\times|T|$ in which the row sum vector is m_S and the column sum vector is m_T. Let $\mathcal{M}(m_S,m_T)$ denote the set of these matrices.

Ryser [12] defined the **term rank** of a $(0,1)$-matrix A by the maximum number of independent $1'$-s which is the matching number of the corresponding bipartite graph. Ryser developed a formula for the maximum term rank of matrices in $\mathcal{M}(m_S,m_T)$.

The maximum term rank problem is equivalent to finding the maximum matching number of bipartite graphs in $\mathcal{G}(m_S,m_T)$ which, in turn, is equivalent to the following.

Theorem 4.2 (Ryser)**.** *Let $\ell\leq|T|$ be an integer. Suppose that $\mathcal{G}(m_S,m_T)$ is non-empty, that is, (10) holds. For an integer $\ell\leq|T|$, $\mathcal{G}(m_S,m_T)$ has a member G with $\nu(G)\geq\ell$ if and only if*

$$\widetilde{m}_S(X)+\widetilde{m}_T(Z)-|X||Z|+(\ell-|X\cup Z|)\leq\gamma \text{ for every } X\subseteq S,\ Z\subseteq T. \tag{13}$$

Moreover, (13) follows from its special case when X consists of the i largest values of m_S and Z consists of the j largest values of m_T ($i=1,\dots,|S|$, $j=1,\dots,|T|$).

Proof. Necessity. Let G be a bipartite graph with the requested properties. Since G is simple, it has at least $\widetilde{m}_S(X)+\widetilde{m}_T(Z)-|X||Z|$ edges having at least one end-node in $X\cup Z$. Moreover, since G has a matching of ℓ edges, there are at least $(|X\cup Z|-\ell)$ edges connecting $S-|X|$ and $S-Z$. Therefore the total number γ of edges is at least $\widetilde{m}_S(X)+\widetilde{m}_T(Z)-|X||Z|+\ell-|X\cup Z|$, that is, (13) is indeed necessary.

To prove sufficiency, define a set-function p on T by $p(Y):=|Y|-(|T|-\ell)$ if Y is non-empty and $p(\emptyset)=0$. Then p is fully supermodular. If there is a simple bipartite graph $G=(S,T;E)$ fitting (m_S,m_T) (that is, a member of $\mathcal{G}(m_S,m_T)$ that covers p, then G has a matching of size ℓ by Theorem 4.1, and we are done. If no such a G exists, then Theorem 3.6 implies that there is a subpartition $\mathcal{T}=\{Z,T_1,\dots,T_q\}$ of T and a subset X of S for which

$$\widetilde{m}_T(Z)+\sum_{i=1}^{q}p(T_i)>\widetilde{m}_S(S-X)+(q+|Z|)|X|. \tag{14}$$

Since in the present case p is fully supermodular, we can assume that $q\leq1$. If $q=0$, then

$$\widetilde{m}_T(Z)>\widetilde{m}_S(S-X)+|Z||X|,$$

that is,

$$\widetilde{m}_T(Z)+\widetilde{m}_S(X)-|Z||X|>\gamma,$$

contradicting (10).

If $q=1$, then (14) is equivalent to

$$\widetilde{m}_T(Z)+p(T_1)>\widetilde{m}_S(S-X)+(1+|Z|)|X|.$$

Since p is monotone non-decreasing, we may assume that $T_1 = V - Z$. Furthermore $p(T-Z) = |T-Z| - (|T| - \ell) = \ell - |Z|$ from which

$$\widetilde{m}_T(Z) + \ell - |Z| > \gamma - \widetilde{m}_S(X) + (1+|Z|)|X|,$$

that is,

$$\widetilde{m}_T(Z) + \widetilde{m}_S(X) - |X||Z| + \ell - |X \cup Z| > \gamma,$$

contradicting (13). □

4.2 Wooded hypergraphs

In [8], Lovász gave the following characterization of bipartite graphs in which the Hall condition holds with strict inequality.

Theorem 4.3 (Lovász)**.** *In a bipartite graph $G = (S, T; E)$, there exists a forest for which the degree of every node $t \in T$ is exactly 2 if and only if*

$$|\Gamma_G(X)| \geq |X| + 1 \tag{15}$$

for every non-empty subset $X \subseteq S$.

Lovász's theorem can be reformulated in terms of hypergraphs. A hypergraph $\mathcal{H} = (V, \mathcal{E})$ is called a **wooded** if it can be trimmed to a graph which is a forest, that is, it is possible to select two distinct elements from each hyperedge in such a way that the selected pairs, as graph edges, form a forest. Theorem 4.3 is equivalent to the following.

Theorem 4.4. *A hypergraph $\mathcal{H} = (V, \mathcal{E})$ is wooded if and only if the union of every j hyperedges has at least $j+1$ elements ($j \geq 1$).*

Theorem 3.2 allows us to characterize the existence of a bipartite graph $G \in \mathcal{G}(m_S, m_T)$ satisfying the Hall condition with strict inequality. By Theorem 4.4, this gives the characterization of the existence of a wooded hypergraph in which the degree of each node and the size of each hyperedge is prescribed.

Theorem 4.5. *Let V be a set of n nodes, $m_V : V \to \mathbb{Z}_+$ a degree prescription, $I = \{1, \ldots, m\}$ and $m_\mathcal{E} : I \to \mathbb{Z}_+$ a size prescription such that $m_\mathcal{E}(i) \geq 2$ for $i \in I$ and $\widetilde{m}_\mathcal{E}(I) = \widetilde{m}_V(V) = \gamma$. There exists a wooded hypergraph $\mathcal{H} = (V, \mathcal{E})$ such that such that $d_\mathcal{H}(v) = m_V(v)$ for $v \in V$, $\mathcal{E} = \{e_1, \ldots, e_m\}$ and $|e_j| = m_\mathcal{E}(j)$ for $j \in I$ if and only if*

$$\widetilde{m}_V(X) + \widetilde{m}_\mathcal{E}(Z) - |X||Z| \leq \gamma \tag{16}$$

for every $X \subseteq V, Z \subseteq I$, and

$$\widetilde{m}_V(X) + \widetilde{m}_\mathcal{E}(Z) + |I| + 1 - |X| - |Z| - |X||Z| \leq \gamma \tag{17}$$

for every $\emptyset \neq X \subseteq V, Z \subset I$.

Proof. Necessity of (16) follows from the Gale-Ryser theorem. Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph with the requested properties, and let $G = (V, I; E)$ be the associated bipartite graph. As G is simple, it has at least $\widetilde{m}_V(X) + \widetilde{m}_\mathcal{E}(Z) - |X||Z|$ edges having at least one end-node in $X \cup Z$. As $I - Z \neq \emptyset$ and $\mathcal{H}$ is wooded, $I - Z$ has at least $|I| - |Z| + 1$ neighbours, hence there are at least $|I| - |Z| - |X| + 1$ edges between $I - Z$ and $V - X$. So the total number of edges is at least $\widetilde{m}_V(X) + \widetilde{m}_\mathcal{E}(Z) + |I| + 1 - |X| - |Z| - |X||Z| \leq \gamma$, showing the necessity of (17).

Now we prove sufficiency. Define a set function $p : 2^I \to \mathbb{Z}_+$ as

$$p(Y) = \begin{cases} |Y| + 1, & \text{if } |Y| \geq 2, \\ m_\mathcal{E}(i), & \text{if } Y = \{i\}, \\ 0, & \text{if } Y = \emptyset. \end{cases}$$

As $m_{\mathcal{E}}(i) \geq 2$ $(i \in I)$, p is intersecting supermodular. Moreover, $m_V(v) \leq |I|$ follows from $\widetilde{m}_{\mathcal{E}}(I) = \widetilde{m}_V(V)$ and (16).

We will show that (8) is satisfied, that is,

$$\widetilde{m}_S(X) + \sum_{i=1}^{q} [p(T_i) - |X|] \leq \gamma \tag{18}$$

for every subpartition $\{T_1, \ldots, T_q\}$ of T and $X \subseteq S$.

Let $Z \subseteq T$ denote the set of nodes that appear as singletons in the subpartition. If $|T_i| = 1$ for $i = 1, \ldots, q$, then (18) is equivalent to (16).

If $|X| \geq 1$, then replacing two sets T_i, T_j of cardinality at least 2 by their union may only increase the left hand side. Hence we may assume that there is exactly one such set in the subpartition. Moreover, we may assume that this set is $T - Z$ as adding further nodes to it may only increase the left hand side of (18) while the right hand side does not change. In this case (18) is equivalent to (17).

If $|X| = 0$, then, by $m_{\mathcal{E}}(i) \geq 2$ for $i \in I$, the left hand side is maximal if the subpartition is in fact the partition $\{\{v\} : v \in V\}$. In this case (18) follows from $\widetilde{m}_{\mathcal{E}}(I) = \widetilde{m}_V(V)$.

By Theorem 3.2, there exists a simple bipartite graph $G = (V, I; E)$ satisfying $|\Gamma_G(Z)| \geq |Z| + 1$ for $Z \subseteq I$, $d_G(v) = m_V(v)$ for $v \in V$ and $d_G(i) \geq m_{\mathcal{E}}(i)$ for $i \in I$. As $\widetilde{m}_V(V) = \widetilde{m}_{\mathcal{E}}(I)$, necessarily $d_G(i) = m_{\mathcal{E}}(i)$ for $i \in I$, so G corresponds to a hypergraph satisfying the degree and the hyperedge-size prescriptions. By Theorem 4.4, the hypergraph is wooded, thus concluding the proof. $\square$

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The aim of this minicourse is to develop the basic theory of various notions of fractal dimension, and to present two interesting constructions of Erdős.

We will focus on the topological dimension, the similarity dimension and the Hausdorff dimension.

1 Topological dimension

Definition 1.1. (X,d) is a *metric space* if X is a nonempty set, $d : X\times X \to [0,\infty)$, and

1) $d(x,y)=0 \Leftrightarrow x=y$,
2) $d(x,y)=d(y,x)$,
3) $d(x,y)+d(y,z)\ge d(x,z)$ (triangle inequality).

Example 1.2.

- $(\mathbb{R}^m, d_{eucl})$, where $d_{eucl}((x_1,\dots,x_m),(y_1,\dots,y_m))=\sqrt{\sum_{n=1}^m (x_n-y_n)^2}$,
- $(X, d_{discrete})$, where $d_{discrete}(x,x)=0$ for every x, and $d_{discrete}(x,y)=1$ if $x\neq y$,
- *Subspaces:* Let (X,d) be a metric space and $Y\subset X$, then $(Y, d|_{Y\times Y})$ is also a metric space,
- *Products:* Let (X,d) and (Y,ϱ) be metric spaces, then $(X\times Y, d\times\varrho)$ is also a metric space, where $(d\times\varrho)((x_1,y_1),(x_2,y_2))=\sqrt{d(x_1,x_2)^2+\varrho(y_1,y_2)^2}$. (If you see this for the first time, check that $d\times\varrho$ is indeed a metric.)

Before we arrive at our main example, we need an important inequality.

Theorem 1.3 (Cauchy–Schwarz inequality)**.** *Let $x_n, y_n\in\mathbb{R}$ for all $n\in\mathbb{N}$, then*

$$\sum_{n=1}^{\infty}|x_n y_n| \le \sqrt{\sum_{n=1}^{\infty} x_n^2}\sqrt{\sum_{n=1}^{\infty} y_n^2}.$$

Proof. By taking limits, it suffices to prove a finite version: for every K

$$\sum_{n=1}^{K}|x_n y_n| \le \sqrt{\sum_{n=1}^{K} x_n^2}\sqrt{\sum_{n=1}^{K} y_n^2}.$$

Without loss of generality, we can assume that $x_n, y_n\ge 0$. Consider the following quadratic equation in t:

$$(x_1t+y_1)^2+\cdots+(x_Kt+y_K)^2=\left(\sum_{n=1}^{K}x_n^2\right)t^2+\left(2\sum_{n=1}^{K}x_ny_n\right)t+\left(\sum_{n=1}^{K}y_n^2\right)=0.$$

Since the left-hand side is nonnegative, there is at most one real root. Therefore the discriminant is nonpositive,

$$\left(2\sum_{n=1}^{K}x_ny_n\right)^2-4\left(\sum_{n=1}^{K}x_n^2\right)\left(\sum_{n=1}^{K}y_n^2\right)\le 0,$$

from which the theorem easily follows. $\square$

Example 1.4. The following is the famous *Hilbert space*.

$$\ell^2 = \left\{ \underline{x} = (x_1, x_2, \dots) \;\middle|\; \forall n\; x_n \in \mathbb{R} \text{ and } \sum_{n=1}^{\infty} x_n^2 < \infty \right\},$$

with the metric

$$d_2(\underline{x}, \underline{y}) = \sqrt{\sum_{n=1}^{\infty} (x_n - y_n)^2}.$$

Notation 1.5. The ℓ^2*-norm* of an element $\underline{x}$ is denoted by $\|\underline{x}\|_2 = \sqrt{\sum_{n=1}^{\infty} x_n^2}$, then $d_2(\underline{x}, \underline{y}) = \|\underline{x} - \underline{y}\|_2$.

Claim 1.6. *(ℓ^2, d_2) is a metric space.*

Proof. 1 & 2 are obvious.

Now we first prove that

$$\|\underline{x} + \underline{y}\|_2 \le \|\underline{x}\|_2 + \|\underline{y}\|_2 \text{ for every } \underline{x}, \underline{y} \in \ell^2. \tag{1}$$

We claim that $\sum_{n=1}^{\infty} x_n y_n$ is convergent, since it is absolutely convergent by the following computation.

$$\sum_{n=1}^{\infty} x_n y_n \le \sum_{n=1}^{\infty} |x_n y_n| \le \sqrt{\sum_{n=1}^{\infty} x_n^2} \sqrt{\sum_{n=1}^{\infty} y_n^2} = \|\underline{x}\|_2 \|\underline{y}\|_2$$

by the Cauchy–Schwarz inequality. Hence in the following we can freely rearrange the terms.

$$\begin{aligned} \|x + y\|_2^2 = \sum_{n=1}^{\infty} (x_n + y_n)^2 = \sum_{n=1}^{\infty} x_n^2 + 2 \sum_{n=1}^{\infty} x_n y_n + \sum_{n=1}^{\infty} y_n^2 \le \\ \le \|\underline{x}\|_2^2 + 2\|\underline{x}\|_2 \|\underline{y}\|_2 + \|\underline{y}\|_2^2 = (\|\underline{x}\|_2 + \|\underline{y}\|_2)^2. \end{aligned}$$

Hence we are done with (1).

On the one hand, applying this to $\underline{x}$ and $-\underline{y}$ we obtain that d_2 is well-defined. On the other hand, in order to check 3, we need to show that $\|\underline{a} - \underline{b}\|_2 + \|\underline{b} - \underline{c}\|_2 \ge \|\underline{a} - \underline{c}\|_2$. Set $\underline{x} = \underline{a} - \underline{b}$, $\underline{y} = \underline{b} - \underline{c}$, then $\underline{x} + \underline{y} = \underline{a} - \underline{c}$, and we are done by (1). □

Remark 1.7. Note that $|x_n - y_n| \le \|\underline{x} - \underline{y}\|_2$ for every n and every $\underline{x}, \underline{y} \in \ell^2$.

We need some more definitions.

