SUMMER SCHOOL IN MATHEMATICS

EÖTVÖS LORÁND UNIVERSITY

BUDAPEST, HUNGARY

10. July – 14. July 2023

Paradoxical Decompositions, Fractals and Dynamics

Zoltán Buczolich (ELTE):
Introduction to dynamical systems, fractals and ergodic theory

Márton Elekes (Rényi Institute – ELTE):
Introduction to measure theory, geometric measure theory, geometric decompositions and descriptive set theory

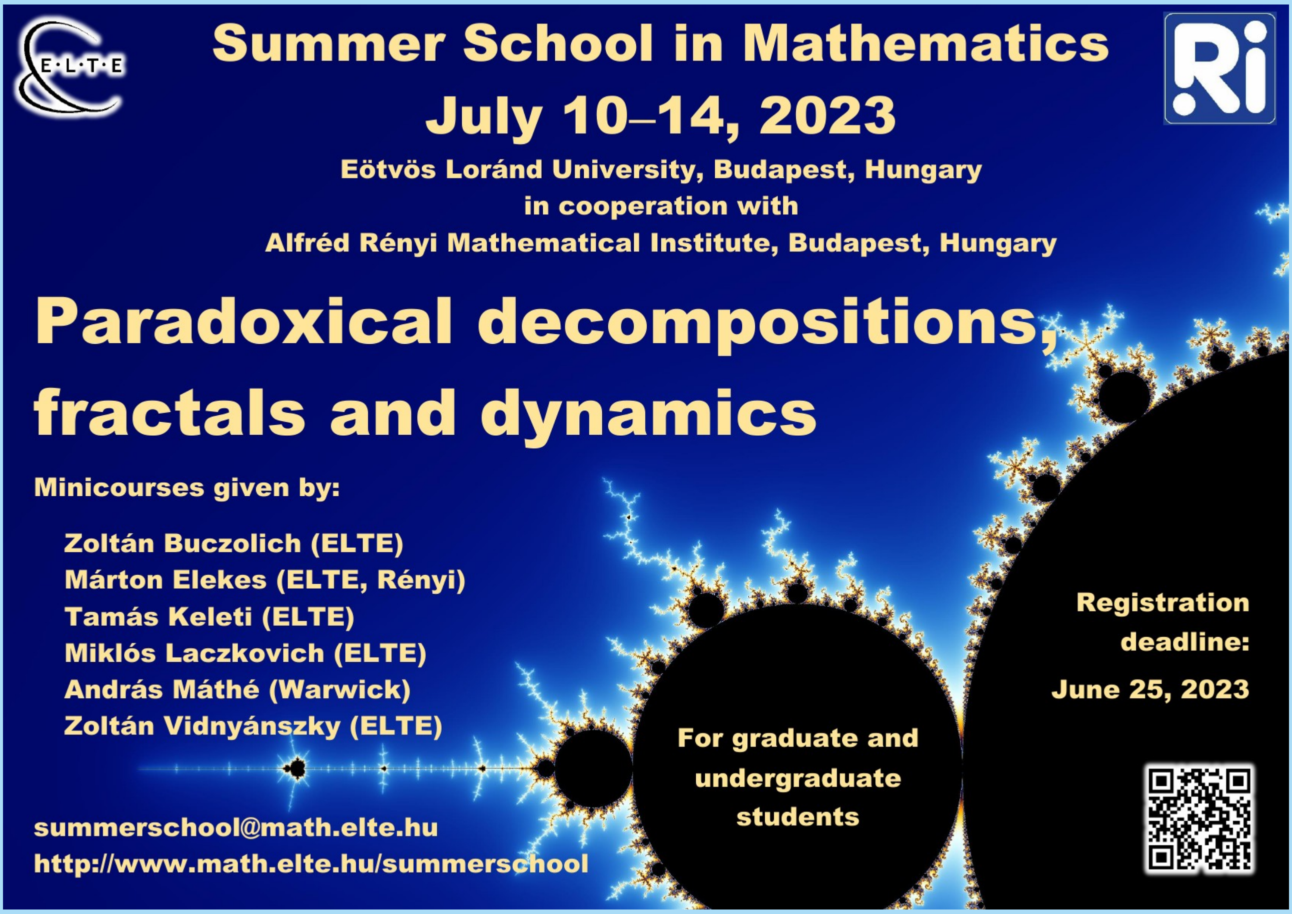
Tamás Keleti (ELTE):
The Kakeya problem

Miklós Laczkovich (ELTE):
The Banach–Tarski paradox

András Máthé (University of Warwick):
Tarski's circle squaring problem

Zoltán Vidnyánszky (ELTE):
Finite and infinite: connections between distirbuted computing and Borel combinatorics

Budapest, July 2023



SSM 2023	PARADOXICAL DECOMPOSITIONS, FRACTALS AND DYNAMICS				ELTE, BUDAPEST
	Monday, July 10	Tuesday, July 11	Wednesday, July 12	Thursday, July 14	Friday, July 14
9.00 -- 10.00	Laczkovich - 1	Buczolich - 1	Buczolich - 2	Buczolich - 3	Vidnyánszky - 3
10.00 -- 10.30	C O F F E E / R E F R E S H M E N T				
10.30 -- 11.30	Elekes - 1	Laczkovich - 2	Laczkovich - 3	Vidnyánszky - 2	Keleti
11.30 -- 11.45	C O F F E E				
11.45 -- 12.45	Elekes - 2	Vidnyánszky - 1	Máthé - 1	Máthé - 2	Máthé - 3
12.45 -- 14.00	L U N C H				
14.00 -- 15.30	Tutorial 1	Tutorial 2	CAVE TOUR	Tutorial 3	Tutorial 4
15.30 -- 17.00		PIZZA PARTY		BIKE TOUR	
17.00 -- 19.00					

Zoltán Buczolich (ELTE)	Introduction to dynamical systems, fractals and ergodic theory
Márton Elekes (Rényi / ELTE)	Introduction to measure theory, geometric measure theory, geometric decompositions and descriptive set theory
Tamás Keleti (ELTE)	The Kakeya problem
Miklós Laczkovich (ELTE)	The Banach–Tarski paradox
András Máthé (Warwick)	Tarski's circle-squaring problem
Zoltán Vidnyánszky (ELTE)	Finite and infinite: connections between distributed computing and Borel combinatorics

CONTENTS

LIST OF REGISTERED PARTICIPANTS

1. **Bakay, Diana** – Ivan Franko National University of Lviv (*Ukraine*)
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3. **Brodbelt, Alexander** – university of Edinburgh (*United Kingdom*)
4. **Carvalho Carneiro, Nuno Gabriel** – Instituto Superior Técnico, University of Lisbon (*Portugal*)
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6. **Forman, Balázs Attila** – Eötvös Loránd University, Budapest (*Hungary*)
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15. **Pigler, Donát** – Eötvös Loránd University, Budapest (*Hungary*)
16. **Richmond, Lorian** – University of Edinburgh (*United Kingdom*)
17. **Sewell, Benedict** – Alfréd Rényi Mathematical Institute, Budapest (*Hungary*)
18. **Urbonaite, Neringa** – Vilnius University (*Lithuania*)

PREFACE

The first international summer school in mathematics, organized by the Institute of Mathematics at Eötvös Loránd Universty in Budapest, Hungary, took place in 2013. Since then a series of similar one week events was organized each year (with the exception of the two COVID-years, i.e. 2020 and 2021). Starting from the second year the schools were concentrating on one particlular topic (general discrete mathematics, algorithms, graph limits, algebraic geometry and topology, number theory etc.) A large portion of related materials of these schools can be found at the archives of the website of the series:

`http://www.math.elte.hu/summerschool/?page=download`

The summer school organized in 2023 was the 9th in this series. It took place at the Lágymányos Campus of Eötvös Loránd University in Budapest between July 10 and 14, 2023. The title of the school was *Paradoxical decompositions, fractals and dynamics.* Many of the lectures were related to questions about unusual geometrical decompositions like the ones appearing in the 100 years old Banach–Tarski paradox, cutting and rearranging 3 dimensional objects or the deocmpositions in the result of Laczkovich from the end of 1980's, rearranging finitely many pieces of a circle to make a square. Besides these topics, many other wonders of analysis – like fractals, ergodic theory, dynamical systems and Borel combinatorics – appeared in the lectures.

The lecturers were Zoltán Buczolich (Eötvös Loránd University), Márton Elekes (Rényi Institute and (Eötvös Loránd University), Tamás Keleti (Eötvös Loránd University), Miklós Laczkovich (Eötvös Loránd University), Adrás Máthé (University of Warwick) and Zoltán Vidnyánszky (Eötvös Loránd University). The practice classes were led by Richárd Balka (Rényi Institute), Márton Borbényi (Eötvös Loránd University), Tamás Kátay (Eötvös Loránd University) and Máté Pálfy (Eötvös Loránd University).

Course notes were made available to the participants for some of the lectures, either before or after the lectures. The present booklet is a somewhat brushed up version of these notes, put together into one volume. They appear together with the set of exercises discussed in the practice classes. Of course it is no way complete: for some of the lectures where easily accessible literature exists, only the abstract is inserted as a reminder of the topic. This volume can be downloaded from the same website as the notes of the previous schools.

We wish to thank Eötvös Loránd University and the Alfréd Rényi Mathematical Institute for financial support. We would also like to express our gratitude to all lecturers and contributors of this volume but also to the audience whose active participation makes the whole series of summerschools meaningful.

Budapest, August 20, 2023

István Ágoston
organizer

July 2023 ELTE

Introduction to Dynamical Systems, Fractals and Ergodic Theory

PART 1

Zoltán Buczolich

Eötvös University
Budapest, Hungary,

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$f(x) = \sqrt{x}$

$f(2) = \sqrt{2} = 1.41421356...$
$f \circ f(2) = f^2(2) = \sqrt{\sqrt{2}} = 2^{1/4} = 1.189207...$
$f^3(2) = 1.090507...$
$f^4(2) = 1.04427...$
...
$f^{100}(2) = 1.0000... + \varepsilon$

$f^n(2) = 2^{1/2^n} \to 1,$ and $f(1) = \sqrt{1} = 1,$
1 is a fixed point of f.

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Recall Banach's fixed point theorem:

D.: Suppose (X, d) is a metric space. If for $f : X \to X$ there exists $\gamma \in (0, 1)$ such that for all $x, y \in X$
$d(f(x), f(y)) \leq \gamma d(x, y)$ then f is a contraction.

T.: *Suppose that f is a contraction defined on the complete metric space (X, d). Then f has exactly one fixed point, x_∞ and for any $x_0 \in X$ we have $f^n(x_0) \to x_\infty$.*

$f(x) = \sqrt{x}$ is NOT a contraction on $X = (0, +\infty)$. (Prove it.)

Find an interval $I \subset (0, +\infty)$ such that f maps I into itself, $2 \in I$ and f is a contraction on I.

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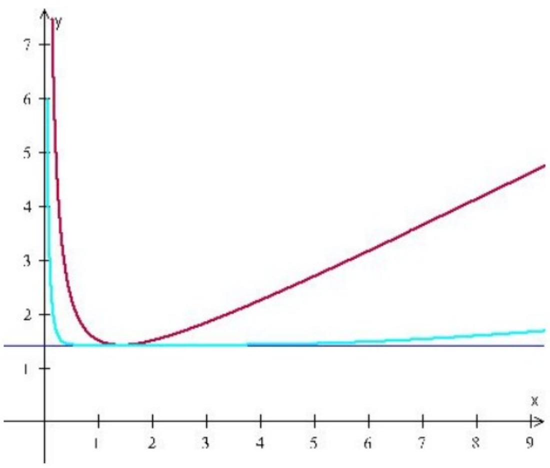

Greek method of computing $\sqrt{2} \approx 1.414213562...$:
$1.2^2 = 1.44 < 2 < (\frac{2}{1.2})^2 = 2.77...$
$\Rightarrow \dfrac{1.2 + \frac{2}{1.2}}{2} = 1.433...$
is a better approximation,
$\dfrac{(1.433...) + \frac{2}{1.433...}}{2} = 1.414341085$
is even better.
We take the sequence $x_1 = 1.2$,
$x_{k+1} = f(x_k) = \dfrac{x_k + \frac{2}{x_k}}{2}$.
$x_k \to \sqrt{2}$.
$f(\sqrt{2}) = \sqrt{2}$ is a globally attracting fixed point in $(0, +\infty)$.
(On the figure
$f(x)$, $y = \sqrt{2}$, $f^3(x)$.)