Definition 1.8. Let (X, d) be a metric space and $A \subset X$. Then $\operatorname{diam} A = \sup\{d(x, y) | x, y \in A\}$. The set A is *bounded* if $\operatorname{diam} A < \infty$. *Open balls* are defined as $B(x, r) = \{y \in X | d(x, y) < r\}$. The metric spaces (X, d) and (Y, ρ) are *isometric* if there exists a bijection $f : X \to Y$ such that $\rho(f(x), f(y)) = d(x, y)$ for all $x, y \in X$. The set A is *dense* if $\forall x \in X\, \forall r > 0\; A \cap B(x, r) \ne \emptyset$. (E.g. $\mathbb{Q} \subset \mathbb{R}$ is dense.) The metric space (X, d) is *separable* if it has a countable dense subset (e.g. $\mathbb{R}$ is separable). A subset $U \subset X$ is *open* if $\forall u \in U\, \exists r > 0$ such that $B(u, r) \subset U$. We say that $x_n \to x$ iff $d(x_n, x) \to 0$. A subset $F \subset X$ is *closed* if $x_n \in F, x_n \to x$ implies $x \in F$. (If you see this for the first time, check that U is open iff $X \setminus U$ is closed.) A subset $A \subset X$ is *clopen* iff it is closed and open at the same time. The *boundary* of A is $\partial A = \{x \in X \mid \exists x_n \in A, \exists y_n \notin A, x_n \to x, y_n \to x\}$. Let $\mathcal{U}$ be a set of open subsets of X. Then $\mathcal{U}$ is a *basis of* X, if $\forall x \in X\, \forall r > 0\, \exists U \in \mathcal{U}$ such that $x \in U \subset B(x, r)$.

Example 1.9.

- $\mathcal{U} = \{(p, q) \mid p, q \in \mathbb{Q}, p < q\}$ is a basis of $\mathbb{R}$.
- $\mathcal{U} = \{B(x, r) \mid x \in X, r \in \mathbb{Q}, r > 0\}$ is a basis of (X, d).

Now we are ready to define the topological dimension.

IDEA: "Why is our World three-dimensional? Because the prison walls are two-dimensional!"

And similarly, to contain a bug living on a prison wall we only need a string, and to contain a smaller bug living on the string we only need two points, and to contain a virus living on these points we do not need anything.

Definition 1.10. The *topological dimension* of a metric space X is defined recursively as follows:

$\dim_t \emptyset = -1$,

$\dim_t X \le n \Leftrightarrow X$ has a basis $\mathcal{U}$ such that $\dim_t \partial U \le n-1$ for every $U \in \mathcal{U}$,

$\dim_t X = n \Leftrightarrow \dim_t X \le n$ but $\dim_t X \nleq n-1$.

$(\dim_t X = \infty \Leftrightarrow$ for every n $\dim_t X \nleq n.)$

Example 1.11.

- $\dim_t \mathbb{R} = 1$, **HOMEWORK #1**,
- $\dim_t \mathbb{R}^m = m$ (Nontrivial, we omit the proof.),
- $\dim_t \mathbb{Q} = 0$ (why?),
- If $\emptyset \ne H \subset \mathbb{R}$ then $\dim_t H = 0$ iff H contains no intervals (why?),
- $\dim_t \ell^2 = \infty$ (since it contains an isometric copy of every $\mathbb{R}^m$).

Remark 1.12. The topological dimension is monotone, that is, if $X \subset Y$ then $\dim_t X \le \dim_t Y$. Indeed, if $A \subset Y$ and $\partial_Y A$ denotes the boundary of A when considered as a subset of Y then it is easy to see that $\partial_Y A \subset \partial A$. A special case we will use is that a nonempty subset of a zero-dimensional space is itself zero-dimensional.

The following problem arose very naturally in the early days of dimension theory.

THE PRODUCT PROBLEM: Is it true that

$$\dim_t (X \times Y) = \dim_t X + \dim_t Y$$

for every X and Y?

HOMEWORK #2: Yes, if $\dim_t Y = 0$.

However, Erdős showed that the answer is in the negative in general!

Theorem 1.13 (P. Erdős, 1940)**.** *There exists a metric space E such that $\dim_t(E \times E) = \dim_t E = 1$.*

Proof.

Definition 1.14. $E = \{(x_1, x_2, \dots) \in \ell^2 \mid \forall n\, x_n \in \mathbb{Q}\}$.

Remark 1.15. This is the famous *Erdős space*.

Lemma 1.16. *E and $E \times E$ are isometric.*

Proof. Define $f((x_n)_{n=1}^\infty) = \left((x_{2n-1})_{n=1}^\infty, (x_{2n})_{n=1}^\infty\right)$. It is easy to see that $f : E \to E \times E$ is a bijection. Moreover,

$$d(\underline{x}, \underline{y}) = \sqrt{\sum_{n=1}^\infty (x_n - y_n)^2} = \sqrt{\sum_{n=1}^\infty (x_{2n-1} - y_{2n-1})^2 + \sum_{n=1}^\infty (x_{2n} - y_{2n})^2} =$$
$$= \sqrt{d((x_{2n-1})_{n=1}^\infty, (y_{2n-1})_{n=1}^\infty)^2 + d((x_{2n})_{n=1}^\infty, (y_{2n})_{n=1}^\infty)^2},$$

hence f is an isometry. $\square$

Therefore $\dim_t E = \dim_t E \times E$. Hence it suffices to prove that $\dim_t E = 1$.

First we show that $\dim_t E > 0$. By the monotonicity of the topological dimension it suffices to show this for a subspace of E. Let

$$E_0 = \left\{ \left(\frac{1}{k_1}, \frac{1}{k_2}, \dots \right) \in \ell^2 \;\middle|\; \forall n\, k_n \in \mathbb{N} \setminus \{0\} \right\}.$$

We need to show that there is no basis $\mathcal{U}$ of E_0 such that $\partial_{E_0} U = \emptyset$ for every $U \in \mathcal{U}$.

Let us choose $\underline{x} = (\frac{1}{k_1}, \frac{1}{k_2}, \dots)$, where $k_n \geq 2$, $k_n \in \mathbb{N}$ for all n. It suffices to show that if $U \subset E_0$ is open with $U \subset B(\underline{x}, \frac{1}{2})$ then $\partial_{E_0} U \neq \emptyset$. We pick two sequences of points by induction. Let

$$m_1 = \min \left\{ m \;\middle|\; \left(\frac{1}{m}, \frac{1}{k_2}, \frac{1}{k_3}, \dots \right) \in U \right\}$$

(k_1 works here, so there exists such an m_1). Then $m_1 \geq 2$, since

$$d\left(\left(\frac{1}{k_1}, \frac{1}{k_2}, \dots \right), \left(\frac{1}{1}, \frac{1}{k_2}, \dots \right) \right) = \left| \frac{1}{k_1} - 1 \right| \geq \frac{1}{2},$$

which contradicts $U \subset B(\underline{x}, \frac{1}{2})$. Set

$$\underline{x}_1 = \left(\frac{1}{m_1}, \frac{1}{k_2}, \frac{1}{k_3}, \dots \right) \text{ and } \underline{y}_1 = \left(\frac{1}{m_1 - 1}, \frac{1}{k_2}, \frac{1}{k_3}, \dots \right),$$

then $\underline{y}_1 \notin U$.

In a similar vein, let

$$m_2 = \min \left\{ m \;\middle|\; \left(\frac{1}{m_1}, \frac{1}{m}, \frac{1}{k_3}, \frac{1}{k_4}, \dots \right) \in U \right\},$$

then $m_2 \geq 2$ as before. Let

$$\underline{x}_2 = \left(\frac{1}{m_1}, \frac{1}{m_2}, \frac{1}{k_3}, \frac{1}{k_4}, \dots \right) \text{ and } \underline{y}_1 = \left(\frac{1}{m_1}, \frac{1}{m_2 - 1}, \frac{1}{k_3}, \frac{1}{k_4}, \dots \right).$$

Continuing in the same manner we obtain $\underline{x}_i \in U$ and $\underline{y}_i \in E_0 \setminus U$. As U is bounded, there exists $K > 0$ such that $\|\underline{x}_i\|_2 \leq K$ for every i. This easily implies $\sum_{n=1}^{\infty} \frac{1}{m_n^2} \leq K$, hence $\underline{y} = \left(\frac{1}{m_1}, \frac{1}{m_2}, \dots \right) \in \ell^2$, from which we obtain

$$m_n \to \infty \text{ as } n \to \infty. \tag{2}$$

Now we prove that $\underline{x}_i \to \underline{y}$. By (1) we obtain that $\underline{x} - \underline{y} \in \ell^2$, hence

$$\sum_{n=1}^{\infty} \left(\frac{1}{k_n} - \frac{1}{m_n} \right)^2 < \infty.$$

Therefore,

$$\|\underline{x}_i - \underline{y}\|_2 = \sqrt{\sum_{n=i+1}^{\infty} \left(\frac{1}{k_n} - \frac{1}{m_n} \right)^2} \to 0 \text{ as } i \to \infty.$$

Next we prove $\underline{y}_i \to \underline{y}$. By (2) we have $\|\underline{x}_i - \underline{y}_i\|_2 = \left| \frac{1}{m_i} - \frac{1}{m_i - 1} \right| \to 0$ as $i \to \infty$. Therefore $\underline{y}_i \to \underline{y}$. This shows that $\underline{y} \in \partial_{E_0} U$, hence $\dim_t E_0 > 0$, thus $\dim_t E > 0$.

Now we turn to the proof of $\dim_t E \leq 1$. It suffices to show that $\dim_t \partial_E B(\underline{x}, r) = \emptyset$ for every $\underline{x} \in E$ and $r \in \mathbb{Q}$, $r > 0$.

We will show this for $\underline{x} = \underline{0}$ and $r = 1$. (All the other cases are similar, or alternatively, the maps $\underline{x} \mapsto r\underline{x} + \underline{x}_0$, preserve balls, boundary, rational coordinates, etc, and every above mentioned ball is the image of the unit ball.)

Let $S = \partial_E B(\underline{0}, 1)$, then it is easy to check that

$$S = \left\{ \underline{x} = (x_1, x_2, \dots) \in E \;\middle|\; \sum_{n=1}^{\infty} x_n^2 = 1 \right\}.$$

Let $\underline{y} \in S$, and $\varepsilon > 0$, $\varepsilon \notin \mathbb{Q}$ be fixed, note that $|y_n| \leq 1$ for every n. Then there exists a $K > 0$ such that

$$\sum_{n=K+1}^{\infty} y_n^2 < \varepsilon. \tag{3}$$

Define

$$U_\varepsilon = \left\{ \underline{x} = (x_1, x_2, \dots) \in S \;\middle|\; \sum_{n=1}^{K} y_n x_y > 1 - \varepsilon \right\}.$$

Clearly, $\underline{y} \in U_\varepsilon$. Moreover, $U_\varepsilon \subset S$ is open in S, since if $\underline{x} \in U_\varepsilon$, $\sum_{n=1}^{K} y_n x_n = 1 - \varepsilon + \delta$ and $\|\underline{x}' - \underline{x}\| < \frac{\delta}{K}$ then $\sum_{n=1}^{K} y_n x_n' > 1 - \varepsilon$ ($|x_n' - x_n| < \frac{\delta}{K}$ and $|y_n| \leq 1$ for every n).

We claim that $\partial_S U_\varepsilon = \emptyset$. Suppose on the contrary that $\underline{x} = (x_1, x_2, \dots) \in \partial_S U_\varepsilon$. Using the above argument it is easy to see that

$$\sum_{n=1}^{K} y_n x_n = 1 - \varepsilon.$$

But this is impossible, since the left-hand side is a rational number but the right-hand side is irrational. Therefore $\partial_S U_\varepsilon = \emptyset$.

It remains to prove that $\operatorname{diam} U_\varepsilon \to 0$ as $\varepsilon \to 0$.

Let $\underline{x} = (x_1, x_2, \dots) \in U_\varepsilon$. Then

$$\begin{aligned}\|\underline{x} - \underline{y}\|_2^2 = \sum_{n=1}^{\infty} y_n^2 + \sum_{n=1}^{\infty} x_n^2 - 2\sum_{n=1}^{K} y_n x_n - 2\sum_{n=K+1}^{\infty} y_n x_n &\leq \\ \leq 1 + 1 - 2(1-\varepsilon) + 2\sqrt{\varepsilon} = 2\varepsilon + 2\sqrt{\varepsilon}&,\end{aligned}$$

where we used $\sum_{n=1}^{K} y_n x_n > 1 - \varepsilon$ and also that by Cauchy-Schwarz

$$\left| \sum_{n=K+1}^{\infty} y_n x_n \right| \leq \sum_{n=K+1}^{\infty} |y_n x_n| \leq \sqrt{\sum_{n=K+1}^{\infty} y_n^2} \sqrt{\sum_{n=K+1}^{\infty} x_n^2} \leq \sqrt{\varepsilon}\sqrt{1} = \sqrt{\varepsilon}.$$

Then $2\varepsilon + 2\sqrt{\varepsilon} \to 0$ as $\varepsilon \to 0$, finishing the proof. □

Concluding remarks

One might actually wonder if $\dim_t A$ is really the "right" definition of dimension? There are in fact numerous other definitions, and it is quite reassuring that all these notions coincide for separable metric spaces! Let us only mention one more.

Definition 1.17 (Lebesgue covering dimension)**.** Let (X, d) be a metric space. $\dim_{LC} X \leq n \Leftrightarrow$ for every open cover $\mathcal{U}$ of X there exists a refinement $\mathcal{V}$ of $\mathcal{U}$ (i.e. another open cover satisfying $\forall V \in \mathcal{V} \exists U \in \mathcal{U}$, $V \subset U$) such that $|\{V \in \mathcal{V} \mid x \in V\}| \leq n + 1$ for every $x \in X$.

Theorem 1.18. $\dim_{LC} X = \dim_t X$ *for every separable metric space* X.

It is therefore good to know that E is separable: elements of the form

$$(x_1, x_2, \ldots, x_K, 0, 0, \ldots)$$

form a countable dense subset.

In addition,

$$\left\{(x_1, x_2, \ldots) \in \ell^2 \;\middle|\; \forall n\, x_n \in \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}\right\}$$

is also a counterexample to the Product Problem, but it is moreover so called *complete*.

Last, but not least, one can actually find such counterexamples in $\mathbb{R}^3$ as well.

Theorem 1.19. *If X is a separable metric space with $\dim_t X \leq n$ then there exists $H \subset \mathbb{R}^{2n+1}$ and a homeomorphism $f : X \to H$ (i.e. a bijection such that f and f^{-1} are continuous, hence open sets, boundaries, etc. are all preserved).*

Corollary 1.20. *There exists $H \subset \mathbb{R}^3$ with $\dim_t H = \dim_t(H \times H) = 1$.*

Remark 1.21. E is totally disconnected (i.e. all connected subsets are singletons), but not zero-dimensional. These notions are equivalent for compact or even locally compact sets, hence $H \subset \mathbb{R}^3$ above cannot be closed.

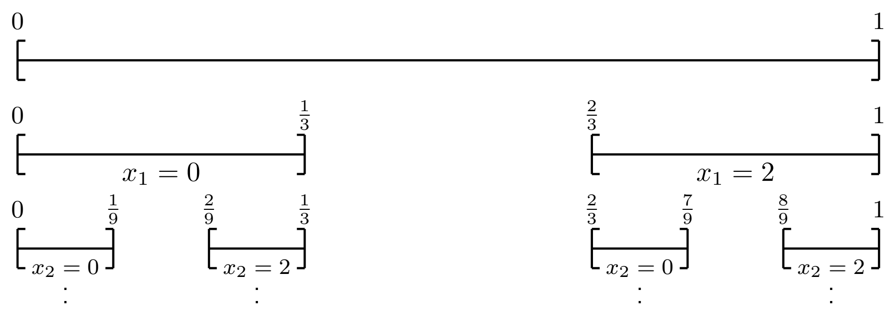
2 Similarity dimension

IDEA: "If a d-dimensional object is enlarged by a factor of 2 then its volume is multiplied by 2^d."

Let us try to apply this to fractal sets (a set is called a fractal if it is built up from pieces that are similar to the whole set).

Definition 2.1. Let $C = \left\{\sum_{n=1}^{\infty} \frac{x_n}{3^n} \;\middle|\; \forall n\; x_n \in \{0, 2\}\right\}$. This is the classical *middle-thirds Cantor set*.

It is not hard to see that it can also be obtained by removing the open middle third of the unit interval, and then recursively removing the open middle third of every remaining interval.