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Given X a "phase space", "state space" and
a transformation of X into itself "law of nature" $f : X \to X$
we would like to understand the dynamics, the long term behavior of this (dynamical) system.

If f is a contraction of a complete metric space everything is very simple. Every point converges to this fixed point. We will see that things can get much more complicated.

i.) If X is a differentiable manifold and T is a (sufficiently smooth) diffeomorphism (or at least a differentiable transformation) then we speak about differentiable (smooth) dynamics.
ii.) If X is a topological, or metric space and T is a homeomorphism (or at least a continuous transformation) then we speak about topological dynamics.
iii.) If X is a measure space $(X, \mathcal{B}, \mu)$ and T is a measure preserving transformation ($\mu(T^{-1}A) = \mu(A)$) then we speak about Ergodic theory.
ergod+odos=energy-path

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ergod+odos=energy-path

Sometimes the same system can be an example of all three types.

Example: Circle rotations: Let $X = \mathbb{T}$ be the circle of unit length $= [0, 1)$ $= \mathbb{R}/\mathbb{Z} =$ the reals modulo 1.
Given $\alpha \in \mathbb{R}$ let $T_\alpha : \mathbb{T} \to \mathbb{T}$, or $[0, 1) \to [0, 1)$ be
$T_\alpha(x) = x + \alpha$ (modulo 1) $= \{x + \alpha\}$.
If we think of $\mathbb{T}$ as the normalized unit circle in $\mathbb{C}$ then $T_\alpha e^{2\pi i\phi} = e^{2\pi i(\phi+\alpha)}$.
T_α clearly smooth (and hence continuous) on the manifold $\mathbb{T}$.
If we consider $(\mathbb{T}, \mathcal{L}, \lambda)$, where λ=Lebesgue-measure and $\mathcal{L}$=Lebesgue measurable sets, then
T_α is measure preserving ($\lambda(T_\alpha^{-1}(A)) = \lambda(A - \alpha) = \lambda(A)$ for $\forall A \in \mathcal{L}$).

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Ergodic theory = study of actions of (semi)groups on measure spaces

$T : X \to X$, we consider $\{T^n : n \in \mathbb{Z}_{\geq 0}\}$ the semi-group action of $\mathbb{Z}_{\geq 0}$.
We have $(\star)\quad T^{n+m} = T^n T^m, \ \forall n, m \in \mathbb{Z}_{\geq 0}$.
If T is invertible we can consider the group action $\{T^n : n \in \mathbb{Z}\}$, having $(\star)$ for all $n, m \in \mathbb{Z}$.
In these cases we have "discrete time", "snapshots" of the system. We work with discrete dynamical systems.

One can consider continuous dynamical systems, flows (coming usually from autonomous differential equations).
These are semigroup actions of $\mathbb{R}_{\geq 0}$, or in the invertible case of $\mathbb{R}$:
$T_t : X \to X$, $T_t : t \in \mathbb{R}$, $T_{s+t} = T_s T_t$ for all $s, t \in \mathbb{R}$.

One can consider other group actions
for example Z^2-actions, $\{T_g : g \in \mathbb{Z}^2\}$,
or in general Z^d-actions, $\{T_g : g \in \mathbb{Z}^d\}$.
If $\alpha \neq \beta$ one can consider the $\mathbb{Z}^2$-action, $T_{\alpha,\beta}^{(n,m)} : \mathbb{T} \to \mathbb{T}$, $(n, m) \in \mathbb{Z} \times \mathbb{Z}$
$T_{\alpha,\beta}^{(n,m)} x = \{x + n\alpha + m\beta\}$.

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Origin from Physics

k particles in $\mathbb{R}^3$,
positions (in generalized coordinates) q_i, momenta p_i, $i = 1, ..., k$.
Phase space $X = \mathbb{R}^{6k}$.
The Hamilton function $H(p, q) = K(p) + U(q)$
where $K(p)$ is the kinetic energy, and $U(q)$ is the potential energy.
Hamilton's equations:

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}.$$

Connection to high-school Physics:

$$mv = \frac{\partial \frac{1}{2} m v^2}{\partial v} = \frac{\partial K(p) + U(q)}{\partial p}, \text{ and } F = ma = \frac{\partial mv}{dt} = -\frac{\partial K(p) + U(q)}{\partial q}.$$

Energy surface $H^{-1}(e)$, Hamiltonian H is constant on solution curves (preservation of energy).
Liouville's theorem: The Hamiltonian flow, T_t (the solution flow from the H. equations) preserves the Lebesgue-measure on $\mathbb{R}^{6k}$.
$(\lambda(T_t^{-1}(A)) = \lambda(A)$, for all $A \in \mathcal{L}$.)

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Boltzmann's ergodic hypothesis: "$\{T_t(x) : t \in \mathbb{R}\}$ "equals" the energy surface $H^{-1}(e)$."
Boltzmann gave the name Ergodic, recall ergon=work, energy, odos=path in Greek.
Boltzmann's ergodic hypothesis is false.
We can only hope for density of $\{T_t(x) : t \in \mathbb{R}\}$ on the energy surface.
Boltzmann also conjectured the hypothesis for
the equality of time means and phase (space) means.

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D.: A measure space $(X, \mathcal{B}, \mu)$ is the triple consisting of the phase space X, the σ-algebra of the measurable sets $\mathcal{B}$ and a probability measure μ, (this means $\mu(X) = 1$).
Sometimes we work with finite measures $\mu(X) < +\infty$, but these can always be normalized.
Infinite Ergodic theory is different in that case σ-finite measure spaces like $(\mathbb{R}, \mathcal{L}, \lambda)$ can be considered.

D.: Given two measure spaces $(X_1, \mathcal{B}_1, \mu_1)$ and $(X_2, \mathcal{B}_2, \mu_2)$ the transformation $T : X_1 \to X_2$ is measure preserving if
$\mu_1(T^{-1}A) = \mu_2(A)$ holds for all $A \in \mathcal{B}_2$.

T.: (Poincaré's Recurrence Theorem) Let $T : X \to X$ be meas. pres. on the prob. space $(X, \mathcal{B}, \mu)$. If $\mu(A) > 0$ then μ almost every $x \in A$ returns to A.

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T.: (Poincaré's Recurrence Theorem) Let $T : X \to X$ be meas. pres. on the prob. space $(X, \mathcal{B}, \mu)$. If $\mu(A) > 0$ then μ almost every $x \in A$ returns to A.

Proof.: $F \stackrel{\text{def}}{=} A \setminus \bigcup_{k=1}^{\infty} T^{-k}A$
(these are those points which never return to A).
$F = A \cap T^{-1}(X \setminus A) \cap T^{-2}(X \setminus A) \cap \ldots$
$F \cap T^{-n}F = \emptyset$ for all $n \geq 1$ $\Rightarrow T^{-k}F \cap T^{-(n+k)}F = \emptyset$ for all $n \geq 1$, $k \geq 0$
$\Rightarrow F$, $T^{-1}F$, $T^{-2}F$, ... are pairwise disjoint.
T is measure preserving $\Rightarrow \mu(T^{-k}F) = \mu(F)$
$\mu(X) < +\infty \Rightarrow \mu(F) = 0$. ■

$(\mathbb{R}, \mathcal{L}, \lambda)$ with $Tx = x + 1$ gives an example that Poincaré's Recurrence Theorem is not true on σ-finite measure spaces.
No point returns to say $A = [0, 1)$.

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The three and n-body problem

The problem of finding the general solution to the motion of more than two orbiting bodies in the solar system known originally as the three-body problem and later the n-body problem ($n \geq 2$). In honour of his 60th birthday, Oscar II, King of Sweden, advised by Gösta Mittag-Leffler, established a prize for anyone who could find the solution to the problem. The prize was finally awarded to Poincaré, even though he did not solve the original problem. (The first version of his contribution even contained a serious error). The version finally printed contained many important ideas which led to the theory of chaos. He found that there can be orbits that are nonperiodic, and yet not forever increasing nor approaching a fixed point. (source Wikipedia, see also https://www.mittag-leffler.se/about-us/history/prize-competition/)
Poincaré called the recurrence theorem: "the stability theorem à la Poisson".

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Ex.1.: Suppose our space X is the disjoint union of two circles $X = \mathbb{T}_1 \cup \mathbb{T}_2$.
We consider the normalized Lebesgue measure on the union.
Suppose $\alpha, \beta \in \mathbb{R} \setminus \mathbb{Q}$. Define $Tx = \begin{cases} x+\alpha & \text{if } x \in \mathbb{T}_1 \\ x+\beta & \text{if } x \in \mathbb{T}_2. \end{cases}$
Then $T^{-1}(\mathbb{T}_1) = \mathbb{T}_1$ is a (strongly) invariant set of measure 1/2.
Ex.2.: Suppose $X = \mathbb{T} \times [0,1]$ with the Lebesgue measure on the product.
Let $T(x, \alpha) = (x + \alpha, \alpha)$.
Then we have continuum many T invariant sets.
The invariant sets $X_\alpha = \{(x, \alpha) : x \in \mathbb{T}\}$ are all of zero measure,
but one can find invariant sets of positive but not of full measure as well, for example $X^* = \mathbb{T} \times [0, 1/2]$ is also invariant and is of measure 1/2.

D.: Suppose $(X, \mathcal{B}, \mu)$ is a prob. space. A meas. pres. tr. T of $(X, \mathcal{B}, \mu)$ is ergodic if for all $A \in \mathcal{B}$, $T^{-1}A = A$ implies $\mu(A) = 0$, or $\mu(A) = 1$. (i.e. only trivial sets are invariant).

Example 1 is not an ergodic tr. but the space can be split into two components on which T is ergodic.

Example 2 is more delicate. This space splits into continuum many "ergodic" components each of measure zero and one needs to "disintegrate" the original masure to obtain suitable ergodic measures on the components.

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T.: (L^p Ergodic Thm. of von Neumann) *Let $1 \le p < \infty$, T be a meas. pres. tr. on the prob. space $(X, \mathcal{B}, \mu)$. If $f \in L^p(\mu)$ then there exists $f^* \in L^p(\mu)$ such that $f^* \circ T = f^*$ a.e. and*

$$\left\| \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) - f^*(x) \right\|_p \to 0.$$

T.: (Birkhoff's Ergodic Theorem) *Suppose $(X, \mathcal{B}, \mu)$ is a prob. meas. space and $T : X \to X$ is a meas. pres. tr., moreover $f \in L^1(\mu)$. Then*

$$\frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) \to f^*(x) \in L^1(\mu) \text{ a.e.}$$

$f^* \circ T = f^*$ a.e. and $\int_X f^* d\mu = \int f d\mu$.

If T is ergodic then $(\star)$ $f^* = \int f d\mu$ *a.e.*

In the ergodic case $(\star)$ means that the "Boltzmann time average" $\frac{1}{n} \sum_{k=0}^{n-1} f(T^k x)$ converges a.e. to the "space average" $\int_X f(x) d\mu(x)$.

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Next we suppose that T is an invertible measure preserving transformation on the prob. meas space $(X, \mathcal{B}, \mu)$.
For invertible transfomations $\mu(T(A)) = \mu(T^{-1}(T(A))) = \mu(A)$, which means that T^{-1} is also measure preserving.

D.: Let T be a meas. pres. tr., and $A \in \mathcal{B}$ with $\mu(A) > 0$ be fixed.
By Poincaré's recurrence theorem
$n_A(x) = \inf\{n \geq 1 : T^n x \in A\}$ is finite for μ a.e. $x \in A$.
Consider $(X, \mathcal{B}|_A, \mu|_A)$ where $\mu_A(B) = \frac{\mu(B)}{\mu(A)}$,
for any $B \in \mathcal{B}|_A = \{B' \cap A : B' \in \mathcal{B}\}$.
The induced ("derivative") transformation
$T_A : A \to A$ is given by
$T_A(x) = T^{n_A}(x)$.