Imagine there is some notion of volume $V(A)$ for subsets of C, and assume that C is d-dimensional in some sense. Then, since $C = \left(C \cap \left[0, \frac{1}{3}\right]\right) \cup \left(C \cap \left[\frac{2}{3}, 1\right]\right)$ and the two parts are both similar to C with ratio $\frac{1}{3}$ we obtain

$$V(C) = 2\left(\frac{1}{3}\right)^d V(C),$$

from which an easy calculation yields $d = \frac{\log 2}{\log 3}$.

In more generality, suppose we have finitely many similarities $f_1, \ldots, f_m$ of ratios $0 < q_1, \ldots, q_m < 1$ such that $K = \dot{\bigcup}_{n=1}^{m} f_n(K)$. Then the same heuristic argument reads as follows

$$V(K) = \sum_{n=1}^{m} q_n^d V(K),$$

hence

$$1 = \sum_{n=1}^{m} q_n^d.$$

So we arrive at the following definitions.

Definition 2.2. A compact metric space K is called *self-similar* if there are $m \geq 2$ and similarities $f_1, \dots, f_m : K \to K$ with ratios $0 < q_1, \dots, q_m < 1$ such that $K = \cup_{n=1}^{m} f_n(K)$.

Definition 2.3. Then $\dim_{sim} K$ is the unique solution d to the equation $1 = \sum_{n=1}^{m} q_n^d$. It is called the *similarity dimension of K*.

Remark 2.4. Indeed, the function $g(d) = \sum_{n=1}^{m} q_n^d$ is continuous, strictly increasing, $g(0) = m > 1$ and $\lim_{d\to\infty} g(d) = 0$, hence there is a unique solution to the equation $g(d) = 1$.

Note that $\dim_{sim} K$ depends on the choice of $f_1, \dots, f_m$, so we only define it when $f_1, \dots, f_m$ are also given.

Example 2.5. $\dim_{sim} C = \frac{\log 2}{\log 3}$.

A huge drawback to this notion is that it is only defined for self-similar sets. However, the above argument suggest that it would be beneficial to have a notion of s-dimensional volume for every s. This leads us to the most important notion of fractal dimension.

3 Hausdorff dimension

IDEA: "A d-dimensional object has zero s-dimensional volume iff $s > d$."

The classical *Lebesgue measure* on $\mathbb{R}^m$ is defined by

$$\lambda(A) = \inf\left\{ \sum_{n=1}^{\infty} \alpha_m r_n^m \;\middle|\; \exists x_1, x_2, \dots \in \mathbb{R}^m, A \subset \bigcup_{n=1}^{\infty} B(x_n, r_n) \right\},$$

where α_m denotes the volume of the m-dimensional unit ball.

We may generalise this as follows.

Definition 3.1. Let X be a metric space and $A \subset X$. Define

$$\mathcal{S}_\infty^s(A) = \inf\left\{ \sum_{n=1}^{\infty} (2r_n)^s \;\middle|\; \exists x_1, x_2, \dots \in X, A \subset \bigcup_{n=1}^{\infty} B(x_n, r_n) \right\}.$$

This is called the *s-dimensional spherical Hausdorff premeasure*.

Remark 3.2. The constants α_m and 2^s are not important for us here. The subscript ∞ will also play no role in these notes. Let us explain where it comes from. We could define $\mathcal{S}_\delta^s$ similarly to $\mathcal{S}_\infty^s$ but requiring that $r_n < \delta$ for every n. The main advantage would be that then $\lim_{\delta\to 0} \mathcal{S}_\delta^s(A)$ exists for every A, and $\mathcal{S}^s = \lim_{\delta\to 0} \mathcal{S}_\delta^s$ is a so called measure. However, $\mathcal{S}^s(A) = 0$ iff $\mathcal{S}_\infty^s(A) = 0$, therefore when we are only interested in the dimensions, we do not need these technical variations.

Moreover, one is often interested in covers by more general sets than just balls. Replacing $B(x_n, r_n)$ by an arbitrary set H_n and replacing $(2r_n)^s$ by $\operatorname{diam}(H_n)^s$ we could define $\mathcal{H}_\infty^s$, $\mathcal{H}_\delta^s$ and $\mathcal{H}^s$ as well. The last one is called the *s-dimensional Hausdorff measure*, and it is the most fundamental object of fractal geometry. However, $\mathcal{H}^s(A) = 0$ iff $\mathcal{H}_\infty^s(A) = 0$ iff $\mathcal{S}_\infty^s(A) = 0$, so we do not need these variations either.

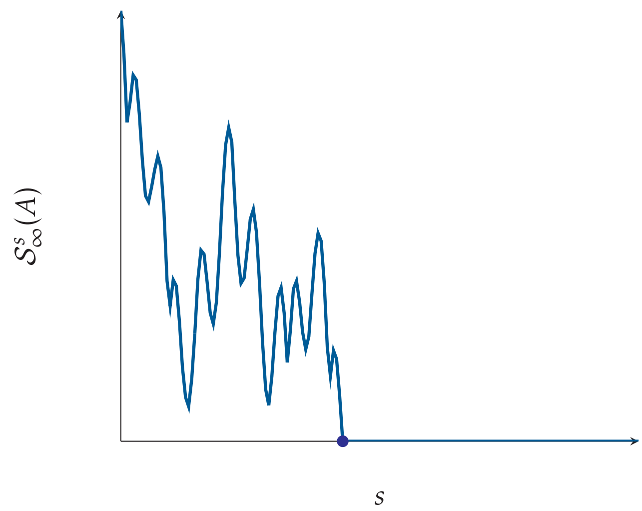
Lemma 3.3. *Let $0 \leq s < t$. Then $\mathcal{S}_\infty^s(A) = 0$ implies $\mathcal{S}_\infty^t(A) = 0$.*

Proof. Fix $0 < \varepsilon \le 1$. There exist $x_n \in X$ and $r_n > 0$ such that $A \subset \bigcup_{n=1}^{\infty} B(x_n, r_n)$ and $\sum_{n=1}^{\infty}(2r_n)^s < \varepsilon$. Note that $2r_n \le 1$ for every n. Now

$$\sum_{n=1}^{\infty}(2r_n)^t = \sum_{n=1}^{\infty}(2r_n)^s \cdot (2r_n)^{t-s} \le \sum_{n=1}^{\infty}(2r_n)^s < \varepsilon,$$

yielding $\mathcal{S}_{\infty}^t(A) = 0$. □

Hence the s-dimensional volume of a set A typically looks like this:



Definition 3.4.

$$\dim_H A = \inf\{s \mid \mathcal{S}_{\infty}^s(A) = 0\} = \sup\{s \mid \mathcal{S}_{\infty}^s(A) > 0\}.$$

Theorem 3.5. $\dim_H[0,1] = 1$.

Proof. First we prove that $\mathcal{S}_{\infty}^s([0,1]) = 0$ for every $s > 1$. For every M

$$[0,1] \subset \bigcup_{n=1}^{M}\left[\frac{n-1}{M}, \frac{n}{M}\right],$$

hence

$$\mathcal{S}_{\infty}^s([0,1]) \le M\left(\frac{1}{M}\right)^s = \left(\frac{1}{M}\right)^{s-1}.$$

But as $M \to \infty$ the right-hand side converges to 0 (since $s-1>0$), hence $\mathcal{S}_{\infty}^s([0,1]) = 0$.

So it suffices to prove that $\mathcal{S}_{\infty}^1([0,1]) > 0$. We will actually show that $\mathcal{S}_{\infty}^1([0,1]) \ge 1$.

Every $B(x_n, r_n)$ is an interval (a_n, b_n) and $2r_n = b_n - a_n$. Assume towards a contradiction that there exists a_n, b_n such that

$$[0,1] \subset \bigcup_{n=1}^{\infty}(a_n, b_n) \text{ and } \sum_{n=1}^{\infty}(b_n - a_n)^1 < 1.$$

By the compactness of $[0,1]$ there exist $(a_1, b_1), \dots, (a_N, b_N)$ covering $[0,1]$, and of course $\sum_{n=1}^{N}(a_n - b_n) < 1$.

However, an easy induction shows that a finite system of intervals cannot cover an interval whose length exceeds the sum of the lengths in the system, a contradiction. □

Now we determine the Hausdorff dimension of the Cantor set.

Theorem 3.6. $\dim_H C = \frac{\log 2}{\log 3}$.

Let (X, d) and (Y, ρ) be metric spaces and $A \subset X$.

Remark 3.7. If $f : A \to Y$ is Lipschitz (i. e. there exists a constant M such that $\rho(f(x), f(y)) \leq Md(x, y)$ for every $x, y \in A$) then $\dim_H f(A) \leq \dim_H A$.

We prove this in more generality.

Lemma 3.8. *Let $(X, d), (Y, \rho)$ be metric spaces, $A \subset X$ and $\alpha > 0$. If $f : A \to Y$ is Hölder-α (i. e. there exists a constant M such that $\rho(f(x), f(y)) \leq Md(x, y)^\alpha$ for every $x, y \in A$) then $\dim_H f(A) \leq \frac{1}{\alpha} \dim_H A$.*

Proof. Let $s > \frac{1}{\alpha} \dim_H A$. We need to show that $\mathcal{S}^s_\infty(f(A)) = 0$. Since $\alpha s > \dim_H A$, for every $\varepsilon > 0$ there exist $x_n \in X, r_n > 0$ such that $A \subset \bigcup_{n=1}^\infty B(x_n, r_n)$ and $\sum_{n=1}^\infty (2r_n)^{\alpha s} < \varepsilon$. Therefore,

$$f(A) \subset f\left(\bigcup_{n=1}^\infty B(x_n, r_n)\right) = \bigcup_{n=1}^\infty f(B(x_n, r_n)) \subset \bigcup_{n=1}^\infty B(f(x_n), M r_n^\alpha).$$

Thus,

$$\mathcal{S}^s_\infty(f(A)) \leq \sum_{n=1}^\infty (2M r_n^\alpha)^s = 2^{s-\alpha s} M^s \sum_{n=1}^\infty (2r_n)^{\alpha s} < 2^{s-\alpha s} M^s \varepsilon.$$

Letting $\varepsilon \to 0$ shows $\mathcal{S}^s_\infty(f(A)) = 0$. □

Proof of Theorem 3.6. In order to show that $\dim_H C \leq \frac{\log 2}{\log 3}$ it suffices to check that if $s > \frac{\log 2}{\log 3}$ then $\mathcal{S}^s_\infty(C) = 0$. For every n there is a cover of C by 2^n many intervals of length $\frac{1}{3^n}$, hence

$$\mathcal{S}^s_\infty(C) \leq 2^n \left(\frac{1}{3^n}\right)^s = 2^n \left(\frac{1}{3^n}\right)^{\frac{\log 2}{\log 3}} \left(\frac{1}{3^n}\right)^{s - \frac{\log 2}{\log 3}} =$$

$$= 2^n \frac{1}{2^n} \left(\frac{1}{3^n}\right)^{s - \frac{\log 2}{\log 3}} = \left(\frac{1}{3^n}\right)^{s - \frac{\log 2}{\log 3}} \to 0 \text{ as } n \to \infty \ \left(\text{since } s - \frac{\log 2}{\log 3} > 0\right).$$

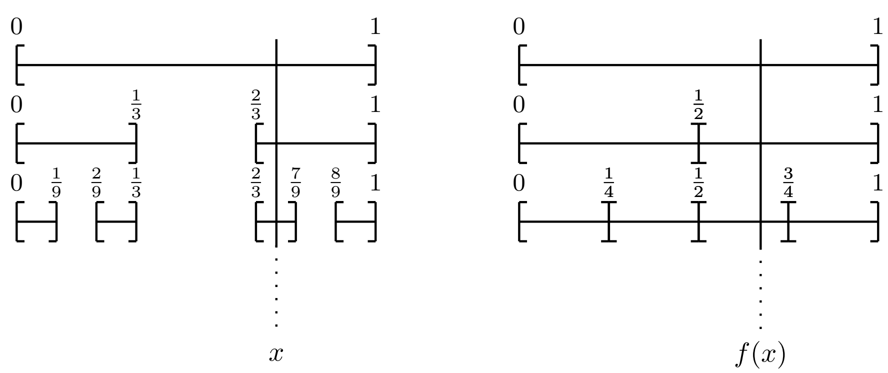
Now we apply Lemma 3.8 to prove $\dim_H C \geq \frac{\log 2}{\log 3}$. Let us define a map $f : C \to [0, 1]$ as follows.

There is a natural correspondence between the intervals defining C (called the *basic intervals*) and the dyadic subintervals of $[0, 1]$:

$$[0,1] \to [0,1]$$

$$\left[0, \tfrac{1}{3}\right] \to \left[0, \tfrac{1}{2}\right], \left[\tfrac{2}{3}, 1\right] \to \left[\tfrac{1}{2}, 1\right]$$

$$\vdots$$



If $x \in C$ then there is a unique sequence of basic intervals I_n with $x \in I_n$ and $|I_n| = \frac{1}{3^n}$.

Let J_n be the sequence of corresponding intervals. Note that $|J_n| = \frac{1}{2^n}$ and this is a sequence of nested intervals. By Cantor's intersection theorem there is a unique $y \in [0,1]$ with $\{y\} = \bigcap_{n=1}^{\infty} J_n$. Let $f(x) = y$.

(Alternatively, we could define this map as

$$f\left(\sum_{n=1}^{\infty} \frac{x_n}{3^n}\right) = \sum_{n=1}^{\infty} \frac{x_n}{2^{n+1}},$$

where for every n we have $x_n \in \{0,2\}$.)

It is easy to check that $f : C \to [0,1]$ is an onto map. Note that the image of every basic interval of length $\frac{1}{3^n}$ is an interval of length $\frac{1}{2^n}$.

We now prove that f is Hölder-$\frac{\log 2}{\log 3}$. Let $x, y \in C$ be distinct. There is a minimal n such that they are not contained in the same basic interval of length $\frac{1}{3^n}$. Then $|x - y| \geq \frac{1}{3^n}$, since the gaps between the basic intervals of length $\frac{1}{3^n}$ are all at least $\frac{1}{3^n}$. Moreover, by the minimality of n we have a basic interval of length $\frac{1}{3^{n-1}}$ containing x and y, therefore $|f(x) - f(y)| \leq \frac{1}{2^{n-1}}$. Hence

$$|f(x) - f(y)| \leq \frac{1}{2^{n-1}} = 2\frac{1}{2^n} = 2\left(\frac{1}{3^n}\right)^{\frac{\log 2}{\log 3}} \leq 2|x - y|^{\frac{\log 2}{\log 3}},$$

so we are done.

Now Lemma 3.8 gives

$$1 = \dim_H [0,1] = \dim_H f(C) \leq \frac{1}{\frac{\log 2}{\log 3}} \dim_H C,$$

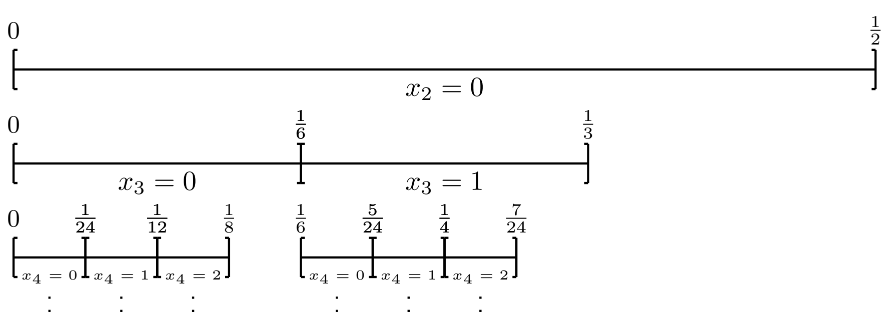
hence $\dim_H C \geq \frac{\log 2}{\log 3}$. □

Remark 3.9. We have seen that for $H \subset \mathbb{R}$ we have $\dim_t H = 1$ iff H contains an interval. The following interesting construction of Erdős and S. Kakutani shows that this is not the case for the Hausdorff dimension.

Definition 3.10.

$$C_{EK} = \left\{ \sum_{n=2}^{\infty} \frac{x_n}{n!} \;\middle|\; \forall n, x_n \in \{0, \ldots, n-2\} \right\}.$$

It is not very hard to check that it can be obtained as follows. First remove the second half of the unit interval. Then remove the last third of the remaining interval. Then remove the last fourth of every remaining interval, etc.



Remark 3.11. This set has many interesting properties. For example it is a compact set of length (= Lebesgue measure) zero that contains a similar copy of every finite set.

It is indeed easy to see that it is compact, since it is obtained as an intersection of compact sets. Moreover, for every n it can be covered by $(n-1)!$ many intervals of length $\frac{1}{n!}$. Therefore,

$$\lambda(C_{EK}) \le (n-1)!\frac{1}{n!} = \frac{1}{n} \to 0 \text{ as } n \to \infty,$$

hence $\lambda(C_{EK}) = 0$. In particular, C_{EK} contains no intervals.

Theorem 3.12. $\dim_H C_{EK} = 1$.

Proof. As in Theorem 3.6, there is a natural correspondence of the basic intervals as follows.

$$\left[0, \frac{1}{2}\right] \to [0,1]$$

$$\left[0, \frac{1}{6}\right] \to \left[0, \frac{1}{2}\right], \left[\frac{1}{6}, \frac{2}{6}\right] \to \left[\frac{1}{2}, 1\right]$$

etc.

Note that an interval of length $\frac{1}{n!}$ is mapped on an interval of length $\frac{1}{(n-1)!}$. Using this correspondence we obtain an onto map $f : C_{EK} \to [0,1]$ as above.

(Alternatively, we could define this map as

$$f\left(\sum_{n=2}^{\infty} \frac{x_n}{n!}\right) = \sum_{n=2}^{\infty} \frac{x_n}{(n-1)!},$$

where for every n we have $x_n \in \{0,1,\dots,n-2\}$.)

Definition 3.13. f is *nearly Lipschitz* if it is Hölder-α for every $0 < \alpha < 1$.

Lemma 3.14. *If $f : A \to Y$ is nearly Lipschitz then $\dim_H f(A) \le dim_H A$.*

Proof. $\dim_H f(A) \le \frac{1}{\alpha}\dim_H A$ for every $0 < \alpha < 1$ and $\alpha \to 1$ yields the result. □

Hence it suffices to check that the above f is nearly Lipschitz. Let $0 < \alpha < 1$. Let $x, y \in C_{EK}$ be distinct and let n be minimal such that they cannot be covered by two consecutive basic intervals of length $\frac{1}{n!}$. Then $|x-y| > \frac{1}{n!}$, and by the minimality of n, $|f(x) - f(y)| \le \frac{2}{(n-2)!}$.

Hence it suffices to show that there exists $M = M_\alpha$ such that for every n

$$\frac{2}{(n-2)!} \le M_\alpha \left(\frac{1}{n!}\right)^\alpha.$$

In order to show this we need to check that

$$\frac{2}{(n-2)!}(n!)^\alpha$$

is bounded. But this is clear, since it equals

$$2\frac{[(n-2)!]^\alpha}{(n-2)!}(n-1)^\alpha n^\alpha = \frac{2(n-1)^\alpha n^\alpha}{[(n-2)!]^{1-\alpha}},$$

which clearly tends to 0 as $n \to \infty$, (the numerator is polynomial, and the denominator is factorial type) hence it is bounded. □

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In 1993 University Eötvös Loránd gave the degree of an honorary doctor to the 80 year old Paul Erdős and asked him to give a talk on "The Actual Problems of Mathematics". The biggest auditorium was completely full on this occasion. The lecture was recorded, and now evoking the beginning of it we can get an impression also about the fascinating personality of Erdős.

"Can you hear me well? Also in the back rows? If you cannot hear me, please protest.

Well, the title of the talk is a bit arrogant, but it was not my formulation; it cannot be said that the questions I will talk about are the actual problems of mathematics. The last such lecture was held by Hilbert at the Paris mathematical congress in 1900, and it is not sure that now we could find a human being capable to perform such a talk. Anyway, it would need years of preparations, and a mathematical congress would be the suitable scene for it. I cannot undertake this task, partly because of my high age, but also because I know nothing about many areas, for example I am not an expert in algebraic topology, algebraic geometry or logic. Thus a more suitable title of my talk is "My favourite problems", and since some people in the audience are not mathematicians, I will speak about elementary geometry and number theory.

Let us start with elementary number theory. I will tell you now two problems. I raised the first one in 1931, so long ago, that I am not certain whether it was before or after Christ. By the way an old joke of mine is that I am two and a half billion years old. To prove it, the age of Earth was two billion years when I was a child, and now it is well known to be 4.6 billion years. Obviously, the difference is my age, and once I gave a talk in Los Angeles with the title "My First Two Billion Years in Mathematics", and the students made a figure with a diagram "Earth born, Erdős born, dinosaur born", and drew a picture where I was riding a dinosaur.

But putting the joke aside, the problem is the following, I pay 500 dollars for a proof or disproof, maybe there is some chalk around, can I get some chalk please, because I am captured by the wire [of the microphone], thank you very much, thus here is the problem:

Let be given a sequence of integers: $a_1 < a_2 < \cdots < a_k \le n$, and assume that all subset sums

$$\sum_{j=1}^{k} \varepsilon_j a_j, \qquad \varepsilon_j = 0 \text{ or } 1,$$

are distinct. Such numbers are for example the powers of two: 1, 2, 4, 8, 16, ..., since every baby knows that each number has a unique representation as the sum of [distinct] powers of two. Now the 500 dollar problem is to determine $\max k$, i.e. maximally how many numbers can be given up to n so that all these sums should be distinct."

Erdős sketched the proof of an upper bound for this maximum (see Exercise 1), we shall discuss rather another interesting question concerning the sequences having this property (i.e. when all subset sums are distinct) in Section 1. In Section 2 we shall investigate the slightly related Sidon sets, when we require that all two term sums should be distinct. Finally, in Section 3 we shall deal with covering congruences, which was the second problem mentioned in his 1993 talk. These three favourite problems of Erdős give a flavour of this field rich both in interesting questions and in approaches from the elementary level up to advanced applications of other areas of mathematics. We shall conclude the material with a list of exercises about various related questions proposed and/or investigated by Erdős.

1 A problem of Erdős and two proofs from "The Book"

Let A be a finite set of positive integers so that adding up any number of distinct elements of A we never get equal sums. How large can be the sum of reciprocals of the elements of A?

Taking suitably many powers of two this sum can be arbitrarily close to 2. Erdős conjectured that the sum is less than 2 for any A. Ryavec verified this by a very tricky proof, where it is not clear why it works. Much later Bruen and Borwein, and independently Frenkel (being a high school student at that time) found two different simple and natural arguments, so these nice proofs definitely form part of "The Book".

Ryavec's proof: If $A = \{a_1 < a_2 < \cdots < a_k\}$, then performing the multiplication in $P = (1 + x^{a_1}) \ldots (1 + x^{a_k})$ we obtain terms x^m, where m is the sum of some a_i-s. By assumption all these values of m are distinct, hence $P < 1 + x + x^2 + \cdots = 1/(1-x)$ if $0 < x < 1$. Now take the (natural) logarithm of both sides, divide by x, and integrate from 0 to 1:

$$\sum_{i=1}^{k} \int_0^1 \frac{\log(1+x^{a_i})}{x} dx < -\int_0^1 \frac{\log(1-x)}{x} dx. \tag{1}$$

Substituting on the LHS: $x^{a_i} = y$; $dy = a_i x^{a_i - 1} dx$; $dx = dy/(a_i x^{a_i-1})$, we obtain

$$\int_0^1 \frac{\log(1+x^{a_i})}{x} dx = \int_0^1 \frac{\log(1+y)}{x a_i x^{a_i-1}} dy = \frac{1}{a_i} \int_0^1 \frac{\log(1+y)}{y} dy.$$

Thus (1) turns into

$$\left(\sum_{i=1}^{k} \frac{1}{a_i}\right) \int_0^1 \frac{\log(1+y)}{y} dy < -\int_0^1 \frac{\log(1-x)}{x} dx.$$

We compute these numerical integrals by using power series:

$$\frac{-\log(1-x)}{x} = 1 + \frac{x}{2} + \cdots + \frac{x^{j-1}}{j} + \ldots,$$

hence

$$\int_0^1 \frac{-\log(1-x)}{x} dx = \left[x + \frac{x^2}{4} + \cdots + \frac{x^j}{j^2} + \ldots\right]_0^1 = \sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6}.$$

Similarly, we obtain

$$\int_0^1 \frac{\log(1+y)}{y} dy = \left[y - \frac{y^2}{4} + \frac{y^3}{9} - \ldots\right]_0^1 = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j^2} = \sum_{j=1}^{\infty} \frac{1}{j^2} - 2\sum_{t=1}^{\infty} \frac{1}{(2t)^2} = \frac{\pi^2}{12}.$$

These imply $\left(\sum_{i=1}^{k} \frac{1}{a_i}\right) \frac{\pi^2}{12} < \frac{\pi^2}{6}$, i.e. $\sum_{i=1}^{k} (1/a_i) < 2$.

Frenkel's proof: For any $1 \le i \le k$ the $2^i - 1$ sums formed from $a_1, \ldots, a_i$ are distinct, hence the largest sum (*) $a_1 + a_2 + \cdots + a_i \ge 2^i - 1$. We show that (*) implies $\sum_{j=1}^{k} (1/a_j) < 2$.

If (*) holds with equality for all i, then clearly $a_i = 2^{i-1}$. Otherwise we change the value of one or two a_i-s so that (*) remains valid, and the sum of reciprocals increases. The procedure terminates when everywhere we have equality.

Let r be the smallest value when equality does not hold in (*), i.e.

$$\begin{aligned} &a_1 + a_2 + \cdots + a_i = 2^i - 1, \quad i = 1, \ldots, r-1, \quad \text{and} \\ &a_1 + a_2 + \cdots + a_r > 2^r - 1. \end{aligned} \tag{1}$$

If we have $>$ for all $i > r$, then take $a'_r = a_r - 1$, this clearly works. If

$$\begin{aligned} &a_1 + a_2 + \cdots + a_i = 2^i - 1, \quad i = 1, \ldots, r-1 \\ &a_1 + a_2 + \cdots + a_i > 2^i - 1, \quad i = r, \ldots, s-1, \quad \text{and} \\ &a_1 + a_2 + \cdots + a_s = 2^s - 1, \end{aligned} \tag{2}$$

then take $a'_r = a_r - 1$, $a'_s = a_s + 1$. The sum of reciprocals increases: $\frac{1}{a_r} + \frac{1}{a_s} < \frac{1}{a'_r} + \frac{1}{a'_s} \iff \frac{a_r + a_s}{a_r a_s} < \frac{(a_r - 1) + (a_s + 1)}{(a_r - 1)(a_s + 1)} \iff$ $a_r a_s > (a_r - 1)(a_s + 1) \iff a_r - 1 < a_s$, which is true.

The proof by Bruen and Borwein: Taking $b_i = 2^{i-1}$, we can write (*) as $a_1 + \cdots + a_i \geq b_1 + \cdots + b_i$. We show that this implies (**) $(1/a_1) + \cdots + (1/a_k) \leq (1/b_1) + \cdots + (1/b_k)$ for any *real* numbers $0 < a_1 < \cdots < a_k$, $0 < b_1 < \cdots < b_k$.

By (*) we have $c_i = a_1 - b_1 + \cdots + a_i - b_i \geq 0$. Now

$$\frac{1}{b_1} - \frac{1}{a_1} + \cdots + \frac{1}{b_k} - \frac{1}{a_k} = \frac{a_1 - b_1}{a_1 b_1} + \frac{a_2 - b_2}{a_2 b_2} + \cdots + \frac{a_k - b_k}{a_k b_k} =$$
$$\frac{c_1}{a_1 b_1} + \frac{c_2 - c_1}{a_2 b_2} + \cdots + \frac{c_k - c_{k-1}}{a_k b_k} = c_1\left(\frac{1}{a_1 b_1} - \frac{1}{a_2 b_2}\right) + c_2\left(\frac{1}{a_2 b_2} - \frac{1}{a_3 b_3}\right) + \cdots + \frac{c_k}{a_k b_k}$$

which is non-negative, thus proving (**).

2 Sidon sets

A (finite or infinite) sequence of positive integers $a_1 < a_2 < \ldots$ is called a *Sidon set*, if the sums $a_i + a_j$ $(i \leq j)$ are pairwise distinct. Our aim is to give upper and lower bounds for the maximal size $k = s(n)$ of a Sidon set contained in the interval $[1, n]$. The best results show that $\lim_{n\to\infty} s(n)/\sqrt{n} = 1$. Erdős offers \$1000 for determining whether $|s(n) - \sqrt{n}|$ is bounded, or not.

Upper bounds:

(U1) There are $\binom{k}{2} + k = k(k+1)/2$ sums $a_i + a_j$, all contained in the interval $[2, 2n]$. Since each sum is a different integer, therefore $k(k+1)/2 \leq 2n - 1$, which implies $k^2 < 4n$, i.e. $s(n) < 2\sqrt{n}$.

(U2) Since $a_i + a_j = a_r + a_s \iff a_i - a_r = a_s - a_j$, also the differences $a_i - a_j$ $(i > j)$ are pairwise distinct, and each difference is in the interval $[1, n-1]$. Thus $\binom{k}{2} \leq n - 1$, hence $(k - 1/2)^2 \leq 2n - 7/4$, i.e. $s(n) < \sqrt{2n} + 1/2$.

(U3) Erdős and Turán proved by elementary methods, using the Cauchy-inequality, that $s(n) \leq \sqrt{n} + \sqrt[4]{n} + 1$.

Lower bounds:

(L1) The powers of 2 clearly form a Sidon set, hence $s(n) \geq 1 + \lfloor \log_2 n \rfloor$.

(L2) We construct a Sidon set using the greedy algorithm: we pick always the first number available $(1, 2, 4, 8, 13, \ldots;$ we cannot take 3, because $3 + 1 = 2 + 2$, we cannot take 5, because $5 + 1 = 4 + 2$, etc.). If we have already selected $a_1, \ldots, a_{k-1}$, then the solutions x of $x + a_s = a_r + a_t$, i.e. $x = a_r + a_t - a_s$ (*) $(r, s, t \leq k - 1)$ are the forbidden values for a_k (these include also the numbers $x = a_r = a_r + a_s - a_s$). In (*) there are at most $\binom{k-1}{2} + (k-1)$ choices for a_r and a_t, and at most $k - 1$ choices for a_s. Hence at most $k^3/2$ values are forbidden, which means that we certainly have a suitable a_k, as long as $k^3/2 < n$, i.e. $k < \sqrt[3]{2n}$. Therefore $\max k = s(n) \geq \sqrt[3]{2n}$.

(L3) If p is an odd prime, then for $n = 2p^2$ we construct a Sidon set of size $p = \sqrt{n/2}$. For general n we can use the largest $2p^2 \leq n$, hence we obtain asymptotically $\sqrt{n/2}$ as a lower bound for $s(n)$.

The construction: $a_i = 1 + 2pi + [i^2]$ $(i = 0, 1, \ldots, p-1)$, where $[i^2]$ means the (least non-negative) residue of i^2 mod p: $a_0 = 1$, $a_1 = 2p + 2$, etc.

To prove the Sidon property, assume $a_i + a_j = a_r + a_s$. We have to show that either $i = r, j = s$, or $i = s, j = r$. By the definition of the a-s, $0 = 2p(i + j - r - s) + ([i^2] + [j^2] - [r^2] - [s^2]) = 2pA + B$. Here $2p \mid B$, but $|B| < 2p$, hence $B = 0$, and also $A = 0$. Rearranging these equalities, we obtain $i - r = s - j$ and $i^2 - r^2 \equiv s^2 - j^2 \pmod{p}$. We are done, if $i - r = s - j = 0$. Otherwise we can divide the congruence by the common value $i - r = s - j$ (*), and obtain $i + r \equiv s + j \pmod{p}$ (**). Adding and subtracting (*) and (**), and dividing by 2, we arrive at $i = s, r = j$.