Most of the time we ignore sets of measure zero so it is not a problem that T_A is defined only $\mu|_A$ a.e.
E.g. the a.e. version of the definition of ergodicity is this:
D.: Suppose $(X, \mathcal{B}, \mu)$ is a prob. space. A meas. pres. tr. T of $(X, \mathcal{B}, \mu)$ is ergodic if for all $A \in \mathcal{B}$,
$\mu(T^{-1}A \Delta A) = \mu((T^{-1}A \setminus A) \cup (A \setminus T^{-1}A)) = 0$
implies $\mu(A) = 0$, or $\mu(A) = 1$.

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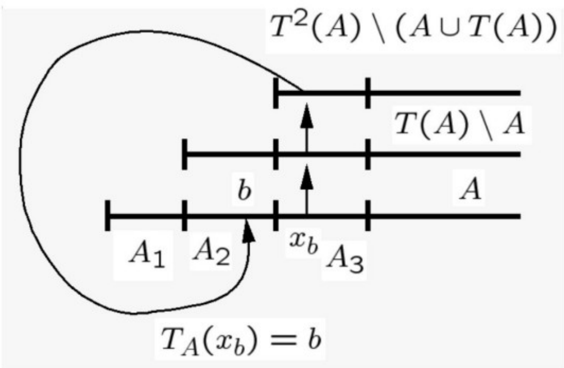

Prop.: T_A is measure preserving.

Proof.: $A_n \stackrel{\text{def}}{=} \{x \in A : n_A(x) = n\}$
Suppose $B \subset A$,
for a.e. $b \in B$ select x_b such that $T^{n_A(x_b)}(x_b) = b$,
since T is meas. pres. and invertible T^{-1} is also meas pres. and invertible,
hence for a.e. $b \in B$ there is x_b.
Set $B_n = \{b \in B : n_A(x_b) = n\}$.
Then $\mu(T_A^{-1}(B_n)) = \mu(T^{-n}(B_n)) = \mu(B_n)$ for all n.
If $n \neq m$ then $T_A^{-1}(B_n)$ and $T_A^{-1}(B_m)$ are disjoint.
$\mu(T_A^{-1}(B)) = \mu(T_A^{-1}(\cup B_n)) = \sum \mu(T_A^{-1}(B_n)) = \sum \mu(B_n) = \mu(B)$. ■

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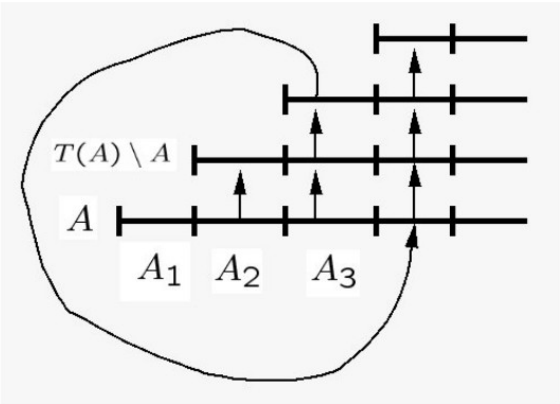

T.: Kac Lemma Suppose $(X,\mathcal{B},\mu)$ is a prob. meas. sp. and T is an invertible ergodic meas. pres. tr. If $A\in\mathcal{B}$ with $\mu(A)>0$ then $\int_A n_A(x)d\mu(x)=1$.

Remark: The expected recurrence time of a point to A: $\int_A n_A d\mu|_A = \frac{1}{\mu(A)}\int_A n_A d\mu = \frac{1}{\mu(A)}$.

Proof.: We use again the Kakutani skyscraper. Let $A_\infty \overset{\text{def}}{=} \cup_{k=0}^{\infty} T^n(A) = A\cup(TA\setminus A)\cup(T^2A\setminus(TA\cup A))\cup\ldots$.

Obviously, $TA_\infty\subset A_\infty$, since T^{-1} is meas. pres. $\mu(TA_\infty)=\mu(A_\infty)$

$\Rightarrow A_\infty = TA_\infty$, (modulo set of meas. zero) $\Rightarrow T^{-1}A_\infty = A_\infty$ a.e.

Since $\mu(A_\infty)>\mu(A)>0$, by ergodicity $A_\infty = X$ a.e.

$A_n=\{x\in A : n_A(x)=n\}$.

$\int_A n_A d\mu = \sum_{n=1}^{\infty} n\cdot\mu(A_n)=\mu(X)=1.$ ■

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Next we turn to topological dynamical systems.

Suppose X is a metric (or a topological) space and $T: X\to X$ is a homeomorphism, (or continuous in the non-invertible case).

D.: The T-orbit, or trajectory of $x\in X$ is $\mathcal{O}_T(x)\overset{\text{def}}{=}\{T^n x : n\in\mathbb{Z}\}$.

In case of non-invertible T we can talk about the positive semiorbit $\mathcal{O}_T^+(x)\overset{\text{def}}{=}\{T^n x : n\in\mathbb{Z}_{\geq 0}\}$. In this case $\mathcal{O}_T^+(x)$ is used in the next definitions instead of $\mathcal{O}_T(x)$.

D.: A $T:X\to X$ topological dynamical system is topologically transitive if $\exists x\in X$ such that its orbit, $\mathcal{O}_T(x)$ is dense in X.

D.: A $T:X\to X$ topological dynamical system is minimal if $\forall x\in X$ its orbit, $\mathcal{O}_T(x)$ is dense in X.

Exercise: Show that for irrartional α the rotation $T_\alpha:\mathbb{T}\to\mathbb{T}$, is minimal.

D.: For $T:X\to X$ denote by $P_n(T)$ the number of the set of those $x\in X$, for which $T^n x = x$. (n is not necessarily the minimal/prime period.)

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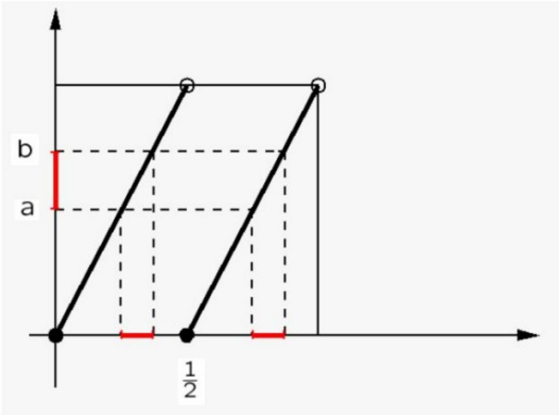
The doubling map



$E_2(x) = \{2x\}$
$= \begin{cases} 2x & \text{if } 0 \le x < \frac{1}{2} \\ 2x - 1 & \text{if } \frac{1}{2} \le x < 1. \end{cases}$

In complex notation $E_2(z) = z^2$, since $(e^{2\pi i x})^2 = e^{2\pi i 2x}$.
E_2 is non-invertible but it preserves the Lebesgue measure, $\lambda(E_2^{-1}(A)) = \lambda(A)$ for all $A \in \mathcal{L}$. One can see it on the figure for intervals, and they generate the σ-algebra $\mathcal{L}$.
Prop.: $P_n(E_2) = 2^n - 1$, the periodic points of E_2 are dense in $\mathbb{T}$ and E_2 is topologically transitive (and obviously non-minimal).
This shows that E_2 has much more complicated dynamics, than T_α.

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Prop.: $P_n(E_2) = 2^n - 1$, the periodic points of E_2 are dense in $\mathbb{T}$ and E_2 is topologically transitive (and obviously non-minimal).

Proof.: Using the complex representation of E_2:
$E_2^n(z) = z \Leftrightarrow z^{2^n} = z \Leftrightarrow z^{2^n-1} = 1$
$\Rightarrow$each $(2^n - 1)$st root of unity corresponds to a point with $z^{2^n} = z$ there are $2^n - 1$ such equally spaced points $\Rightarrow$the result about number and density.
Topological transitivity: Consider $x \in [0, 1) = \mathbb{T}$ in base-2,
$x = \sum_{i=1}^{\infty} a_i 2^{-i} = \Xi[a_1a_2...]$,
where $a_i \in \{0, 1\}$ and $\forall N > 0$, $\exists n > N$ s.t. $a_n = 0$ (this way we have unique repr.).
Then $E_2(x) = \left\{a_1 + \sum_{i=2}^{\infty} a_i 2^{-i+1}\right\} = \sum_{i=1}^{\infty} a_{i+1} 2^{-i} = \Xi[a_2a_3...]$.
$\Rightarrow E_2$ acts on the binary digits of x as the one sided shift: delete the first entry and then move each entry to the left. Notation $\sigma[a_1a_2...] = [a_2a_3...]$.
(From this approach one can see the periodic points as well, there are $2^p - 1$ many $0-1$-sequences of length p which are allowed, $[\underbrace{1...1}_{p}...]$ is not allowed.)

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Topological transitivity: Consider $x \in [0,1) = \mathbb{T}$ in base-2,
$x = \sum_{i=1}^{\infty} a_i 2^{-i} = \Xi[a_1 a_2 ...]$,
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Then $E_2(x) = \left\{ a_1 + \sum_{i=2}^{\infty} a_i 2^{-i+1} \right\} = \sum_{i=1}^{\infty} a_{i+1} 2^{-i} = \Xi[a_2 a_3 ...]$.

$\Rightarrow E_2$ acts on the binary digits of x as the one sided shift: delete the first entry and then move each entry to the left. Notation $\sigma[a_1 a_2 ...] = [a_2 a_3 ...]$.

For the top. transitivity we need x with a dense orbit:

$x \overset{\text{def}}{=} \Xi[\underbrace{0\,1}_{\text{len.}1}\ \underbrace{0\,0\,0\,1\,1\,0\,1\,1}_{\text{all str. of length }2}\ \underbrace{0\,0\,0\,0\,0\,1 \ldots 1\,1\,1}_{\text{all strings of length }3} \ldots] = \Xi[\omega]$, this x is allowed and for any binary "base interval" $J = \Xi[a_1 ... a_j *)$, $\exists k$ s.t. the first j entries of $E_2^k(x) = \Xi(\sigma^k[\omega])$ equal $a_1 ... a_j$, i.e. $E_2^k(x) \in J$.■

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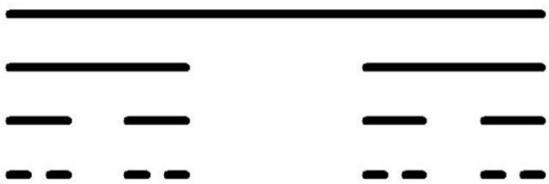
D.: Given a $T : X \to X$ top. dyn. sys. and $x \in X$
the ω-limit set of x (and the α-limit set) is

$$\omega(x) \overset{\text{def}}{=} \{y \in X : \exists n_i \to +\infty \text{ s.t. } T^{n_i} x \to y\} = \bigcap_{n=0}^{\infty} \text{cl}\Big(\bigcup_{m \geq n} T^m x\Big)$$

$$\alpha(x) \overset{\text{def}}{=} \{y \in X : \exists n_i \to -\infty \text{ s.t. } T^{n_i} x \to y\} = \bigcap_{n=0}^{-\infty} \text{cl}\Big(\bigcup_{m \leq n} T^m x\Big).$$

It is clear that $\omega(x)$ and $\alpha(x)$ are closed.
So far we have seen examples when $\omega(x)$ is
one point when we have an attracting fixed point;
union of finitely many points if x is a periodic point;
the whole space X if T is topologically transitive and $\mathcal{O}^+(X)$ is dense in X.

22



Other cases are also possible: It is possible that $\omega(x)$ is the Cantor-triadic set C_3.
Exercise: $x \in C_3$ iff x has an expansion in base-3 that do not contain the digit 1, in fact $\forall x_3 \in C_3$ there is unique triadic expansion $0.a_1a_2...$ with $a_i \in \{0, 2\}$.