(L4) If p is an odd prime, then for $n = p^2 - 1$ we construct a Sidon set of size $p = \lceil \sqrt{n} \rceil$. For general n we use the largest $p^2 - 1 \leq n$, hence we obtain asymptotically $\sqrt{n}$ as a lower bound for $s(n)$. Moreover, using deep number theoretical results about the difference of the consecutive primes, we have $s(n) \geq \sqrt{n} - n^{0.27}$.

In fact, we shall prove more for $n = p^2 - 1$: we construct elements $a_1, \ldots, a_p$, such that the differences $a_i - a_j$ $(i \neq j)$ are pairwise incongruent mod n. (We cannot have more numbers with this property, since $(p+1)p > p^2 - 2$.)

For the construction we use the field $F = F_{p^2}$ of p^2 elements. Let α be a generator of the multiplicative group of F, i.e. each non-zero element of F is of the form α^j, where j is uniquely determined mod $p^2 - 1$. Let c_i be the elements of $\mathbf{Z}_p \subset F$, and define a_i as the exponent of α representing the element $\alpha + c_i$, i.e. $\alpha^{a_i} = \alpha + c_i$, $i = 1, 2, \ldots, p$.

Now, if $a_i + a_j \equiv a_r + a_s$ (mod $p^2 - 1$), then

$$(\alpha + c_i)(\alpha + c_j) = \alpha^{a_i + a_j} = \alpha^{a_r + a_s} = (\alpha + c_r)(\alpha + c_s),$$

i.e. $(c_i + c_j - c_r - c_s)\alpha + (c_i c_j - c_r c_s) = 0$. Since $\deg \alpha = 2$, this implies $c_i + c_j - c_r - c_s = c_i c_j - c_r c_s = 0$, and hence the (unordered) pairs $\{i, j\}$ and $\{r, s\}$ are the same.

We note the following variants:

(A) For $n = p^2 + p + 1$ we can construct elements $a_1, \ldots a_{p+1}$, such that the differences $a_i - a_j$ $(i \neq j)$ are pairwise incongruent mod n (hence each non-zero residue has a unique representation as $a_i - a_j$). We use the fact, that two non-zero elements, α^i and α^j, are linearly dependent in F_{p^3} iff $i \equiv j$ (mod $p^2 + p + 1$).

(B) For $n = p^2 - p$ we can construct elements $a_1, \ldots, a_{p-1}$, such that the differences $a_i - a_j$ $(i \neq j)$ are pairwise incongruent mod n. Here we use a primitive root g mod p, and a_i is the solution of the system of congruences $x \equiv i$ (mod $p - 1$), $x \equiv g^i$ (mod p), $i = 1, 2, \ldots, p - 1$.

3 Covering congruences

We want to cover the non negative integers by arithmetic progressions (AP-s) having distinct differences:

$$\{0, 1, 2, \ldots, n, \ldots\} = \{a_1, a_1 + m_1, a_1 + 2m_1, \ldots\} \cup \cdots \cup \{a_k, a_k + m_k, a_k + 2m_k, \ldots\},$$

where $1 < m_1 < m_2 < \cdots < m_k$.

If $0 \leq a < m$, then the AP $a, a + m, a + 2m, \ldots$ contains those numbers whose remainder is a when divided by m, hence this AP is (the non negative part of) a residue class (RC). This RC is denoted by a (mod m), and m is called the modulus. If the integers b and c give the same remainder when divided by m, i.e. $b - c$ is divisible by m, then we say that b is congruent to c modulo m, and denote this relation in the form $b \equiv c$ (mod m).

Thus we can restate our problem as to cover the integers with RC-s belonging to distinct moduli: Every integer t is the element of at least one RC

$$a_1 \text{ (mod } m_1), \quad a_2 \text{ (mod } m_2), \quad \ldots, \quad a_k \text{ (mod } m_k),$$

i.e. there is an i, for which $t \equiv a_i$ (mod m_i).

How can we construct such a congruence covering system? The remainders of the numbers are periodic modulo the least common multiple (lcm) of the moduli, hence it is sufficient to check the covering for the numbers between 1 and the lcm. A good candidate for the lcm is 12, since it has relatively many divisors, which all can be used as moduli. Then the even numbers are covered by 0 (mod 2), the multiples of three are covered by 0 (mod 3). Thus we have to ensure the covering of 1, 5, 7 and 11 using the moduli 4, 6 and 12; e.g. 1 (mod 4), 1 (mod 6), and 11 (mod 12) are okay. Hence these altogether five congruences form a covering system. Another example is 0 (mod 2), 0 (mod 3), 1 (mod 4), 7 (mod 8), 11 (mod 12), 19 (mod 24).

A long standing unsolved 1000 dollar problem was whether the smallest modulus of a covering system can be arbitrarily large or not. The record for this minimum was 40, established by P. Nielsen in 2008 in a 26 page paper. But in 2013 the question was settled in the negative, i.e. the minimum cannot be arbitrarily large, however its best value is not known. This result was announced on the conference in Budapest organized in the honour of the 100th anniversary of Erdős' birth. The other famous problem, whether all moduli can be odd, is still unsolved.

A nice result states that exact covering is impossible, i.e. there is no covering system where every number satisfies exactly one of the congruences (see Exercise 5).

And now let us turn to the question which led to the covering congruences: Can every odd integer large enough be represented as the sum of a prime and a power of two?

This was asked by Romanov in 1934, and Erdős gave a negative answer in 1950 using covering congruences: There exists an infinite AP of odd numbers such that no element of it can be represented as such a sum.

As a preparation we show that if q is an odd prime then the modulo q remainders of the powers of 2 are periodic.

It is sufficient to show that there exists a $t > 0$, for which the remainder of 2^t is 1, since then the pairs 2^{t+1} and 2, 2^{t+2} and 2^2 etc. have the same remainder.

There are infinitely many powers of two, but only q possible remainders, therefore the pigeon hole principle guarantees some $i < j$, for which $q \mid 2^j - 2^i = 2^i(2^{j-i} - 1)$, hence q being odd yields $q \mid 2^{j-i} - 1$, i.e. $t = j - i$ works.

The smallest such period is called the order of 2 mod q. E.g. the remainders of the powers of two mod 17 are $1, 2, 4, 8, 16, 15, 13, 9, 1, 2, \dots$, i.e. the order of 2 is 8 mod 17.

We can similarly show that if h and v are coprime, then the remainders of the powers of h mod v are periodic (and the general notion of the order can be introduced, as well).

We shall need also the Chinese Remainder Theorem: If we prescribe arbitrary remainders modulo finitely many, pairwise coprime moduli, then there is a number which satisfies all these conditions simultaneously, and all such numbers form a RC modulo the product of the moduli (i.e. we obtain an AP where the difference is the product of the moduli). or expressing this with formulae: If the integers $v_1, \dots, v_r$ are pairwise coprime, then the simultaneous system of the congruences (SSC)

$$x \equiv b_i \pmod{v_i}, \quad i = 1, \dots, k$$

is solvable for every $b_1, \dots, b_k$, and the solutions form a RC mod $v_1 \dots v_k$.

Now we are ready to prove the result of Erdős.

Assume that the order of 2 is m modulo q, and $n \equiv a \pmod{m}$. Since the remainders of the powers of 2 get repeated with period m, therefore $2^n \equiv 2^a \pmod{q}$.

Hence if $c = 2^n + p$, where p is a prime, and $c \equiv 2^a \pmod{q}$, then $p = c - 2^n \equiv c - 2^a \equiv 0 \pmod{q}$, so only $p = q$ is possible.

This means that if using some primes q_i we can take care for the c-s belonging to all n, then only the numbers $2^n + q_i$ have to be excluded (which will not be too hard).

Thus consider a covering system $a_i \pmod{m_i}$, $i = 1, 2, \dots, k$, and let $q_1, q_2, \dots, q_k$ be primes such that mod q_i the order of 2 is just m_i.

It can be shown, that if $m_i \neq 6$, then we can always find such distinct primes q_i. Let us find these in our second example of the covering system $m_1 = 2$, $m_2 = 3$, $m_3 = 4$, $m_4 = 8$, $m_5 = 12$, $m_6 = 24$ (and this will be sufficient to the proof of our theorem). We need that the order of 2 should be m_i mod q_i, i.e. $2^{m_i} \equiv 1 \pmod{q_i}$, or $q_i \mid 2^{m_i} - 1$, and here m_i is the smallest such (positive) exponent. Then for $m_1 = 2$ we have $q_1 \mid 2^2 - 1 = 3$, thus $q_1 = 3$. We obtain similarly $q_2 = 7$. Further, for $m_3 = 4$ we need $q_3 \mid 2^4 - 1 = 15 = 3 \cdot 5$, but 3 is bad, since the order of 2 mod 3 is 2 (and not 4), so only $q_3 = 5$ is possible, which really works. For $m_4 = 8$ consider $q_4 \mid 2^8 - 1 = (2^4 - 1)(2^4 + 1)$, but q_4 cannot divide the first factor, because then the order of 2 would be at most 4, hence $q_4 = 17$ which is okay. For

$m_5 = 12$ we take $q_5 \mid 2^{12} - 1 = (2^6 - 1)(2^6 + 1) = (2^6 - 1)(2^2 + 1)(2^4 - 2^2 + 1)$, here the first two factors are out of question (why?), whence $q_5 = 13$. Finally for $m_6 = 24$ we have $q_6 \mid 2^{24} - 1 = (2^{12} - 1)(2^4 + 1)(2^8 - 2^4 + 1)$, and again the last factor gives $q_6 = 241$. Hence the suitable primes q_i are $3, 7, 5, 17, 13, 241$.

In accordance with the idea sketched at the beginning of the proof, we consider the SSC $c \equiv 2^{a_i} \pmod{q_i}$, and insert one more congruence $c \equiv 1 \pmod{2^s}$, where s satisfies $2^{s-1} > q_i$ for every i. (This last congruence will guarantee that c is odd, and also the impossibility of representations of type $c = 2^n + q_i$.)

This SSC is solvable since the moduli are pairwise coprime, and the soluions form a RC mod M, where M is the product of the moduli. Thus we obtain an odd AP of difference M.

We show that no element of this sequence is the sum of a prime and a power of two.

Assume the converse: $c = 2^n + p$. Then by the covering property $n \equiv a_i \pmod{m_i}$ for some i. By the arguments used at the beginning of the proof we get that p can be only some q_i.

We show that even this is impossible. If $n \le s - 1$, then $1 < c = 2^n + q_i < 2^{s-1} + 2^{s-1} = 2^s$, which contradicts $c \equiv 1 \pmod{2^s}$. And if $n \ge s$, then $c = 2^n + q_i \equiv q_i \pmod{2^s}$, which is a contradiction, as well.

4 Exercises

1. We choose $A \subseteq \{1, 2, \ldots, n\}$ so that adding up the elements of the subsets of A we never get equal sums. Prove that any such A has at most $\log_2 n + \log_2 \log_2 n + 1$ elements.
2. Let $1 \le a_1 < a_2 < \cdots < a_k \le n$ be integers such that all threefold sums $a_i + a_j + a_m$ are distinct $(1 \le i \le j \le m \le k)$. Show that there exist positive constants c and d such that $c\sqrt[3]{n} > \max k > d\sqrt[5]{n}$.

   The lower bound can be improved to $\sqrt[3]{n}$: If $n = p^3 - 1$, where p is prime, then $k = p$ is achievable.
3. Show that a covering system must contain at least five congruences.
4. Construct a covering system where the smallest modulus is 3.
5. Prove that there is no exact covering system.
6. How many numbers can be selected from $\{1, 2, \ldots, 6n\}$ so that

   (a) no two should be relatively prime;

   (b) no three should be pairwise relatively prime;

   Generalize (a) and (b); formulate a conjecture!
7. How many numbers can be selected from $\{1, 2, \ldots, 2n\}$ so that none should be a multiple of another one?
8. Given n, find the minimal k for which any set of k integers contains

   (a) a nonempty subset having a sum divisible by n;

   (b) ** a subset of n elements having a sum divisible by n.
9. Can we assemble a cube from 2015 (suitably chosen and not necessarily congruent) cubes?
10. How many integers can be selected from $\{1, 2, \ldots, n\}$ so that (a) the sum; (b) the difference of two elements is never a (non-zero) square? (The difference problem is far from being solved.)

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1 Markov chains and mixing times

1.1 Preliminaries

The very first example of a stochastic process is a series of independent, identically distributed (i.i.d.) variables. However, there is no chance to see an evolution, a trend as we study such a process as it is just a collection of variables nothing to do with each other.

The simplest concept adding some dependence among the variables is that of *Markov chains*. Broadly speaking, we have a series of variables $X(t),\ t = 0,1,2\ldots$, and each variable might depend on the preceding one, but not on the whole history. In a formal way:

Definition 1.1. The series of random variables $X(t),\ t = 0,1,2$ form a Markov chain if for any $t \geq 1$ the distribution conditioned on the past simplifies as

$$P(X(t) \mid X(t-1),\ldots,X(0)) = P(X(t) \mid X(t-1)).$$

From now on, we will also assume that $X(t)$ is discrete valued, in particular, $X(t) \in [n] = \{1,2,\ldots,n\}$. Another convenient assumption we make is time invariance:

Definition 1.2. A Markov chain is *time invariant* if for any times t,t' and states i,j we have

$$P(X(t) = j \mid X(t-1) = i) = P(X(t') = j \mid X(t'-1) = i).$$

The usual interpretation of a Markov chain is a random walk: from $X(t-1)$ we use some independent noise to choose where to step next. Consider the following example to illustrate this concept.

Example 1.3. [13], [7], [18] Let $d_1,d_2,\ldots,d_n$ be a series of non-negative integers such that there exists a simple graph with this degree sequence. Let $X(0)$ be one of these (a deterministic one). To obtain $X(t),\ t \geq 1$ we perform the following steps:

- Randomly uniformly draw 4 different nodes v_1,v_2,v_3,v_4 of the graph $X(t-1)$.
- Check if $(v_1,v_2),(v_3,v_4)$ are edges and if $(v_1,v_3),(v_2,v_4)$ are not.
- If this is the case, perform a "swap", replace the edges $(v_1,v_2),(v_3,v_4)$ by $(v_1,v_3),(v_2,v_4)$ to get $X(t)$. Otherwise set $X(t) = X(t-1)$.

It can be shown that $X(t)$ will have a limiting distribution, which is uniform on all possible simple graphs with the given degree sequence. In turn, this is a practical way of generating a uniform sample on this set of graphs which could otherwise be quite difficult. From now on, we want to analyze Markov chains with similar behavior, where the distribution of $X(t)$ converges to some unique stationary distribution. This can be ensured using a few simple properties.

Definition 1.4. A Markov chain is *aperiodic* if the possible cycle lengths k

$$\{k\ :\ \exists i\ P(X(k) = i \mid X(0) = i) > 0\}$$

have greatest common divisor 1.

Definition 1.5. A Markov chain is *irreducible* if it is possible to reach any state starting from any state, that is, for any i,j we have

$$\exists k\ P(X(k) = j \mid X(0) = i) > 0.$$

Theorem 1.6. *Consider an irreducible aperiodic Markov chain. For any starting distribution of $X(0)$ the distribution of $X(t)$ converges to a unique distribution Π. This distribution is stationary, that is,*

$$P(X(1) \mid X(0) \sim \Pi) = \Pi.$$

For the rest of this note we always assume that the Markov chain is irreducible and aperiodic. For the following calculations it is convenient to introduce linear algebra notations. Distributions will be represented as vectors $x \in \mathbb{R}_+^n$ and transitions as matrices $P \in \mathbb{R}_+^{n\times n}$ such that

$$\begin{aligned} x &= (x_1, x_2, \ldots, x_n)^T & x_i &= P(X = i), \\ P &= (p_{ij}) & p_{ij} &= P(X(1) = i \mid X(0) = j). \end{aligned}$$

With this notation the effect of a time step simplifies to

$$x(t) = Px(t-1).$$

Moreover, the claim of Theorem 1.6 can be reformulated as

$$P^t x(0) \to \pi,$$

for any starting distribution $x(0)$, and π denoting the vector corresponding to the stationary distribution Π.

1.2 Convergence speed

Once convergence is established, we want to quantify the speed of this convergence. In order to do this, we first have to define a distance on probability measures.

Definition 1.7. The total variation distance of two probability measures on $[n]$ is defined as

$$\|\mu - \nu\|_{\mathrm{TV}} = \max_{A \subseteq [n]} |\mu(A) - \nu(A)|.$$

Using the vector notation, this simplifies to the l_1 distance of the two corresponding vectors, scaled by a factor of $1/2$. Once a distance is defined, the speed of convergence can be quantified:

Definition 1.8. For a Markov chain with transition matrix P we define the mixing time for any $\varepsilon > 0$ as

$$t_{\mathrm{mix}}(P, \varepsilon) = \max_{x \in \mathbb{R}_+^n,\ 1^T x = 1} \min \left\{ t : \|P^t x - \pi\|_{\mathrm{TV}} \le \varepsilon \right\}.$$

We omit the parameters P and/or ε if it is clear from the context.

In the next subsection we present tools to bound the mixing time.

1.3 Bounding the mixing time

The first bound is based on the eigenvalue structure of P. The Perron-Frobenius theorem for this matrix can be applied as follows.

Proposition 1.9. *The transition matrix P has a single eigenvalue at 1 provided by $P\pi = \pi$, all other eigenvalues are strictly inside the unit disc of the complex plane.*

Therefore the convergence to the stationary distribution is determined by how the components of x other than π disappear and this is limited by the second largest eigenvalue of P. Let us define

$$\Lambda := \max\{|\lambda| \mid \lambda \neq 1 \text{ is an eigenvalue of } P\}.$$

With this notation we get

Proposition 1.10. *For any starting distribution x we have*

$$\limsup_{t\to\infty} \frac{1}{t} \log \|P^t x - \pi\|_{\mathrm{TV}} \le \log \Lambda \le \Lambda - 1.$$

This gives us insight on the asymptotic behavior, but we can also deduce finite time bounds for Markov chains with some symmetry properties.

Definition 1.11. A Markov chain is *reversible* if for any $1 \le i, j \le n$ we have

$$p_{ij}\pi_j = p_{ji}\pi_i.$$

Intuitively this means that the frequency of $i \to j$ transitions is the same as that of $j \to i$ transitions. For such matrices the mixing time can be bounded as follows.

Proposition 1.12. *[15], [9] For a reversible Markov chain and $\varepsilon > 0$ we have*

$$\frac{-\Lambda \log(2\varepsilon)}{1-\Lambda} \le t_{\mathrm{mix}}(P,\varepsilon) \le \left\lceil \frac{-\log \varepsilon \pi_{min}}{1-\Lambda} \right\rceil,$$

where $\pi_{min} = \min_i \pi_i$.

The other tool we introduce is based on the worst bottleneck we can find between two subsets of the state space.

Definition 1.13. The *conductance* of a Markov chain is

$$\Phi = \min_{\emptyset \ne S \subsetneq [n]} \Phi(S) = \min_{\emptyset \ne S \subsetneq [n]} \frac{Q(S,S^C)}{\pi(S)\pi(S^C)} = \min_{\emptyset \ne S \subsetneq [n]} \frac{\sum_{i\in S, j \in S^c} p_{ij}\pi_j}{\pi(S)\pi(S^C)},$$

where $S^C = [n] \setminus S$, the complement of the set S.

Using the conductance it is possible to get both lower and upper bounds on the mixing time. However, they do not provide a sharp characterization as there is a square factor between the lower and upper bounds.

Theorem 1.14. *[20] For a reversible Markov chain the following bounds hold for the mixing time:*

$$c_1 \frac{1}{\Phi} \le t_{\mathrm{mix}} \le c_2 \frac{1}{\Phi^2} \log\left(\frac{1}{\pi_{min}}\right).$$

One may not want to be restricted to reversible Markov chains, but this condition can be replaced by requiring the Markov chain to be *lazy* instead. A Markov chain is lazy if $p_{ii} \ge 1/2$ for all i.

Theorem 1.15. *[16] For a lazy Markov chain the following bounds hold for the mixing time:*

$$c_1 \frac{1}{\Phi} \le t_{\mathrm{mix}} \le c_2 \frac{1}{\Phi^2} \log\left(\frac{1}{\pi_{min}}\right).$$

Most of these bounds involve some constants, we always assume they are strictly positive and they may change from line to line.

2 Random graphs models

In this section we introduce multiple models of random graphs. As we will see, there are several different ways of generating random graph based on what properties we want to appear.

2.1 The starting point

The fundamental model of random graph is that proposed by Erdős and Rényi [11]. This is treated in detail in the lecture of Ágnes Backhausz, we just recall the definition (of one of the two variants).

Definition 2.1. Given $n \in \mathbb{Z}^+$, $0 \le p \le 1$ we define the $ER(n,p)$ distribution of random graphs as follows. Starting from a vertex set $\{v_1, v_2, \dots, v_n\}$ each possible edge (v_i, v_j) is included independently with probability p.

The Erdős-Rényi random graph model provides a strong form of homogeneity. Each vertex is treated in the same way, and almost independently. In particular, we have

Proposition 2.2. *The degree distribution D is $Bin(n-1,p)$, therefore*

$$\mathbb{E}(D) = (n-1)p,$$
$$\mathbb{D}^2(D) = (n-1)p(1-p).$$

The degrees of different nodes have low correlation:

$$cov(D_i, D_j) = p(1-p).$$

2.2 Sparse connected random graphs

For the Erdős-Rényi graph to be connected, it is known this holds asymptotically almost surely (a.a.s.) if the edge probability is at least $a \log n/n$ with $a > 1$ [12]. For this case the average degree is $a \log n$.

It is natural to search for a model of connected graphs where the average degree does not increase with the size of the graph, for example, when a social network is to be modeled. One convenient way to provide this is to start with a graph that is already connected and then enrich it with some random edges. We present two models implementing this concept.

Definition 2.3. [19] Fix a vertex count $n \in \mathbb{Z}^+$ and $\varepsilon > 0$. The Newman-Watts random graph distribution $NW(n,\varepsilon)$ is defined as follows. Starting from an n node cycle C_n on some vertex set $\{v_1, v_2, \dots, v_n\}$ we also include the edges of an Erdős-Rényi graph $\mathcal{G}(n, \varepsilon/n)$ on the same vertex set.

In some way this model combines connectivity and the randomness of the Erdős-Rényi graph with bounded average degree. Along the same lines another model can be defined with even stronger structure.

Definition 2.4. [4] Fix an even vertex count $n \in \mathbb{Z}^+$. The Bollobás-Chung random graph distribution $BC(n)$ is defined as follows. Starting from an n node cycle C_n on some vertex set $\{v_1, v_2, \dots, v_n\}$ we add a random perfect matching on the vertices.

The two models are very similar by construction. Still, there are some interesting differences. The degree distribution for the Bollobás-Chung model is simply constant 3. For the Newman-Watts model, it is close to $2 + Poisson(\varepsilon)$. This also implies that while there is an extra edge for each vertex for the Bollobás-Chung model, there is a sequence of $c(\varepsilon)\log n$ nodes without any extra edge somewhere in a Newman-Watts graph, a.a.s.

Both models give connected graphs by design. Even more, it turns out that these added random edges cause the graph to have very good connectivity properties as it is demonstrated in the following theorem.

Theorem 2.5. *[4] Fix $\varepsilon > 0$. For both the $BC(n)$ and the $NW(n,\varepsilon)$ random graphs the diameter is $O(\log n)$ a.a.s.*

This is quite remarkable, for the $NW(n,\varepsilon)$ random graph we build it by starting with C_n containing n edges and a having a diameter $n/2$, then by adding only $\approx \varepsilon n$ edges the diameter collapses to $O(\log n)$.

We may view the Newman-Watts graph as a small perturbation added to the cycle graph. Recently the same concept has been applied to any connected graph and lot of results have been successfully adapted to this more general case, including the diameter estimate above and the mixing time bounds from the next section [14].

2.3 Non-homogeneous random graphs

In this subsection two random graph models are presented which lack the homogeneous nature the previous models had.

One of them are the preferential attachment graphs. Here we include the simple example based on the heuristics of Barabási and Albert [2]. Such graphs are built iteratively.

Definition 2.6. The random graph model $BA(n,k)$ random graph on n nodes and average degree k is defined using the following process. Start with the graph $BA(1,k)$ as the graph on 1 nodes and k loops. We get $BA(m+1,k)$ from $BA(m,k)$ by adding a new vertex v^+, and k edges to this vertex. The edges are added one by one to v^+. The other endpoint is chosen with probability proportional to the current degrees, that is,

$$\frac{d(v_i)}{\sum_j d(v_j)},$$

where $d(v)$ is the degree of vertex v. Note that the degrees have to be updated while adding the new edges and the edge being added is already counted contributing one to the degree of v^+. Therefore loop edges or multiple edges might appear in the graph.

For the degree distribution of this model we see that there is a positive feedback: if a vertex already has a high degree, it has an increased probability to get even more edges. Indeed, this results in having a long-tail distribution for the degrees:

Theorem 2.7. *[6] The degree distribution of the $BA(n,k)$ random graph model converges in probability to a power-law distribution as the vertex count grows, in particular,*

$$P(D=m) \to \frac{2k(k+1)}{m(m+1)(m+2)}.$$

The other random graph model to consider here is the configurational model. Here the desired degree sequence has to be explicitly specified. We then generate a uniform random graph among the possible ones with the given degree sequence.

Definition 2.8. [17] Given are $n \in \mathbb{Z}^+$ and a degree sequence $d_1, d_2, \ldots, d_n$ such that $\sum_i d_i$ is even. The configurational model is defined as the random graph distribution $\mathcal{C}(d_1, d_2, \ldots, d_n)$ as follows. Take a vertex set $\{v_1, v_2, \ldots, v_n\}$ and equip vertex v_i with d_i "half edges". Form a uniform random perfect matching on the "half edges" and add an edge for each pair connecting the two vertices they belong to.

Note that this allows loops and multiple edges. One may want to consider the distribution conditioned on the resulting graph being simple. However, this might cause difficulties when sampling from this distribution, if the probability of getting a simple graph is very small (which might happen when high degrees are imposed). Here we refer to Example 1.3 which proposes an alternative way to sample from this distribution.

3 Markov chains on random graphs

We now combine the tools for Markov chains with the observations on random graphs to explore how does such a random process evolve if it is launched on a random graph. In the previous section we have seen that the added random edges cause the graph to have very small diameter. Similarly, it results in the mixing time to drop significantly.

For simplicity we restrict ourselves to the lazy symmetric random walk on this graphs. This means that at every step the Markov chain stays put with probability $1/2$ and otherwise moves along one of the edges with equal probabilities.

Again, we aim to understand the typical behavior of the random graphs and disregard exceptional events that happen with very low probability. Therefore all the results phrase asymptotically almost sure claims.

Theorem 3.1. *[10] Consider the symmetric lazy random walk on $BC(n)$ random graphs. The following bounds hold a.a.s.*

$$c_l \log n \le t_{\text{mix}} \le c_u \log n.$$

Theorem 3.2. *[1] Fix $\varepsilon > 0$ and consider the symmetric lazy random walk on $NW(n,\varepsilon)$ random graphs. The following bounds hold a.a.s.*

$$c_l(\varepsilon)\log^2 n \le t_{\text{mix}} \le c_u(\varepsilon)\log^2 n.$$

Note that despite the similarity of the Newton-Watts and the Bollobás-Chung model, the regularity of the latter results in a significantly lower mixing time.

Let us turn our attention to the other random graph models we discussed. The mixing time is meaningful only for connected graphs, so for the Erdős-Rényi graphs an edge probability of $c \log n/n$ is needed. However, for lower edge probabilities we can still investigate the mixing time on the giant component.

Theorem 3.3. *[3], [8], [10] Fix $\lambda > 1$ and consider the random graphs $ER(n, \lambda \log n/n)$. Recall that such graphs are connected a.a.s. [12]. The following bounds hold a.a.s.*

$$c_l(\lambda)\frac{\log n}{\log\log n} \le t_{\text{mix}} \le c_u(\lambda)\log n.$$

Consider now the graphs $ER(n, \lambda/n)$. Note that such graphs have a giant connected component with size proportional to n. There are constants $0 < c_l'(\lambda) < c_u'(\lambda)$ such that the following bounds hold a.a.s. for the mixing time of the giant component:

$$c_l'(\lambda)\log^2 n \le t_{\text{mix}} \le c_u'(\lambda)\log^2 n.$$

Intuitively we expect that the higher degree increases the size of the connected component but at the same time reinforces its connectivity. The theorem above shows that second effect is stronger eventually resulting in a lower mixing time.

Theorem 3.4. *[5], [10] Fix $k \in \mathbb{Z}^+$ and consider the symmetric lazy random walk on $BA(n,k)$ random graphs. There are $0 < c_l(k) < c_u(k)$ such that a.a.s.*

$$c_l(k)\frac{\log n}{\log\log n} \le t_{\text{mix}} \le c_u(k)\log n.$$

We omit the discussion of the mixing time of the configurational model. Note that this would highly depend anyway on the degree sequence specified.

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1 Introduction

Paul Erdős worked in several branches in mathematics, but his results on and around Ramsey's theorem can be considered one of his favourite topics which pointed to a deep phenomenon in nature. (Another one is his probability method.)

For later reference here we state a very simple case of Ramsey's theorem.

Theorem 1.1 (The pigeon hole principle)**.**

(a) If $a_0,\dots,a_k$ are natural numbers, $n = a_0+\dots+a_k$, the set V with $|V| > n$ is partitioned as $V = V_0 \cup \dots \cup V_k$, then $|V_i| > a_i$ holds for at least one i.
(b) If an infinite set is partitioned into finitely many parts, one of them must be infinite.

Problem 1.2. Prove that if an uncountable set is decomposed into the union of countably many subsets, then one of the subsets is uncountable.

Problem 1.3. Assume that $A \subseteq \mathbb{R}$ is an uncountable set of reals. Show that there is real number a such that $A \cap (-\infty, a)$ and $A \cap (a, \infty)$ are both uncountable.

2 Two colors

We start with a simple word problem.

In a company of six people either there are 3 who mutually like each other or there are 3 so that no two like each other.

This can be mathematically reformulated as the following graph theory statement. In a graph of six vertices either there is a clique of size 3 (a triangle) or an independent set of size 3.

In what follows we define $R(a,b)$ as the least natural number (if there exists one) such that the following holds (R stands for Ramsey). Every graph on $R(a,b)$ vertices contains either an independent set of size a or a clique of size b. It is easy to see that $R(a,b) = R(b,a)$ and $R(a,b)$ is weakly monotonic in either variable.

Theorem 2.1. $R(3,3) = 6$.

Proof. Let G be a graph on V which does not contain 3-element cliques or independent sets. We show that $|V| \le 5$. Let $v \in V$ be an arbitrary vertex. $V - \{v\}$ can be decomposed as $N(v) \cup N^*(v)$ where $N(v)$ denotes the set of vertices joined to v (the *neighborhood of* v) and $N^*(v)$ denotes those vertices which are not joined to v. No two vertices in $N(v)$ are joined, as this would give a triangle. That is, $N(v)$ is indepedent, but then $|N(v)| \le 2$ by assumption. A dual argument gives $|N^*(v)| \le 2$, in toto we have $|V| = 1 + |N(v)| + |N^*(v)| \le 5$.

To show $R(3,3) > 5$ it suffices to exhibit a graph of 5 vertices that does not contain a clique or independent set of size 3. This is C_5, the circuit of length 5. □

Problem 2.2. Prove $R(3,4) \le 9$.

Theorem 2.3. $R(a,b) \le \binom{a+b-2}{b-1}$

Proof. By induction on $\min(a,b)$ (or $a+b$). If $a = 2$, then $R(2,b) = b$ which is $\binom{b}{b-1}$. The case $b = 2$ is similar.