Let $E_3 : \mathbb{T} \to \mathbb{T}$ be given by
$E_3(x) = \{3x\}$.
Prop.: *For E_3, C_3 is E_3 invariant, $E_3(C_3) \subset C_3$, and C_3 contains a dense orbit $\Rightarrow \exists x \in \mathbb{T}$ s.t. $\omega(x) = C_3$.*
Proof.: E_3 acts on the trenary digits of x as the shift $\Rightarrow E_3(C_3) \subset C_3$
$x \overset{\text{def}}{=} 0.\underbrace{0\,2}_{\text{len.}1}\ \underbrace{0\,0\,0\,2\,2\,0\,2\,2}_{\text{all str. of length }2}\ \underbrace{0\,0\,0\,0\,0\,2\ldots2\,2\,2}_{\text{all strings of length }3}\ldots$ ■

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Symbolic Dynamical Systems
Suppose $N \geq 2$,
$\Omega_N = \{\omega = (..., \omega_{-1}, \omega_0, \omega_1, ...) : \omega_i \in \{0, 1, ..., N-1\},\ i \in \mathbb{Z}\}$,
the space of bi-infinite sequences on N symbols.
$\Omega_N^R = \{\omega = (\omega_0, \omega_1, ...) : \omega_i \in \{0, 1, ..., N-1\},\ i \in \mathbb{Z}_{\geq 0}\}$,
the space of (right)-infinite sequences on N symbols.
Topology on Ω_N (and on Ω_N^R) take $\{0, 1, ..., N-1\}$ with the discrete topology and consider on $\{0, 1, ..., N-1\}^{\mathbb{Z}}$, (or on $\{0, 1, ..., N-1\}^{\mathbb{Z}_{\geq 0}}$) the product topology.
(More structure: If we think of $\{0, 1, ..., N-1\}$ as a finite Abelian group $\mathbb{Z}/N\mathbb{Z}$ then Ω_N and Ω_N^R are compact Abelian (product) topological groups.)
Given $n_1 < n_2 < ... < n_k$ and $\alpha_1, ..., \alpha_k \in \{0, 1, ..., N-1\}$ the sets
$C_{\alpha_1,...,\alpha_k}^{n_1,...,n_k} = \{\omega \in \Omega_N : \omega_{n_i} = \alpha_i,\ i = 1, ..., k\}$ are the cylinder sets,
(similar def. for Ω_N^R).
One can define the topology on Ω_N, (or on Ω_N^R) by saying that the cylinder sets are open and form the base for the topology.
(The cylinder sets are also closed, since their complement is the union of finitely many cylinder sets.)
With $t > 1$, the metric $d_t(\omega, \omega') = \sum_{n=-\infty}^{\infty} \frac{|\omega_n - \omega_n'|}{t^{|n|}}$ generates this top.

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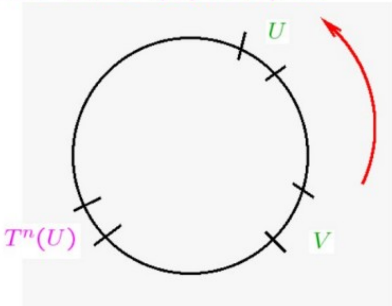
Shift:

$\sigma_N : \Omega_N \to \Omega_N$, $\sigma_N(\omega) = (..., \omega'_0, \omega'_1, ...)$, where $\omega'_n = \omega_{n+1}$ for $\forall n$.
σ_N is one-to-one and cylinders are mapped onto cylinders $\Rightarrow \sigma_N$ is a homeomorphism.
(Ω_N, σ_N) is the topological Bernoulli shift.
The right-N-shift $\sigma_N^R : \Omega_N^R \to \Omega_N^R$ is given by
$\Omega_N^R(\omega_0, \omega_1, ...) = (\omega_1, \omega_2, ...)$.
It is a continuous, but a non-invertible map of Ω_N^R into itself.
D.: A top. dyn. sys. $T : X \to X$ is topologically mixing if for any open (non-empty) $U, V \subset X$ there exists an integer $N = N(U, V)$ such that for $\forall n > N$, $T^n(U) \cap V \neq \emptyset$.



Example 1. Irrational rotations of $\mathbb{T}$ are not top. mixing.

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D.: A top. dyn. sys. $T : X \to X$ is topologically mixing if for any open (non-empty) $U, V \subset X$ there exists an integer $N = N(U, V)$ such that for $\forall n > N$, $T^n(U) \cap V \neq \emptyset$.
Prop.: *The periodic points of σ_N (and of σ_N^R) are dense in Ω_N (or in Ω_N^R), $P_n(\sigma_N) = P_n(\sigma_N^R) = N^n$ moreover σ_N and σ_N^R are top. mixing.*
Proof.: $\sigma_N^n \omega = \omega \Leftrightarrow \omega_{n+m} = \omega_m$ for $\forall m \in \mathbb{Z}$, for $\forall m \in \mathbb{Z}_{\geq 0}$ for σ_N^R.
For the density wee need to find in each cylinder set $C^{n_1,...,n_k}_{\alpha_1,...,\alpha_k}$ a periodic point.
Each cylinder in Ω_N contains symmetric cylinders
$C^{-m,...,m}_{\beta_{-m},...,\beta_m} = C^m_{\underline{\beta}}$ with $\underline{\beta} = \beta_{-m}, ..., \beta_m$.
$\omega = (... \underbrace{...} \underbrace{\beta_{-m}, ..., \beta_m}_{\uparrow 0} \underbrace{\beta_{-m}, ..., \beta_m} \underbrace{...})$ is a periodic point in $C^m_{\underline{\beta}}$.
(The case of Ω_N^R is similar.)
Each ω periodic by n is determined by the entries $\omega_0, ..., \omega_{n-1}$ and these can be chosen N^n many ways.

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Prop.: *The periodic points of σ_N (and of σ_N^R) are dense in Ω_N (or in Ω_N^R), $P_n(\sigma_N) = P_n(\sigma_N^R) = N^n$ moreover σ_N and σ_N^R are top. mixing.*

Topological mixing: Each cylinder contains symmetric cylinders. $\Rightarrow$ it is sufficient to show that for any $\underline{\alpha} = \alpha_{-m}, ..., \alpha_m$ and $\underline{\beta} = \beta_{-m}, ..., \beta_m$ for sufficiently large n we have $\sigma_N^n(C_{\underline{\alpha}}^m) \cap C_{\underline{\beta}}^m \neq \emptyset$.
If $n > 2m+1$, $n = 2m+k+1$ with $k > 0$ then let

$$\omega = (*\underbrace{\alpha_{-m}, ..., \alpha_m}_{-m \quad \overset{\uparrow}{0} \quad m} * \underbrace{\beta_{-m}, ..., \beta_m}_{n-m \quad \overset{\uparrow}{n} \quad n+m} *)$$

Then $\omega_i = \alpha_i$ if $|i| \leq m$ and
$\omega_i = \beta_{i-n}$ if $|i-n| \leq m$, that is $i = m+k+1, ..., 3m+k+1 = n-m, ..., n+m$.
Then $\omega \in C_{\underline{\alpha}}^m$ and $\sigma_N^n(\omega) \in C_{\underline{\beta}}^m$, since $\sigma_N^n(\omega) \in \sigma_N^n(C_{\underline{\alpha}}^m) \Rightarrow$
$\sigma_N^n(C_{\underline{\alpha}}^m) \cap C_{\underline{\beta}}^m \neq \emptyset$.
The argument for σ_N^R is similar. ■

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D.: If $T : X \to X$ are $S : Y \to Y$ are two top. dyn. sys. and there exists a homeomorphism $h : X \to Y$ such that $h \circ T = S \circ h$ then the two systems are called topologically conjugate.

Prop.: (Ω_2^R, σ_2^R) and (C_3, E_3) are topologically conjugate.

Proof.: Set $\phi(0) = 0$ and $\phi(1) = 2$.
For points in C_3 we will use again the triadic expansion.
Define $h : \Omega_2^R \to C_3$ by $h(\omega_0, \omega_1, ...) = 0.\phi(\omega_0)\phi(\omega_1)...$.
It is not difficult to see that h is a homeomorphism and $h \circ \sigma_2^R = E_3 \circ h$.
■

D.: A symbolic dynamical system, or a shift space, is the restriction of σ_N, (or of σ_N^R) onto a closed shift invariant subspace of Ω_N, (or of Ω_N^R).

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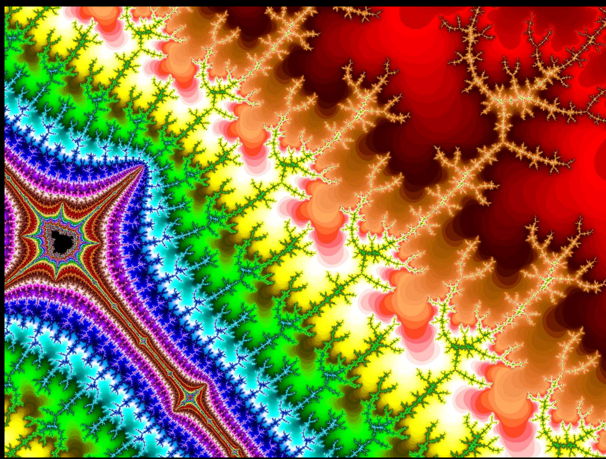
July 2023 ELTE

Introduction to Dynamical Systems, Fractals and Ergodic Theory

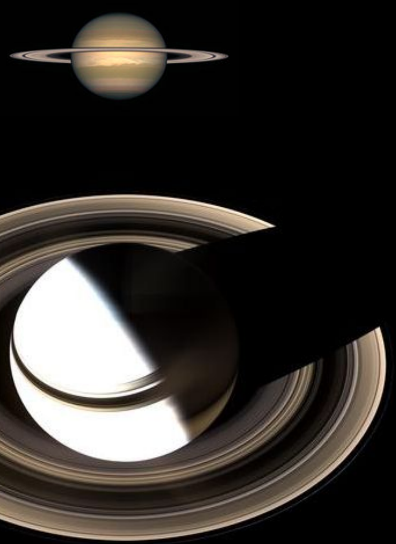
PART 2, DYNAMICAL SYSTEMS AND FRACTALS

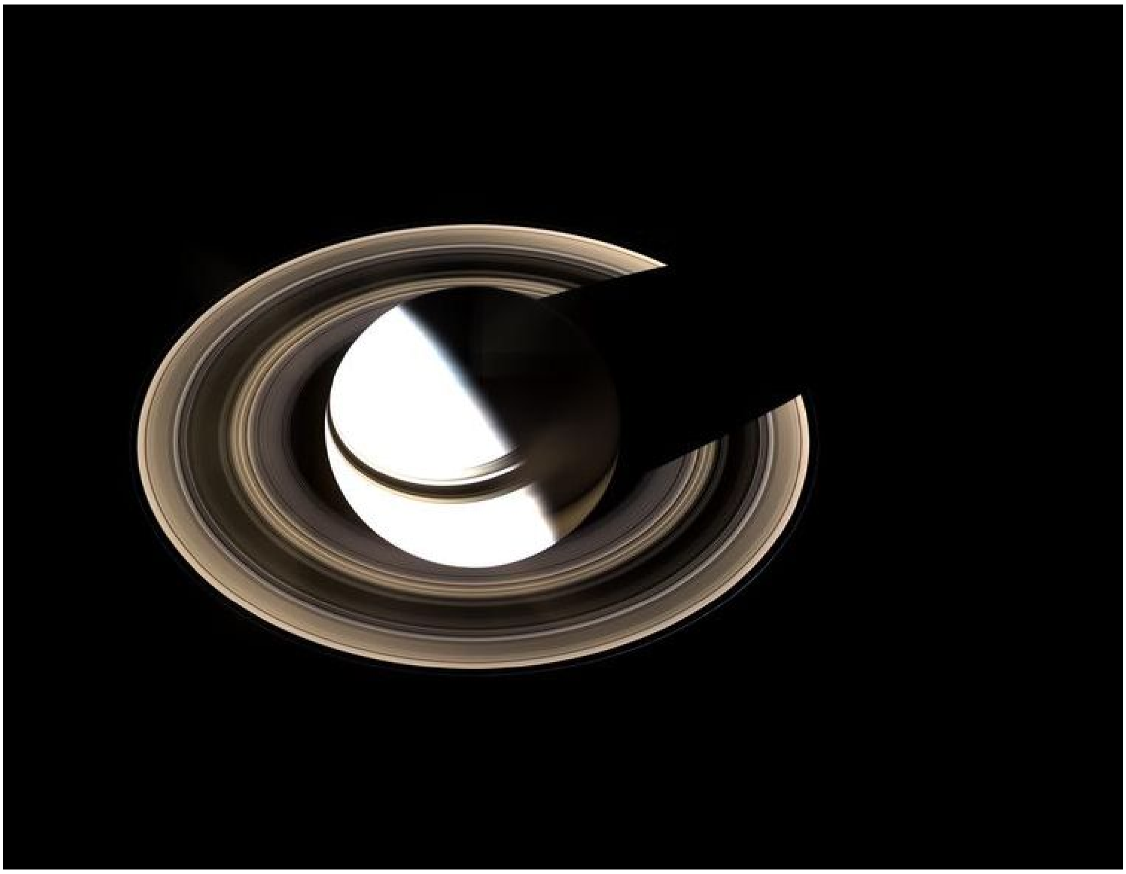


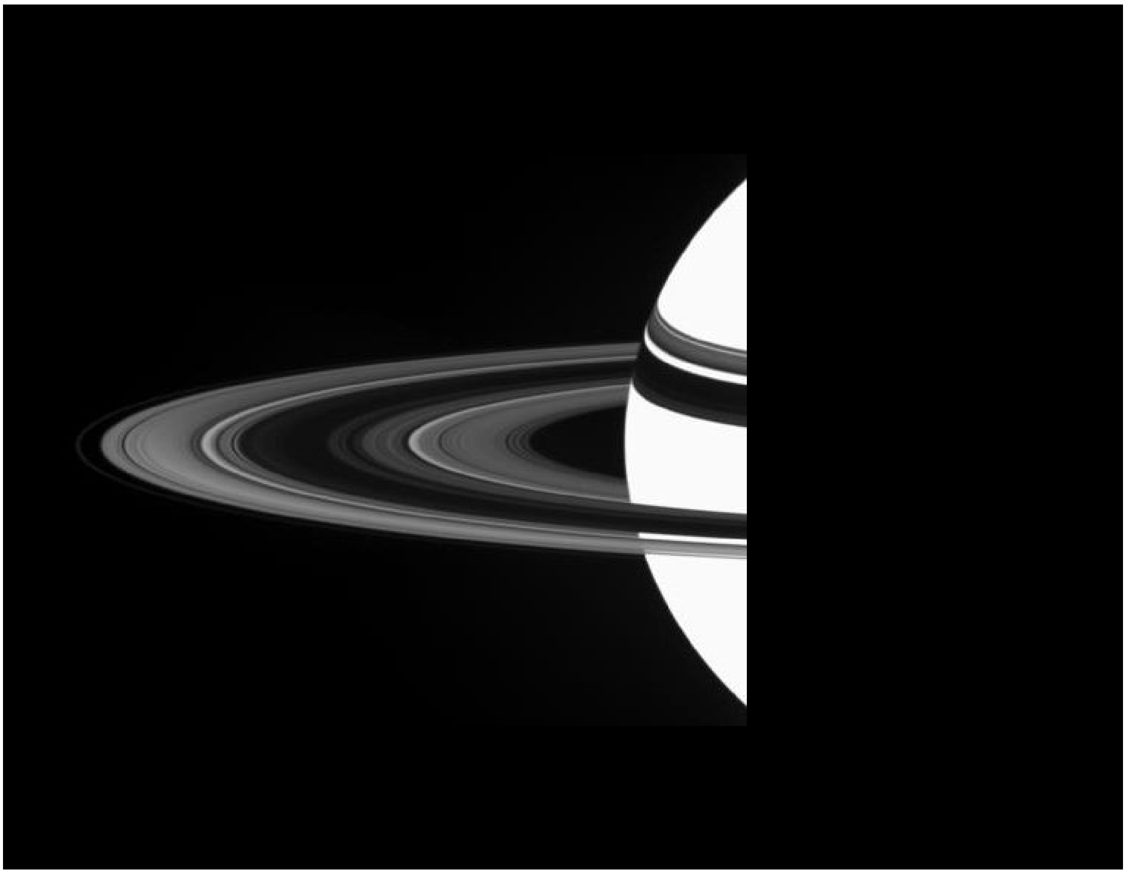
Zoltán Buczolich

(this file contains some embedded videos, the pdf reader should be enabled to play them)

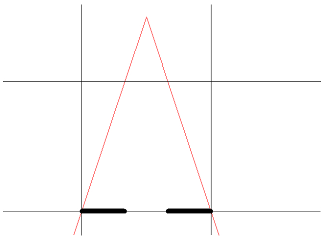
The rings of Saturn (NASA photos):







Let $f(x) = \begin{cases} 3x & \text{if } x \le \frac{1}{2} \\ 3(x - \frac{1}{2}) + \frac{3}{2} & \text{if } x \ge \frac{1}{2}. \end{cases}$



Set $f^2(x) = f(f(x))$, $f^k(x) = f(f^{k-1}x)$.
Then for $x \in (\frac{1}{3}, \frac{2}{3})$ we have $f(x) \notin [0,1]^2$.
These points leave $[0,1]$ for good $\forall k \ge 1$, $f^k(x) \notin [0,1]$.

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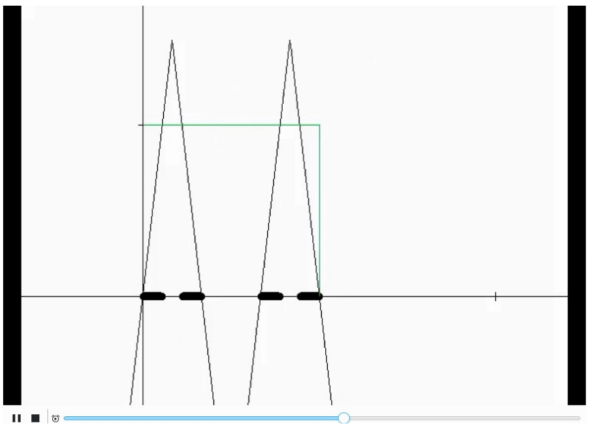

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Which are the points which stay in $[0,1]$ forever?
For the points of the ternary Cantor set, C_3 we have
$\forall x \in C_3$, $\forall k$, $f^k(x) \in C_3 \subset [0,1]$.
This is the repeller of our "dynamical system".

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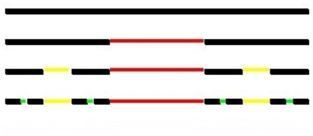

We can color the complement of the repelling Cantor set according to the number of steps a certain point leaves $[0,1]$ for good.
The colored figure is the complement of C_3.
While the "leftover" is the fractal.

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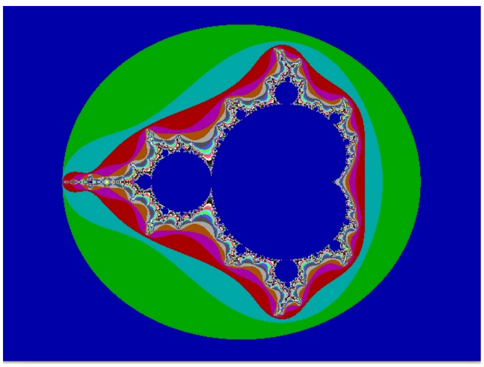
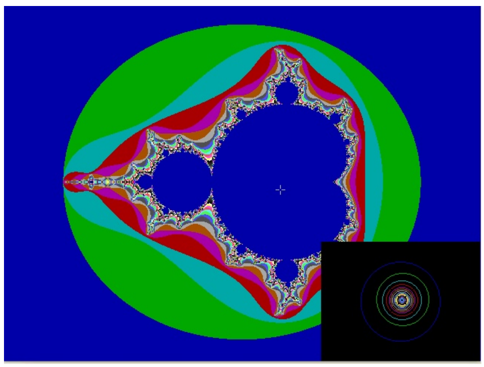
Consider in the complex plane for the mapping
$f_c(z) = z^2 + c$ the orbit of 0:
i.e., the sequence $f_c(0), f_c^2(0), f_c^3(0), \ldots$.
If $c = 0$, then it is a fixed point $\forall k\ f_0^k(0) = 0$.
If $c = 10$, then $f_{10}^1(0) = 10$ and if $|z| \geq 10$, then $|f_{10}(z)| = |z^2 + 10| \geq 10|z| - |z| > 2|z|$, therefore 0 goes to infinity.
The Mandelbrot set consists of those c for which the sequence
$f_c(0), f_c^2(0), f_c^3(0), \ldots$ is bounded.





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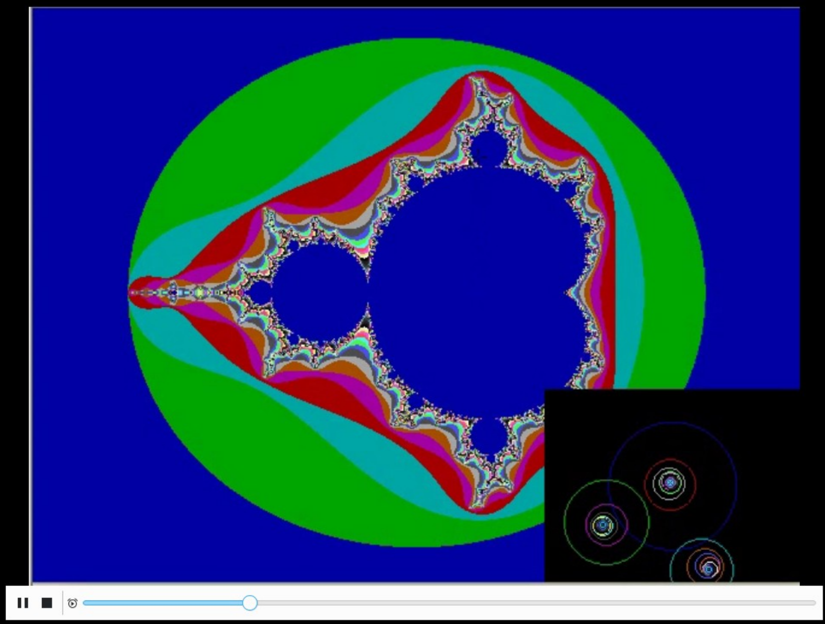

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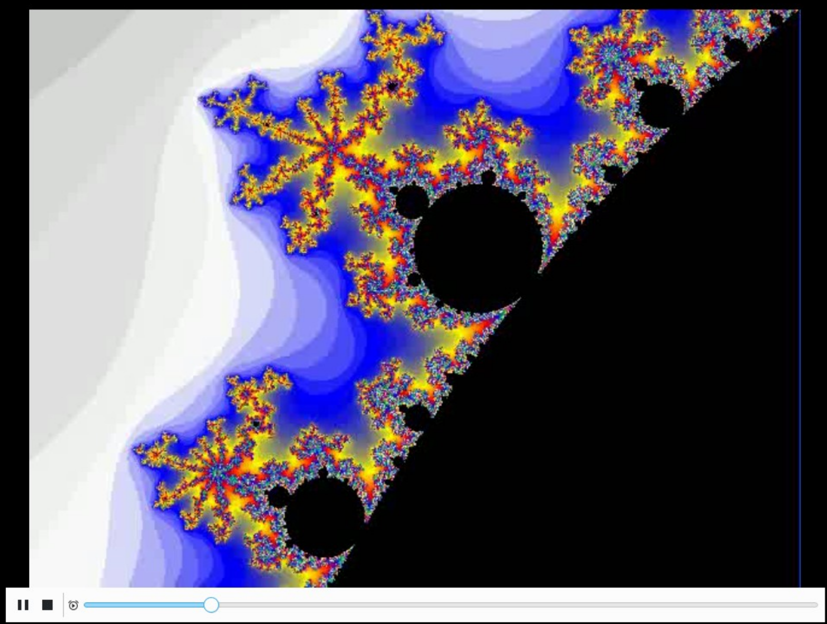
The Mandelbrot set and orbit circles