Assume that $\min(a,b) \ge 3$. Let $G = (V,E)$ be a graph with no cliques of size a or independent sets of size b. Let $v \in V$ be an arbitrary vertex. Decompose V as

$$V = \{v\} \cup N(v) \cup N^*(v)$$

where $N(v) = \{w : \{v, w\} \in E\}$ is the set of neighbors and $N^*(v) = \{w \neq v : \{v, w\} \notin E\}$ is the set of non-neighbors of v. The graph on $N(v)$ has no clique of size $a-1$ (as that would give a clique of size a with the addition of v) and has no independent set of size b (as it is a subgraph of G). Consequently,

$$|N(v)| < \binom{a+b-3}{b-1}.$$

Dually,

$$|N^*(v)| < \binom{a+b-3}{b-2}.$$

Adding them up, we get

$$|V| = 1 + |N(v)| + |N^*(v)| \leq 1 + \left(\binom{a+b-3}{b-1} - 1\right) + \left(\binom{a+b-3}{b-2} - 1\right)$$

which is $\binom{a+b-2}{b-1} - 1$, i.e., we obtained that if $G = (V, E)$ omits cliques of size a and independent sets of size b, then $|V| < \binom{a+b-2}{b-1}$, as stated. □

Theorem 2.4. $R(n, n) \leq 4^n$.

Proof. By the formula in Theorem 1.2

$$R(n, n) \leq \binom{2n-2}{n-1} < 4^n.$$

The last inequality follows from the general bound $\binom{a}{b} \leq 2^a$ which can be seen as

$$\binom{a}{0} + \binom{a}{1} + \cdots + \binom{a}{a} = 2^a.$$

□

Problem 2.5. Use Stirling's formula (i.e., $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$) to deduce

$$\binom{2n}{n} \sim \frac{1}{\sqrt{\pi n}} 4^n.$$

Theorem 2.6. $R(n, n) > \frac{1}{2} 2^{\frac{n}{2}}$

First proof. Let N be a natural number. On a fixed set V of N vertices there are

$$2^{\binom{N}{2}}$$

graphs, as each pair in V can be in or out of the graph and these choices are independent.

We calculate the number of those graphs which contain a clique or independent set of size n:

$$2\binom{N}{n} 2^{\binom{N}{2} - \binom{n}{2}}.$$

If we want that there should be graphs on V with no cliques or independent sets of size n, then this must be less than the total number of graphs:

$$2\binom{N}{n} 2^{\binom{N}{2} - \binom{n}{2}} < 2^{\binom{N}{2}}.$$

This can be simplifies to

$$\binom{N}{n} < 2^{\binom{n}{2} - 1}$$

We increase $\binom{N}{n}$ to N^n. Consequently, if

$$N^n < 2^{\binom{n}{2}-1}$$

then $N < R(n,n)$. Extracting n'th root we get

$$N < 2^{\frac{n-1}{2}-\frac{1}{n}}$$

and we are done. ☐

Second proof. Let V be a set of N points. Let X be the random graph on V, i.e., where each pair of element is an edge of X with probability $1/2$, independently of each other.

If $S \subseteq V$, $|S| = n$, then the probability that S is a clique of X is

$$\mathbf{P}[S \text{ is a clique}] = \frac{1}{2^{\binom{n}{2}}}$$

as this exatly means that any pair in V is an edge of X. The same probability is obtained for the case S is an independent set. We get that

$$\mathbf{P}[\text{some } |S| = n \text{ is a clique/ind. set}] \leq 2\binom{N}{n}\frac{1}{2^{\binom{n}{2}}}$$

and the same computation finishes as in the First Proof. ☐

The above lower and upper bounds give essentially that

$$\sqrt{2} < \sqrt[n]{R(n,n)} < 4.$$

One of Erdős's favorite questions was if $\sqrt[n]{R(n,n)}$ converges to some number and if it does, to what.

Problem 2.7. Prove $R(3,n) \leq \frac{n^2-n+6}{2}$.

n^2 is not the right order of $R(3,n)$. It was suggested by Erdős that

$$c\frac{n^2}{\log n} \leq R(3,n) \leq c'\frac{n^2}{\log n}$$

for some positive constants c and c'. The latter estimate was proved by Ajtai, Komlós, and Szemerédi in 1980, the former by Kim.

Problem 2.8. If $a_1, a_2, \dots, a_{R(k,n)}$ is a sequence of distinct real numbers then either there is an increasing subsequence of length k or there is a decreasing subsequence of length n.

The length in the previous problem seems the right one, but it is not.

Theorem 2.9 (Erdős–Szekeres). *If $a_1, a_2, \dots, a_{kn+1}$ is a sequence of distinct real numbers, then either there is an increasing subsequence of length $k+1$ or there is a decreasing subsequence of length $n+1$.*

Proof. Assume the sequence $a_1, \dots, a_{kn+1}$ does not contain increasing subsequence of length $k+1$.

For each $1 \leq i \leq kn+1$ let $u(i)$ be the length of the longest increasing subsequence starting from a_i. By our indirect assumption, $1 \leq u(i) \leq k$ for every $1 \leq i \leq kn+1$. The mapping $i \mapsto u(i)$ maps the $kn+1$ element set of i's into the k-element set $\{1,2,\dots,k\}$. By the pigeon hole principle some value is attained $n+1$ times: $u(i_1) = u(i_2) = \cdots = u(i_{n+1}) = r$ where $i_1 < \cdots < i_{n+1}$. We claim that

$$a_{i_1} > a_{i_2} > \cdots > a_{i_{n+1}}$$

and so we found a decreasing subsequence of length $n+1$. Indeed, if, say, $a_{i_t} < a_{i_{t+1}}$, then, by placing the r-element increasing subsequence beginning with $a_{i_{t+1}}$ after a_{i_t}, we would obtain an $r+1$-long increasing sequence starting with a_{i_t}, contradicting $u(i_t) = r$. ☐

Problem 2.10. Give an example of a sequence of length kn with no increasing subsequence of length $k+1$ or decreasing subsequence of length $n+1$.

Problem 2.11. Prove that for every natural number k there exists another, n, such that each tournament on n vertices contains a transitive subtournament
(a) using Ramsey's theorem,
(b) without it.
What bound can we obtain for n?
(A *tournament* is a directed graph in which between any two vertices exactly one directed edge goes. A tournament is *transitive* if $u \to v$ and $v \to w$ imply $u \to w$.)

3 More colors

In what follows we are going to use the following notation. If V is a set, r a natural number, then

$$[V]^r = \{X \subseteq V : |X| = r\}.$$

An old problem goes like this. There are seventeen scientists each of them frequently sending letters to each other. For the correspondence, each pair chose one the languages English, French, or German. Prove that there are three corresponding to each other on the same language.

We introduce the notation $R(a_1, a_2, \ldots, a_s)$ being the least natural number N (if exists) satisfying that if the complete graph on N vertices is colored with colors $1, 2, \ldots, s$, then either
there is a monochromatic set of size a_1 in color 1, or
there is a monochromatic set of size a_2 in color 2, or
...
there is a monochromatic set of size a_1 in color s.

This way, the above problem requires to show $R(3,3,3) \le 17$.

In general, $R(3,3,\ldots,3) \le g(k)$ where $g(1) = 3$, $g(2) = 6$, $g(3) = 17$, and the larger values are given by the recursion $g(k+1) = (k+1)g(k) - k + 1$.

Problem 3.1. Prove that

$$g(k) = \frac{k!}{0!} + \frac{k!}{1!} + \cdots + \frac{k!}{k!} + 1 = \lceil ek! \rceil.$$

Here the ceiling function $\lceil x \rceil$ denotes the least integer $\ge x$, e.g., $\lceil 5.45 \rceil = 6, \lceil 8 \rceil = 8$.

Problem 3.2. If $N = \lfloor ek! \rfloor$ and the numbers $\{1, 2, \ldots, N\}$ are colored with k colors, then there is a monochromatic solution of $x + y = z$.

Problem 3.3. The set $\mathbb{R}$ of reals can be colored with countably many colors with no monochromatic solution of $x + y = z$ (except when $x = y = z = 0$).

Theorem 3.4. *The Ramsey number $R(a_1, \ldots, a_s)$ satisfies*

$$R(a_1, \ldots, a_s) \le b_1 + b_2 + \cdots + b_s - s + 2$$

where $b_i = R(a_1, \ldots, a_i - 1, \ldots, a_s)$ $(1 \le i \le s)$.

Theorem 3.5. $R(a_1, \ldots, a_s) \le s^{\sum a_i}$.

Proof. Set $M = \sum a_i$. Let V be a set with $|V| = s^M$ and assume that $f : [V]^2 \to \{1, 2, \ldots, s\}$ is a coloring.

Let $v_1 \in V$ be arbitrary. $V - \{v_1\}$ decomposes as $W_1 \cup \cdots \cup W_s$ where

$$W_i = \{w \in V - \{v_1\} : f(v_1, w) = i\}.$$

As $|W_1 \cup \cdots \cup W_s| = s^M - 1$, one of them has $|W_i| \geq s^{M-1} - 1$. Let A_1 be this W_i and $g(v_1) = i$. What we have now is that if $x \in A_1$ then $f(v_1, x) = g(v_1)$.

Next we continue with the graph on A_1: pick $v_2 \in A_1$, and choose $A_2 \subseteq A_1$ such that $|A_2| \geq s^{M-2} - 1$ and if $f(v_2, x)$ is the same for every $x \in A_2$, we call this common color $g(v_2)$.

Proceeding this way, we obtain the vertices $v_1, \ldots, v_M$ and sets $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_{M-1}$ such that $|A_i| \geq s^{M-i} - 1$ and $f(v_i, x) = g(v_i)$ if $x \in A_i$. $g : \{v_1, \ldots, v_M\} \to \{1, 2, \ldots, s\}$ is a coloring of the M-element set $\{v_1, \ldots, v_M\}$ with s colors. As the set has size

$$M \geq (a_1 - 1) + (a_2 - 1) + \cdots + (a_s - 1) + 1$$

for some $1 \leq j \leq s$ we have $|g^{-1}(j)| \geq a_i$. Now $g^{-1}(j)$ is homogeneous for f in color j, and its size is (at least) a_j, we are done. □

Problem 3.6. Prove Theorem 3.4.

Problem 3.7. Deduce Theorem 3.5 from Theorem 3.4.

Problem 3.8. If the lattice points of the plane (=points with both coordinates integers) are colored with finitely many colors, then there is a monocolored rectangle.

One of Erdős's favorite subtopics was *Euclidean Ramsey Theory*, in which the colorings of the Euclidean spaces were investigated. Here we give just the simplest examples.

Problem 3.9.

(a) If the plane is colored with 3 colors, then there are two monocolored points in distance one.
(b) This does not hold for 7 colors.

Problem 3.10. There is a coloring of the plane with 2 colors with no monocolored equilateral triangle of sides of length one.

4 Coloring triples, etc

Theorem 4.1 (Ramsey)**.** *If $2 \leq r, s \geq 2$ are natural numbers, $a_1, \ldots, a_s$ are finite, then there exists a natural number $R_s^r(a_1, \ldots, a_s)$ with the following property.*

Problem 4.2. Prove that if k is a natural number, then there is a finite set S with the following property. If all nonempty subsets of S are colored with k colors, then there are disjoint subsets X and Y of S, such that X, Y, and $X \cup Y$ get the same color.

Problem 4.3 (Happy ending problem)**.**

(a) Prove that among 5 planar points in general position some 4 form a convex 4-gon.
(b) Apply Ramsey's Theorem to show that for each natural number k there is a natural number $g(k)$ such that among $g(k)$ points in the plane in general position, some k form a convex set.

5 Infinite Ramsey theory

Theorem 5.1. *If $f : [\mathbb{N}]^2 \to \{0, 1, 2, \ldots, s-1\}$ then there is an infinite monocolored $A \subseteq \mathbb{N}$.*

First proof. Let $a_0 = 0$. The coloring $x \mapsto f(\{a_0, x\})$ colors the infinite set $\mathbb{N} - \{a_0\} = \{1, 2, \ldots\}$ with s colors. By the pigeon hole principle, one of the color classes is infinite, so there is an infinite $A_0 \subseteq \{1, 2, \ldots\}$ such that if $x \in A_0$, then $f(\{a_0, x\}) = g(a_0 0)$.

Set $a_1 = \min(A_0)$ and proceed: the coloring $x \mapsto f(\{a_1, x\})$ colors the infinite $A_0 - \{a_1\}$ with s colors, there are an infinite $A_1 \subseteq A_0 - \{a_1\}$, and $g(a_1 1)$ such that $f(\{a_1, x\}) = g(a_1)$ for $x \in A_1$.

Continuing this way, we obtain a sequence $a_0 < a_1 < \cdots$ and a coloring $g : \{a_0, a_1, \dots\} \to \{0, 1, \dots, s-1\}$ such that if $i < j$ then $f(\{a_i, a_j\}) = g(a_i)$.

Applying the pigeon hole principle to g, we find an infinite subset $a_{i_0} < a_{i_1} < \cdots$ such that $g(a_{i_j}) = k$ for every j. Now $\{a_{i_0}, a_{i_1}, \dots\}$ is homogeneous in color k for f. $\square$

Second proof. An alternative proof can be given using the notion of ultrafilter. A *nontrivial ultrafilter* on $\mathbb{N}$, the set of natural numbers is a collection U of subsets of $\mathbb{N}$ such that
(a) if $X \in U$, $X \subseteq Y \subseteq \mathbb{N}$, then $Y \in U$,
(b) if $X, Y \in U$, then $X \cap Y \in U$,
(c) if $X \cup Y = \mathbb{N}$, then either $X \in U$ or $Y \in U$,
(d) if $n \in \mathbb{N}$, then $\{n\} \notin U$.

One can think of an ultrafilter as a decomposition of all subsets of $\mathbb{N}$ into small and large sets such that each set is either small or large but not both, finite sets are small, $\mathbb{N}$ is large, the superset of a large set is large, if a large set is split into finitely many parts, then exactly one of them is large.

Set theoretical methods (e.g., Zorn's lemma) can be used to show that there is a nontrivial ultrafilter on $\mathbb{N}$.

Assume now that is a coloring of the pairs of $\mathbb{N}$ with the colors $0, 1, \dots, s-1$. For each $a \in \mathbb{N}$ there is a unique color $i(a)$ such that

$$K_a = \{b : f(\{a, b\}) = i(a)\} \in U.$$

The map $a \mapsto i(a)$ partitions $\mathbb{N}$ into s pieces, exactly one of them, say L, is in U: $L = \{a : i(a) = i\}$. We can now inductively select the elements $a_0, a_1, \dots \in L$ such that $a_{n+1} \in K_{a_0} \cap \cdots \cap K_{a_n}$ and $a_{n+1} \neq a_0, \dots, a_n$. $\square$

Theorem 5.2. *If* $1 \le r \in \mathbb{N}$, $2 \le s \in \mathbb{N}$, $f : [\mathbb{N}]^r \to \{1, 2, \dots, s\}$ *then there is an infinite monocolored* $A \subseteq \mathbb{N}$.

Proof. By induction on r as in the proof of Theorem 4.1. $\square$

With the help of the Continuum Hypothesis, one can construct a nontrivial ultrafilter U on $\mathbb{N}$ such that if $f : [\mathbb{N}]^r \to \{0, 1, 2, \dots, s-1\}$, then there is a homogeneous set A for f with $A \in U$.

Problem 5.3. Deduce the finite Ramsey theorem from the infinite one.

Theorem 5.4 (Rado)**.**

If f colors the natural numbers with an arbitrary (possibly infinite) number of colors, then there is an infinite set $H \subseteq \mathbb{N}$ on which f is canonical in the sense that one of the following possibilities occur:
(a) for $a, b \in [H]^2$, $f(a) = f(b)$, i.e., H is homogeneous for f*;*
(b) for $a, b \in [H]^2$, $f(a) = f(b)$ iff $\min(a) = \min(b)$*;*
(c) for $a, b \in [H]^2$, $f(a) = f(b)$ iff $\max(a) = \max(b)$*;*
(d) $\{f(a) : a \in [H]^2\}$ are all different.