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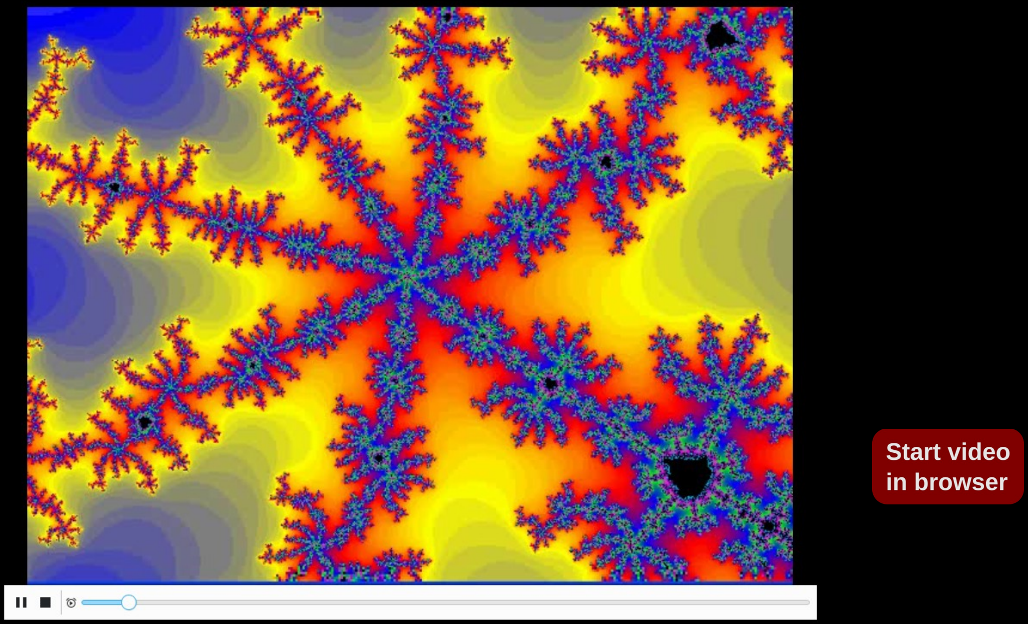
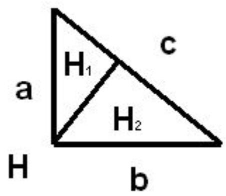
Details of the Mandelbrot set and coloring



More details of the Mandelbrot set and coloring

If we apply similarities of ratio $0 < \lambda < 1$ in $\mathbb{R}^3$ then the lengths are scaled by a factor of λ, the areas are scaled by a factor of λ^2, the volumes are scaled by a factor of λ^3.
If a set H is "visible" according to the n-dimensional measure, i.e. $0 < \mu_n(H) < \infty$, then $\mu_n(\lambda \cdot H) = \lambda^n \mu_n(H)$.
Based on this scaling property one can prove the Pythagorean theorem:



We have $H = H_1 \cup H_2$. $\mu_2(H_1 \cap H_2) = 0$.
H_1 is similar to H.
The similarity ratio equals $\frac{a}{c}$.
Likewise H_2 is similar to H.
The similarity ratio equals $\frac{b}{c}$.
Hence $\mu_2(H) = \mu_2(H_1) + \mu_2(H_2) = \mu_2\left(\frac{a}{c}H\right) + \mu_2\left(\frac{b}{c}H\right) = \left(\frac{a}{c}\right)^2 \mu_2(H) + \left(\frac{b}{c}\right)^2 \mu_2(H)$.
If $0 < \mu_2(H) < \infty$ (that is, we can divide by it) $\Rightarrow$ $1 = \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 \Rightarrow c^2 = a^2 + b^2$.

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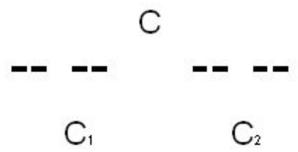

A similar argument applies to the Cantor set. Suppose that for a suitable "s-dimensional measure", μ_s we have:
$\mu_s(\lambda A) = \lambda^s \mu_s(A)$ for all Borel sets A, and $0 < \mu_s(C) < \infty$.
Then according to the figure:
$C = \frac{1}{3}C \cup (\frac{1}{3}C + \frac{2}{3}) = C_1 \cup C_2$.
$\mu_s(C) = \mu_s(C_1) + \mu_s(C_2) =$
$= \left(\frac{1}{3}\right)^s \mu_s(C) + \left(\frac{1}{3}\right)^s \mu_s(C)$,
this implies $1 = 2\frac{1}{3^s}$. Hence $s = \frac{\log 2}{\log 3}$.
We have a good reason to think that C's "measure theoretical dimension" $s = \frac{\log 2}{\log 3}$.
Hence C's dimension is not an integer, C is a fractal.
(Mandelbrot: Latin fractus="broken" or "fractured.")

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We need to define the suitable $\mu_s = \mathcal{H}^s$ Hausdorff-measure:
The diameter of the set $U \subset \mathbb{R}^n$ is given by: $|U| = \sup\{||x-y|| : x, y \in U\}$.
If $\delta > 0$ is given and F is a set in $\mathbb{R}^n$, then the sets U_i form a δ-cover of F, if $F \subset \bigcup_i U_i$ and $|U_i| < \delta$, $(i = 1, 2, ...)$.
Suppose $F \subset \mathbb{R}^n$, $s \geq 0$, and
$\mathcal{H}^s_\delta(F) = \inf\{\sum_{i=1}^{\infty} |U_i|^s : \{U_i\}$ is a δ cover of F $\}$.
(Using only convex, closed or open U_i we get the same value for $\mathcal{H}^s_\delta(F)$.)
Obviously $\delta_1 > \delta_2 > 0 \Rightarrow \mathcal{H}^s_{\delta_1}(F) \leq \mathcal{H}^s_{\delta_2}(F)$
$\mathcal{H}^s(F) \stackrel{\text{def}}{=} \lim_{\delta \to 0+} \mathcal{H}^s_\delta(F) = \sup_{\delta > 0} \mathcal{H}^s_\delta(F)$
is the s-dimensional Hausdorff outer measure of F
($\mathcal{H}^s$ is an outer measure: $\mathcal{H}^s(\emptyset) = 0$, $\mathcal{H}^s(F) \in [0, +\infty]$ and
$F \subset \bigcup_{i=1}^{\infty} F_i \Rightarrow \mathcal{H}^s(F) \leq \sum_{i=1}^{\infty} \mathcal{H}^s(F_i)$ (this needs proof).)
Moreover, $\mathcal{H}^s$, is a metric outer measure:
$(\text{dist}(F_1, F_2) > 0 \Rightarrow \mathcal{H}^s(F_1 \cup F_2) = \mathcal{H}^s(F_1) + \mathcal{H}^s(F_2)$.)
$\Rightarrow$ Borel sets are $\mathcal{H}^s$-measurable.

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The scaling property holds for this outer measure:

T.: *If* $F \subset \mathbb{R}^n$ *&* $\lambda > 0$ *then* $\mathcal{H}^s(\lambda F) = \lambda^s \mathcal{H}^s(F)$, *where* $\lambda F = \{\lambda x : x \in F\}$.

Proof.: We omit it due to lack of time. ■

L.: *If* $\mathcal{H}^s(F) < \infty$ *and* $s < t$ *then* $\mathcal{H}^t(F) = 0$.

Proof.: By $s - t < 0$ we have $|U_i| \leq \delta \Rightarrow |U_i|^{s-t} \geq \delta^{s-t}$.

$$\mathcal{H}^s_\delta(F) = \inf\{\sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta \text{ cover of } F\} =$$

$$\inf\{\sum_{i=1}^{\infty} |U_i|^{s-t}|U_i|^t : \{U_i\} \text{ is a } \delta \text{ cover of } F\} \geq$$

$$\inf\{\sum_{i=1}^{\infty} \delta^{s-t}|U_i|^t : \{U_i\} \text{ is a } \delta \text{ cover of } F\} = \delta^{s-t}\mathcal{H}^t_\delta(F).$$

Thus $\mathcal{H}^s(F) \geq \mathcal{H}^s_\delta(F) \geq \delta^{s-t}\mathcal{H}^t_\delta(F) \Rightarrow$

$\mathcal{H}^t_\delta(F) \leq \mathcal{H}^s(F)\delta^{t-s} \to 0$ if $\delta \to 0+0$, $\mathcal{H}^t(F) = 0$. ■

Therefore $\mathcal{H}^s(F) < \infty \Rightarrow \mathcal{H}^t(F) = 0 \ \forall t > s$.

Hence $\mathcal{H}^t(F) > 0 \Rightarrow \mathcal{H}^r(F) = \infty \ \forall r < t$.

D.: The Hausdorff dimension of F

$\dim_H(F) \overset{\text{def}}{=} \inf\{t > 0 : \mathcal{H}^t(F) = 0\} = \sup\{r \geq 0 : \mathcal{H}^r(F) = \infty\}$

(where $\sup \emptyset \overset{\text{def}}{=} 0$).

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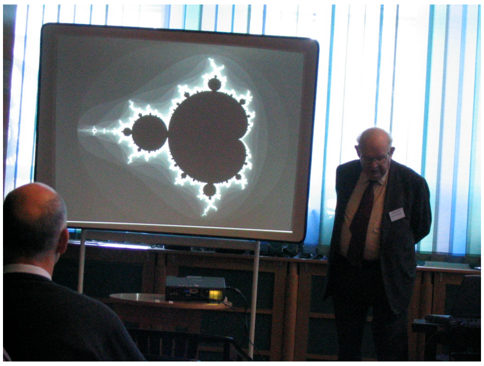
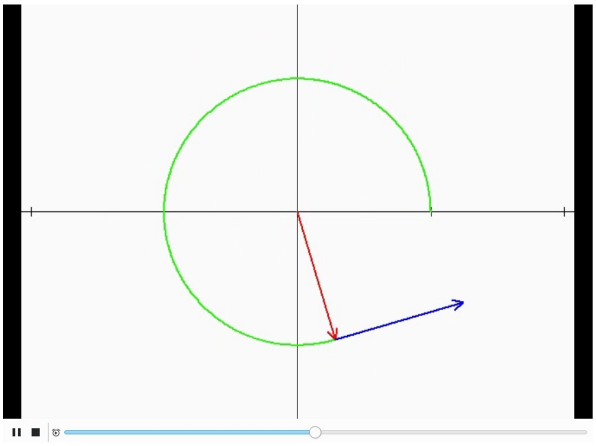


The (two dimensional) area of the Mandelbrot set:
1.506 591 77 ± 0.000 000 08.
Mandelbrot first thought that it is not connected, but
Douady and Hubbard showed that it is connected.
(There is a conformal isomorphism between its complement and the complement of the closed unit disk.)
The set and its boundary are both of Hausdorff dimension 2 (Mitsuhiro Shishikura, 1994).
It is not known whether its boundary is of positive (2 dim.) Lebesgue measure.

16



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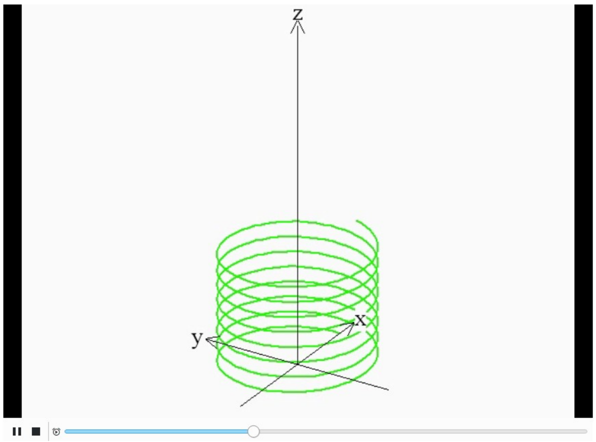
Consider the curve staisfying the differential equation:
$x'(t) = -y(t)$
$y'(t) = x(t).$

The scalar product of the vectors $(x'(t), y'(t)) \cdot (x(t), y(t)) =$
$(-y(t), x(t)) \cdot (x(t), y(t)) = -x(t)y(t) + x(t)y(t) = 0$
The velocity is perpendicular to the position vector.

17



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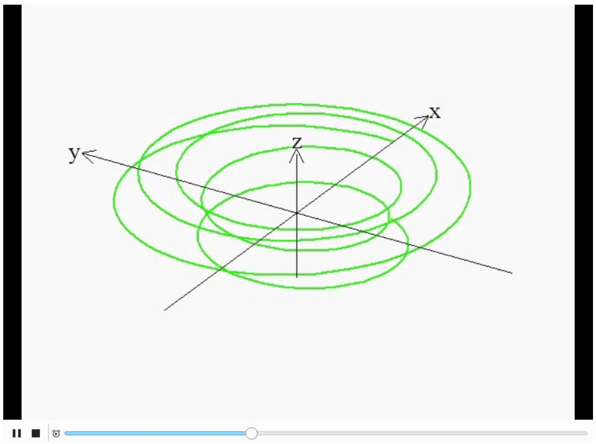
By the Poincaré-Bendixson Theorem if we have a C^1 autonomous dynamical system in the plane ($\mathbb{R}^2$) then the solutions curves are "attracted" to sets, which are either periodic cycles, or contain equilibrium points. So we need to move to higher dimensions. We add one more equation:
$x'(t) = -y(t)$
$y'(t) = x(t)$
$z'(t) = 0.3.$

18



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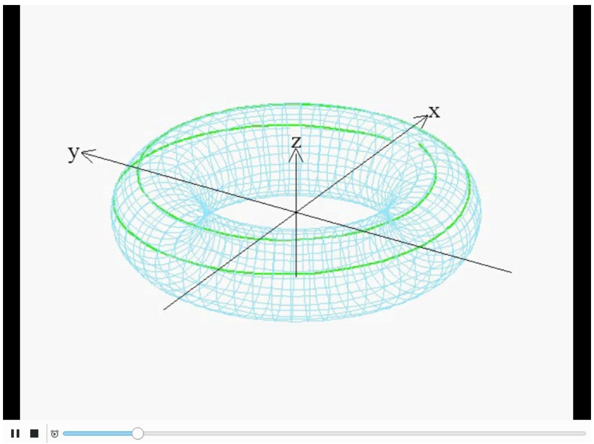
More complicated curves.

19



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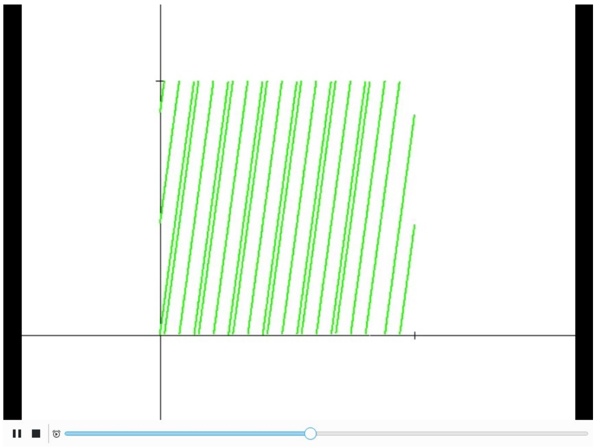
More complicated curves on the torus.

20



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One can obtain the previous curves on the torus by taking the solution lines of the linear system:
$x'(t) = a$
$y'(t) = b.$

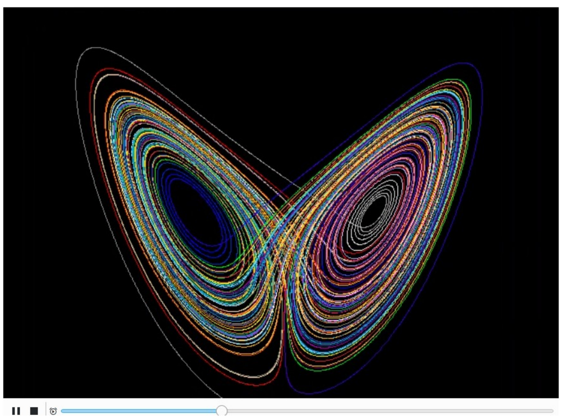
Modulo one. If a/b is rational the solution curve is not dense on the torus. If a/b is irrational then it is.

21

The Lorenz attractor:

$$\frac{dx}{dt} = \sigma(y - x)$$
$$\frac{dy}{dt} = x(\rho - z) - y$$
$$\frac{dz}{dt} = xy - \beta z$$

Edward Lorenz (1963) studied a simplified model of convection rolls arising in the equations describing the atmosphere.
The solutions of this system of differential equations for the choice of parameters ($\rho = 28$, $\sigma = 10$, $\beta = 8/3$) "converge towards" a "fractal attractor". By an estimate of Grassberger (1983) the Hausdorff dimension of the attractor is 2.06 ± 0.01.



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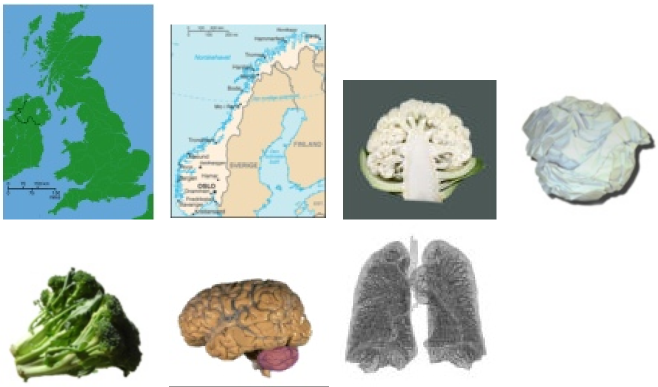
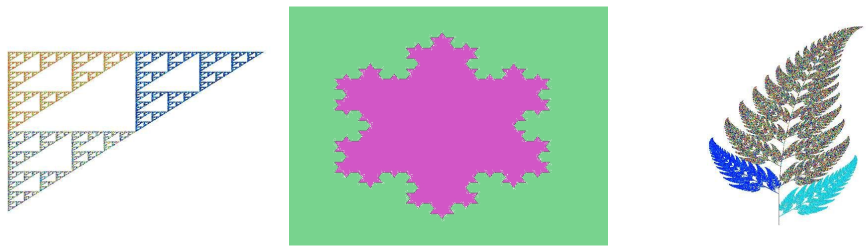


A few more fractals from the "real world" (Source: Wikipedia):
Coastline of Great Britain: "estimated dimension": 1.24
Coastline of Norway: "estimated dimension": 1.52
Cauliflower: $\frac{\log 13}{\log 3} \approx 2.3347$
(on each branch there are 13 branches 3 times smaller)
Balls of crumpled paper: "estimated dimension": 2.5
Broccoli: "estimated dimension": 2.66
Surface of human brain: "estimated dimension": 2.79
Lung surface: "estimated dimension": 2.97.

23



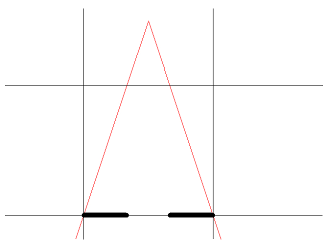
Iterated function systems (IFS) and self-similar sets
Suppose $X \subset \mathbb{R}^n$, $X \neq \emptyset$, is closed (can be $\mathbb{R}^n$).
An IFS consists of a family of contractions $\{F_1, ..., F_m\}$, $(m \geq 2)$ defined on X.
$\forall\ i,\ F_i : X \to X,\ r_i < 1$ and $\forall\ x, y \in X$
$|F_i(x) - F_i(y)| \leq r_i |x - y|$. $\qquad r_{max} = \max_i r_i < 1$.
E.g.: $F_1, F_2 : \mathbb{R} \to \mathbb{R},\ F_1(x) = \frac{1}{3}x,\ F_2(x) = \frac{1}{3}x + \frac{2}{3}$, then
$C_3 = F_1(C_3) \cup F_2(C_3)$, where C_3 is the trenary Cantor set.

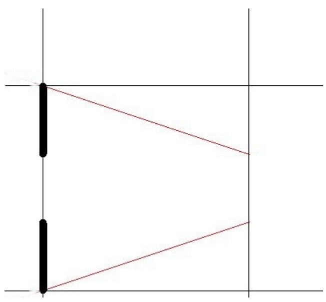
24

$X \subset \mathbb{R}^n$, $X \neq \emptyset$, closed. Denote by $\mathcal{S}$ the system of non-empty compact K subsets of X.
Let $A_\delta \stackrel{\text{def}}{=} \{x \in \mathbb{R}^n : \text{dist}(x, A) < \delta\}$, the "$\delta$-sausage" around A.
The Hausdorff distance: of the sets $A, B \subset \mathbb{R}^n$
$\text{d}_{\text{Hau}}(A, B) \stackrel{\text{def}}{=} \inf\{\delta : A \subset B_\delta \text{ and } B \subset A_\delta\}$
$(\mathcal{S}, \text{d}_{\text{Hau}})$ is a complete metric space.
T.: *If $\{F_1, ..., F_m\}$ is a given IFS on $X \subset \mathbb{R}^n$ then $\exists!$ compact set $E \subset X$, $E \neq \emptyset$ such that $E = \bigcup_{i=1}^{m} F_i(E)$. If we define the map $F : \mathcal{S} \to \mathcal{S}$ for $\forall\ A \in \mathcal{S}$ by $F(A) = \bigcup_{i=1}^{m} F_i(A)$ then $\forall\ A \subset \mathcal{S}$ in the Hausdorff metric $F^k(A) \to E$. If $A \in \mathcal{S}$ and $\forall\ i$, $F_i(A) \subset A$, (e.g. $A = X$, if X is compact) then $E = \bigcap_{k=0}^{\infty} F^k(A)$.*
Proof.: (idea) F is a contraction on $(\mathcal{S}, \text{d}_{\text{Hau}})$ hence one can apply the Banach fixed point theorem. ■
E is the attractor, or invariant set of the IFS.

25

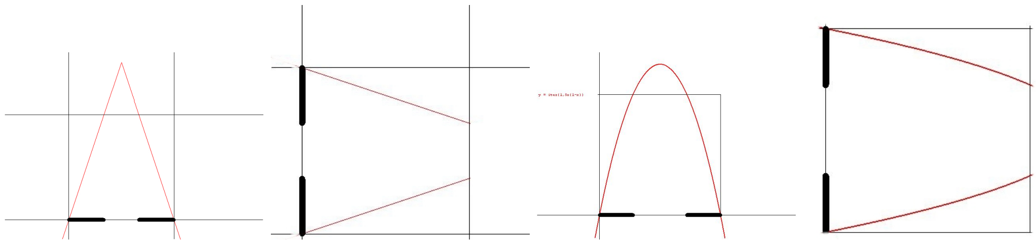
Legyen $f(x) = \begin{cases} 3x & \text{if } x \leq \frac{1}{2} \\ 3(x - \frac{1}{2}) + \frac{3}{2} & \text{if } x \geq \frac{1}{2}. \end{cases}$





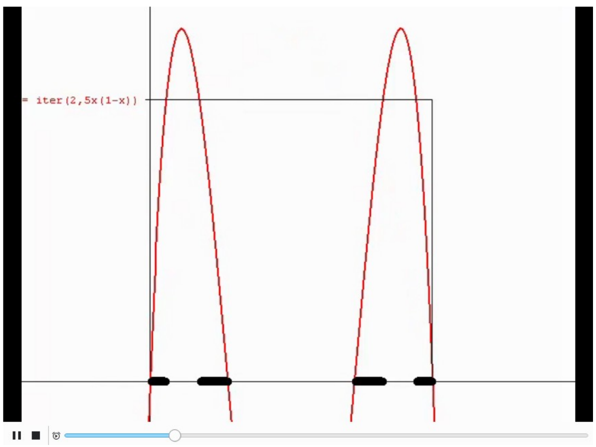
We encountered C_3 first as the repeller of a dynamical system $f : \mathbb{R} \to \mathbb{R}$. Since f is not invertible if we want to travel backwards in time we need to use the inverse branches
$F_1(x) = (f|_{[0,1/3]})^{-1}(x) = \frac{1}{3}x$ and $F_2(x) = (f|_{[2/3,1]})^{-1}(x) = 1 - \frac{1}{3}x$
This way we obtain an IFS and the attractor of this IFS is C_3, the repeller of the original system. $(C_3 = f(C_3) = F_1(C_3) \cup F_2(C_3).)$

26



One can define "nonlinear" Cantor sets:
Take $f(x) = 5x(1-x)$ as our dynamical system.
The inverse $\{F_1, F_2\}$ IFS is a nonlinear system.
D.: If in the $\{F_1, ..., F_m\}$ IFS all contractions F_i, $i = 1, ..., m$ are similarities, then its attractor E satisfying $E = \bigcup_{i=1}^{m} F_i(E)$ is called a self-similar set.

27



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The function $f(x) = 5x(1-x)$ mentined above belongs to the $f_c(x) = cx(1-x)$ logistic family.
For $0 \leq c \leq 4$ the function f_c maps $[0,1]$ into $[0,1]$.
For $c > 4$ some points leave $[0,1]$ for good and the repellers will be (generalized) Cantor sets, which are not self-similar.