Theorem 5.5 (Sierpiński)**.** *There is a coloring* $f : [\mathbb{R}]^2 \to \{0, 1\}$ *with no monocolored uncountable set.*

Proof. By the well ordering theorem, there is a well ordering $<_w$ of $\mathbb{R}$. We define the following coloring $f : [\mathbb{R}]^2 \to \{0, 1\}$. If $a < b$ are reals, we let

$$f(\{a, b\}) = \begin{cases} 1, & a <_w b, \\ 0, & b <_w a. \end{cases}$$

In words, the color of $\{a, b\}$ is 1 exactly if $<$ and $<_w$ agree on a, b.

In order to show the statement of the theorem, assume that $A \subseteq \mathbb{R}$ is homogeneous in color 1. This means, that $<$ and $<_w$ agree on A, specifically, A is a well ordered subset of $\mathbb{R}$. From this, we deduce that A is countable. Assume indirectly, that A is uncountable. For each $x \in A$, except the largest element $\max(A)$ (if exists), we can let $g(x) \in A$ be the least element $y \in A$ with $y > x$. $g(x)$ exists, as $<$ is a well order on A. The intervals

$$\{(x, g(x)) : x \in A - \{\max(A)\}\}$$

are uncountably many pairwise disjoint open intervals in $\mathbb{R}$. Each interval contains a rational number, they are distinct, and this is a contradiction, as there are only countably many rationals.

The case of the uncountable homogeneous set in color 0 goes exactly as the previous case. □

Theorem 5.6 (Gödel). *There is a coloring $f : [\mathbb{R}]^2 \to \{0, 1, \dots\}$ with no monocolored triangle.*

Proof. Enumerate the rational numbers as $\mathbb{Q} = \{q_0, q_1, \dots\}$. Define $f : [\mathbb{R}]^2 \to \mathbb{N}$ as follows. If $x < y$ are real numbers, then set $f(x, y) = i$ iff i is the least natural number such that $x < q_i < y$. Assume that $x < y < z$ and x, y, z form an i-colored triangle. Then $x < q_i < y < q_i < z$, an impossibility. □

We notice the following two fundamental results in infinite Ramsey theory.

Theorem 5.7 (Dushnik–Erdős–Miller). *If A is an infinite set, $f : [A]^2 \to \{0, 1\}$ then either there is a set of size $|A|$, homogeneous in color 0, or else there is an infinite set, homogeneous in color 1.*

Theorem 5.8 (Erdős–Rado). *If A is a set with cardinality greater than continuum, the pairs of A are colored with countably many colors, then there is an uncountable homogeneous set.*

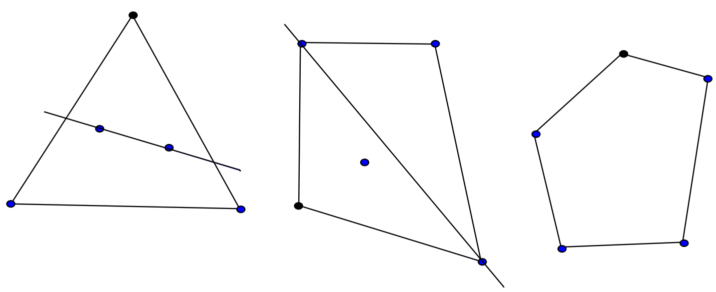
1 A nice problem — boost for combinatorial geometry and Ramsey theory

In 1933, the 20-year-old student Pál Erdős together with his friends in Budapest started to investigate a problem of a geometric flavour.

Problem 1.1 (Klein, Erdős, Szekeres)**.** Given a set of some points in the plane in general position (no collinear triples), is there a subset of n points which are the vertices of a convex k-gon?

Definition 1.2. $K(n)$ is the minimal number of points such that any set of $K(n)$ points in the plane in general position contains the vertices of a convex k-gon.

Proposition 1.3. $K(4) = 5, K(5) = 9, K(6) = 17.$



Exercise 1.4. Prove that $K(5) = 9$.

(Hint: consider the convex hull!)

Conjecture 1.5 (Erdős)**.** $K(n) = 2^{n-2} + 1$.

Theorem 1.6 (Erdős-Szekeres, ′35, [6])**.** $K(n)$ *is well defined: A large enough point set contains an arbitrary large convex subset.*

This statement resembles the well known Ramsey theorem, t.i., a large enough graph contains either a complete or an empty subgraph of a given size. Both express examples for the principle "in a large system, complete disorder is impossible," or more precisely by Mirsky: "there are numerous theorems in mathematics which assert, crudely speaking, that every system of a certain class possesses a large subsystem with a higher degree of organization than the original system."

Erdős and his co-author also discussed some related questions.

Theorem 1.7 (Erdős, Szekeres)**.** *For given* $r, s > 1$*, any sequence of length at least* $(r-1)(s-1)+1$ *contains a monotone increasing subsequence of length* r *or a monotone decreasing subsequence of length* s*.*

Problem 1.8 (Erdős, Szekeres)**.** Given a set of some points in the plane in general position, is there a subset of n points which determine a convex or a concave arc?

1.1 Basic facts about Ramsey theory

Problems in Ramsey theory typically ask a question of the following form. How many elements of some structure must have to guarantee that a particular property will hold?

The starting point to this theory was a result due to Ramsey, which can be formulated in the language of graphs as follows.

Theorem 1.9. *For any given integers k and l there is a positive integer number N such that if the edges of a complete graph of order N are colored with 2 different colors, then it must contain a complete subgraph of order k or l whose edges are colored uniformly with the first or second color, respectively.*

Definition 1.10. *(Ramsey numbers for 2-colored graphs) $R^{(2)}(k,l)$ denotes the least such number N.*

A well known upper bound based on a recursion form was obtained by Erdős and Szekeres:

$$R^{(2)}(k,l) \le \binom{k+l-2}{k-1}.$$

The problem has various generalizations. One may consider some monochromatic subgraphs of a given order belonging to a certain family (instead of considering only cliques). Also, the number of colors can be increased or one may consider colorings of every t-element subsets of a certain set. In this direction, a general statement is

Theorem 1.11. *For every choice of natural numbers t, p and $k_1, \dots, k_p$, there exists a natural number N such that whenever all the t-element subsets of an N-element set is colored by p colors, there exists a k_i element subset whose t-tuples are all colored with the ith color, for some i.*

Definition 1.12 (Ramsey numbers for p-colored t-uniform hypergraphs)**.** $R^{(t)}(k_1, \dots k_p)$ denotes the least such number N.

Exercise 1.13. Prove an upper bound on $R^{(3)}(n,n)$.

We end this subsection by mentioning some of the most celebrated Ramsey-type theorems. Van der Waerden's theorem states that for any given c and n, there is a number N, such that if N consecutive numbers are colored with c different colors, then it must contain an arithmetic progression of length n whose elements are all the same color.
The Hales-Jewett theorem informally states that for any positive integers n and c there is a number H such that if the cells of a H-dimensional $n \times n \times n \times ... \times n$ cube are colored with c colors, there must be one row, column, or certain diagonal of length n all of whose cells are the same color.
Another number theoretic result is Schur's theorem, claiming that for any given c there is a number N such that if the numbers $1, 2, \dots, N$ are colored with c different colors, then there must be a pair of integers x, y such that x, y, and $x+y$ are all the same color.

For further results see [9].

1.2 Proofs for the problems of Erdős and Szekeres

1st proof for the Erdős-Szekeres theorem 1.6, key ideas. Color the quadruples red if they determine a convex quadrilateral, otherwise color them blue. $R^{(4)}(n,5)$ provides an upper bound.

□

2st proof - by Tarsi -, key ideas. Assume that the horizontal direction is not determined by any pair of points. Then any triple determines either a convex or a concave arc. Color the triples red and blue accordingly.

$R^{(3)}(n,n)$ thus provides an upper bound.

□

Definition 1.14. k points in the plane in general position is called a *convex, resp. concave arc* if the points lie on the graph of a convex, resp. concave function. Alternatively, every point must be a point on the convex hull and the convex hull is bounded by a single edge from above, resp. from below.

Large convex polygons obviously determine large convex or concave arcs. This motivates the following problem of Erdős.

Problem 1.15. Let $F(k,l)$ denote the least number N such that any set of N points in the plane contains a convex k-arc or a concave l-arc, if the points are in general position and the vertical direction is not determined.

Proposition 1.16. $K(n) \leq F(n,n)$.

Observe that the second proof for Theorem 1.6 essentially gives an upper bound for $F(n,n)$ as well! Our next goal is to study the function $F(k,l)$.

Theorem 1.17. *Upper bound:* $F(k,l) \leq \binom{k+l-4}{k-2} + 1$.

Proof. We prove by induction. If $k = 2$ or $l = 2$, the statement is obvious. Otherwise, we apply the following inequality: $F(k,l) \leq F(k-1,l) + F(k,l-1) - 1$, which implies immediately the desired result.

To show the inequality, suppose we have a set $\mathcal{S}$ of $N = F(k-1,l) + F(k,l-1) - 1$ points with no concave l-arc. Consider those points for which a convex $k-1$-arc exists ending with point in view. Let the set $\mathcal{P}$ consists of these points.
Observe that
- $\mathcal{S} \setminus \mathcal{P}$ contains neither a convex $k-1$-arc, nor a concave l-arc, thus $|\mathcal{S} \setminus \mathcal{P}| < F(k-1,l)$. Thus
- $|\mathcal{P}| \geq F(k,l-1)$.
- If $\mathcal{P}$ contains a convex k-arc, we are done;
- otherwise $\mathcal{P}$ contains a concave $l-1$-arc. The concave $l-1$-arc, and the convex $k-1$-arc ending in the first point of the concave $l-1$-arc together determine either a concave l-arc or a the convex k-arc. □

Theorem 1.18. *Lower bound: construction,* $F(k,l) > \binom{k+l-4}{k-2}$.

If l or k is 3, the construction (without convex k-arcs and concave l-arcs) with the given cardinality is straightforward. The general construction is recursive, based on a disjoint union of a particular set $\mathcal{S}'$ containing no convex $k-1$-arcs or concave l-arcs, and a set $\mathcal{S}''$ containing no convex k-arcs or concave $l-1$-arcs with the property that $|\mathcal{S}'| = \binom{k+l-5}{k-3}$ and $|\mathcal{S}''| = \binom{k+l-5}{l-3}$

The crucial point is to place these sets on the plane to satisfy the following statement: If a convex arc shares points with both S' and S'', then it cannot share more than 1 with S''. Similarly, if a concave arc shares points with both S' and S'', then it cannot share more than 1 with S'.

Corollary 1.19. $F(k,l) = \binom{k+l-4}{k-2} + 1$.

Corollary 1.20. *There are planar point sets of n points in general position for which the largest convex subset is of order $O(\log n)$.*

Theorem 1.21 (Erdős, Szekeres, [7])**.** $K(n) \geq 2^{n-2} + 1$.

The result is based again on a construction, a really nice but a bit involved one.

So far, the above lower bound is the best known. As for the upper bound it was slightly improved by Géza Tóth and Pawel Valtr.

Proposition 1.22 (G. Tóth, P. Valtr, [15])**.** $K(n) \leq \binom{2n-5}{n-2} + 2$.

Problem 1.23. The Erdős-Szekeres problem makes sense also in higher dimension. Let $K_d(n)$ denote the least number such that, in any set of $K_d(n)$ points in general position in the Euclidean d-space a vertex set of a convex polytope with n vertices is contained, that is, n points in convex position.

Exercise 1.24. Show that $K_d(n) \leq K_2(n) \equiv K(n)$ for $d > 2$.

Imre Bárány, Gyula Károlyi and Pawel Valtr obtained bounds on this general function.

Proposition 1.25 ([1])**.** $K_d(n) \leq \binom{2n-2d-1}{n-d} + d$

Conjecture 1.26 (Füredi)**.** $\log K_d(n) = O(n^{1/(d-1)})$.

2 Empty convex polygons and large angles

In 1978, Erdős raised the following question: What is the smallest integer such that any point set in the plane with the prescribed cardinality contains at least one *empty convex polygon* with k vertices - a polygon which does not contain any further point of the point set in its interior? An empty convex k-gon is also called *convex k-hole*.

Problem 2.1 (Horton sets)**.** Erdős conjectured that for every k there is an n_k for which, if n_k points are given in the plane, no three on a line, one can find among them k which form a convex k-gon that contains none of the points in its interior. $n_4 = 5$ is immediate and Harborth proved that $n_5 = 10$.

Surprisingly, this conjecture turned out to be false.

Theorem 2.2 (Horton [10], Nicolás and Gerken [8,12])**.** *n_k does not exist for $k \geq 7$ (Seven-hole theorem). On the other hand, n_6 exists.*

Horton's proof is based on nice construction built recursively, in a quite similar way to the proof of Erdős and Szekeres in [7]. The latter statement is less then 10 years old. Since this result, the least number of convex k-holes determined by any set of n planar points is also investigated for the remaining cases ($3 \leq k \leq 6$). The bounds are quadratic in n.

The original question naturally raises further extremal problems.

Problem 2.3. Bound the maximal angle determined by a planar point set of N points!

Observation 2.4. This is strongly connected to the Erdős-Szekeres result on convex polygons of point sets, since an angle at least $\pi(1 - \frac{2}{n})$ is contained in a convex n-gon, so we may apply Theorem 1.22.

Erdős and Szekeres essentially solved this problem.

Theorem 2.5 (Szekeres [14])**.** *If $N \geq 2^n+1$, then a point set of N points contains an angle at least $\pi(1-\frac{1}{n}+\frac{1}{n\cdot 2^n+1})$.*

Theorem 2.6 (Szekeres [14])**.** *If $N = 2^n$, $\varepsilon > 0$, then there exists a point set of N points, such that the largest angle is at most $\pi(1 - \frac{1}{n} + \varepsilon)$.*

Theorem 2.7 (Erdős-Szekeres [7])**.** *If $N = 2^n$, then every planar point set of N points contains an angle greater than $\pi(1 - \frac{1}{n})$.*

3 Further related problems

Erdős and Guy posed the following generalization of the Erdős-Szekeres problem: What is the least number of convex k-gons determined by any set of n points in the plane? The trivial solution for the case $k = 3$ is $\binom{n}{3}$.

The problem of finding sets of n points minimizing the number of convex quadrilaterals is equivalent to minimizing the crossing number in a straight-line drawing of a complete graph. The number of quadrilaterals must be proportional to the fourth power of n, but the precise constant is not known [13].

Erdős formulated a conjecture also on possible large angles determined by a point set in the d-dimensional space.

Conjecture 3.1 (Erdős)**.** *Any set $\mathcal{P}$ of points in $\mathbb{R}^d$ with cardinality $|\mathcal{P}| > 2^d$ contains a triple of points which determines a angle greater than $\frac{\pi}{2}$.*

Exercise 3.2. It is easy to see why this bound is natural. Why?

Danzer and Grünbaum confirmed this conjecture [2], and then asked the problem of estimating the size of a point set in the d-dimensional space such that any three points form an acute triangle. (That is, we exclude the occurrence of the right angle as well.)
They also constructed a point set in $\mathbb{R}^d$ of size $2d - 1$ such that no three points form a right angle or an obtuse angle, and conjectured that this size is best possible.
However, this conjecture was disproved by Erdős and Füredi [5]. Surprisingly, the linear lower bound was improved to an exponential lower bound.

Theorem 3.3. *There exists a point set $\mathcal{S} \subset \{0,1\}^d \in \mathbb{R}^d$ of size $|\mathcal{S}| \geq \left\lfloor \frac{1}{2}(\frac{2}{\sqrt{3}})^d \right\rfloor$ which determines only acute triangles.*

(Note that $\left\lfloor \frac{1}{2}(\frac{2}{\sqrt{3}})^d \right\rfloor \approx 0.5 \cdot 1.55^d$)

Their proof is based on a probabilistic argument, a method which was pioneered by Pál Erdős.

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