28

Self-similar sets

Suppose that the $\{F_1, ..., F_m\}$ IFS consists of similarities
$\forall\ i,\ 0 < r_i < 1$ and
$\forall\ x, y \in \mathbb{R}^n,\ |F_i(x) - F_i(y)| = r_i|x - y|$.
The attractor set E satisfies $E = \bigcup_{i=1}^{m} F_i(E)$.
Suppose that the sets $F_i(E)$ are disjoint,
$\dim_H(E) = s$ and $0 < \mathcal{H}^s(E) < \infty$, that is, E is an s-set (in fact, this follows from general theorems).
Then $\mathcal{H}^s(E) = \sum_{i=1}^m \mathcal{H}^s(F_i(E)) = \sum_{i=1}^m r_i^s \mathcal{H}^s(E)$
$\Rightarrow 1 = \sum_{i=1}^m r_i^s$ and from this one can determine s.
In the Sierpinski triangle the sets $F_i(E)$ are not disjoint, but they "do not intersect too much".
Open Set Condition, OSC:
$\exists\ V \neq \emptyset$ open such that $V \supset \cup_{i=1}^m F_i(V)$, and the sets $F_i(V)$ are disjoint.
(In the Sierp. tri. V can be the interior of the large triangle.)

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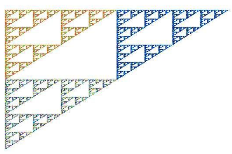
T.: *Suppose that the similarities $F_i : \mathbb{R}^n \to \mathbb{R}^n$, $(i = 1, ..., m)$ of ratio r_i satisfy the OSC and $E = \bigcup_{i=1}^{m} F_i(E)$. Then $\dim_H E = \overline{\dim}_B E = \underline{\dim}_B E = s$, where*
(*) $\sum_{i=1}^{m} r_i^s = 1$ and $0 < \mathcal{H}^s(E) < \infty$.



The nonlinear case is much more difficult. To verify the s-set property and
$\dim_H E = \overline{\dim}_B E = \underline{\dim}_B E = s$
one needs to use "implicit methods".
Formula (*) can be generalized by the Thermodynamical Formalism.

Using (*) for the Sierpinski triangle $3 \cdot (\frac{1}{2})^s = 1$,
that is, $\log 3 = s \log 2 \Rightarrow s = \frac{\log 3}{\log 2}$.

30

The von Koch snowflake At each iteration its perimeter is increased by a factor of $\frac{4}{3}$. In the Hausdorff metric it converges to a fractal.
One can also define this fractal as the attractor of an IFS. Even OSC is satisfied. Why?
The upper edge of the snowflake is the attractor of
4 similarities of ratio $\frac{1}{3}$.
$4\frac{1}{3^s} = 1 \Rightarrow \dim_H E = \frac{\log 4}{\log 3} \approx 1.2619.$

Open Set Condition, OSC
$\exists\ V \neq \emptyset$ open set, such that $V \supset \cup_{i=1}^{m} F_i(V)$, and the $F_i(V)$'s are disjoint.

31

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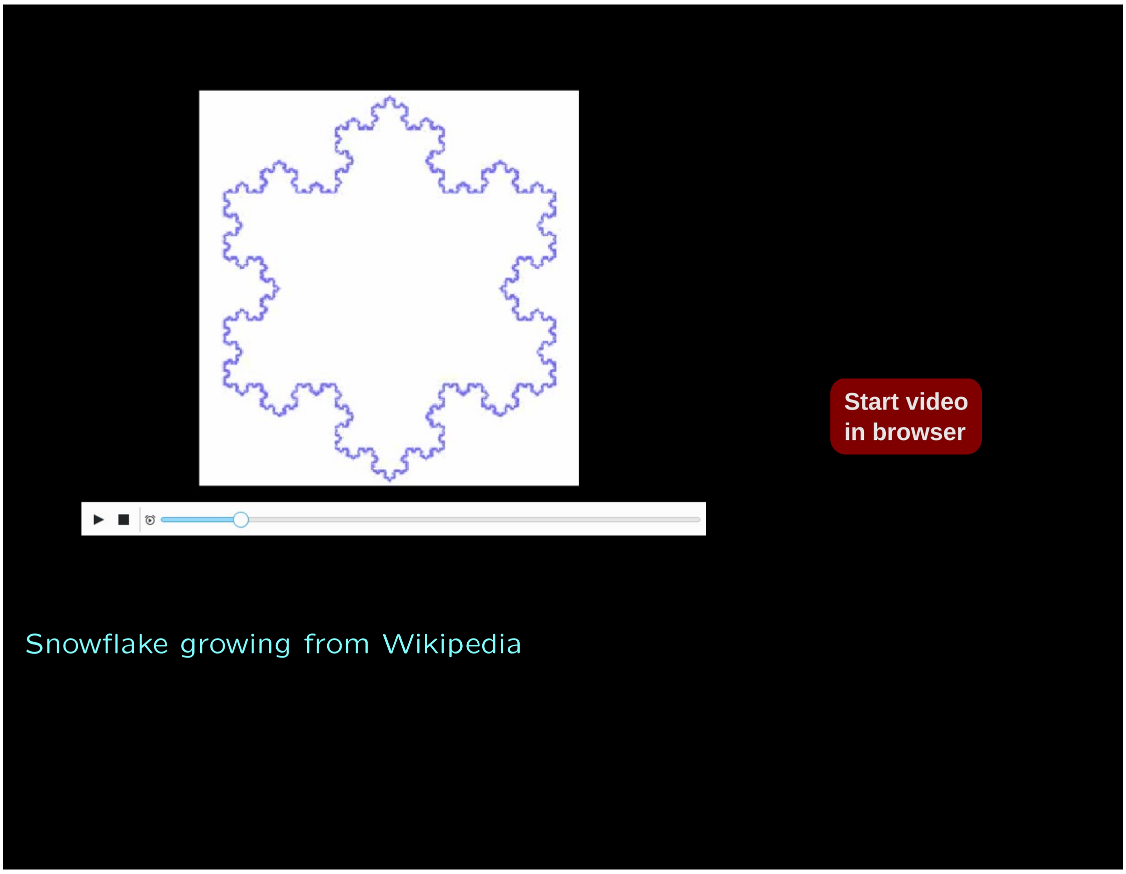
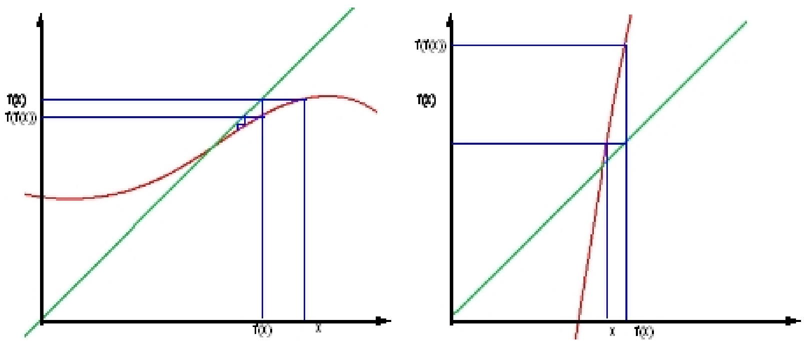
$\exists\ V \neq \emptyset$ open set, such that $V \supset \cup_{i=1}^{m} F_i(V)$, and the $F_i(V)$'s are disjoint.

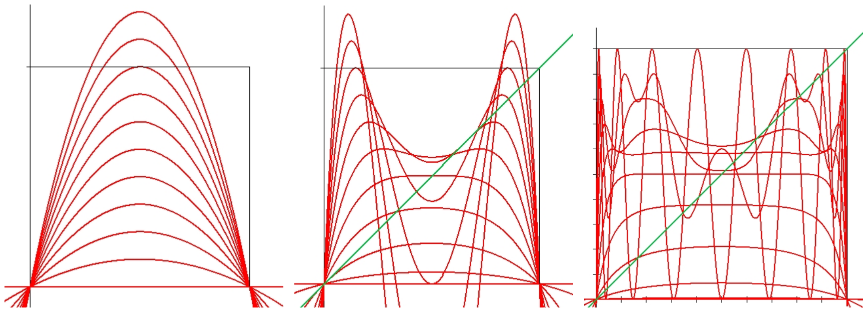
32





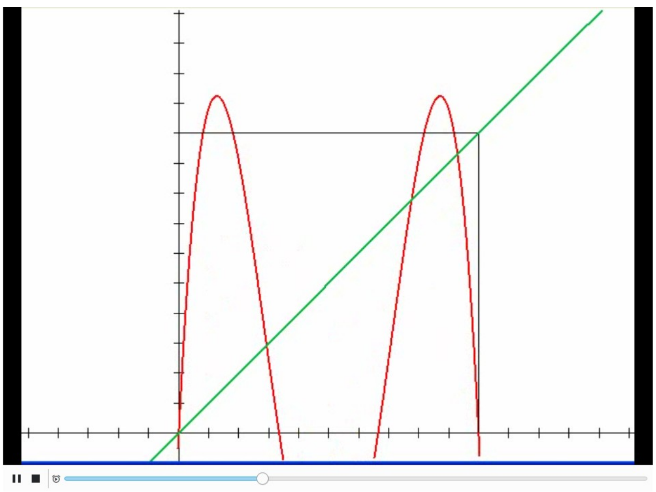
If $f(x_0) = x_0$ and $|f'(x_0)| < 1$, then x_0 is a locally attracting fixed point
if x is sufficiently close to x_0 then $1 > r > \dfrac{|f(x) - f(x_0)|}{|x - x_0|} = \dfrac{|f(x) - x_0|}{|x - x_0|}$
$\Rightarrow |f(x) - x_0| < r|x - x_0|$ and it can be repeated $\Rightarrow$
$|f^k(x) - x_0| < r^k|x - x_0| \to 0$
If $f(x_0) = x_0$ and $|f'(x_0)| > 1$, then x_0 is a locally repelling fixed point.
if x is sufficiently close to x_0, then $1 < r < \dfrac{|f(x) - f(x_0)|}{|x - x_0|} = \dfrac{|f(x) - x_0|}{|x - x_0|}$
$\Rightarrow |f(x) - x_0| > r|x - x_0|$ and this can be repeated for a while
$\Rightarrow |f^k(x) - x_0| > r^k|x - x_0| \to \infty \Rightarrow f^k(x)$ leaves the neighborhood of x_0.

34



$f_c(x) = cx(1-x)$ the logistic family for $c = 0, 0.5, ..., 5$.
The biologist R. May studied a discrete time demographic model:
$x_{n+1} = f(x_n) = cx_n(1-x_n)$,
$x_0 =$ the initial population, x_n the population after n many years.
If x_n is small then $1 - x_n \approx 1$, hence $x_{n+1} \approx cx_n \approx c^{n+1}x_0$
exponential growth.
For large x's the factor $(1-x)$ decr. the growth rate, (starvation factor). c a combined rate for reproduction and starvation. If $c < 1$, then the population will eventually die, for larger c's it stabilizes at a fixed point, for even larger c's it oscillates (period doubling), later it becomes chaotic, unstable. If $c > 4$ for almost all initial values it diverges to $-\infty$.

35



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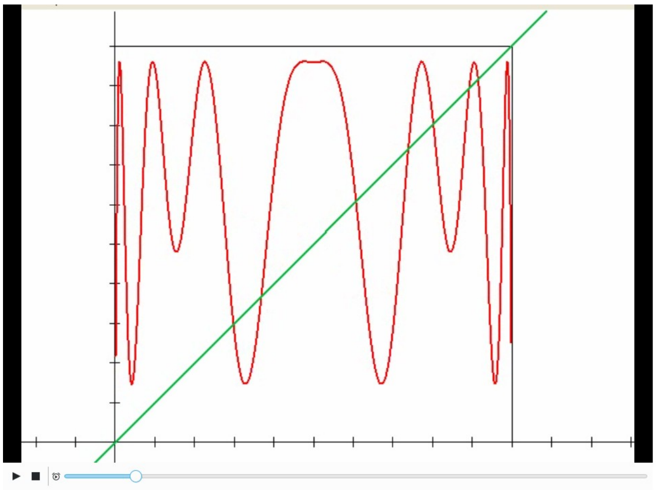
The effect of the change of c on $f_c(x) = cx(1-x)$ and f_c^2, the birth of an attracting fixed point and its evolution into a repelling one, bifurcation

36



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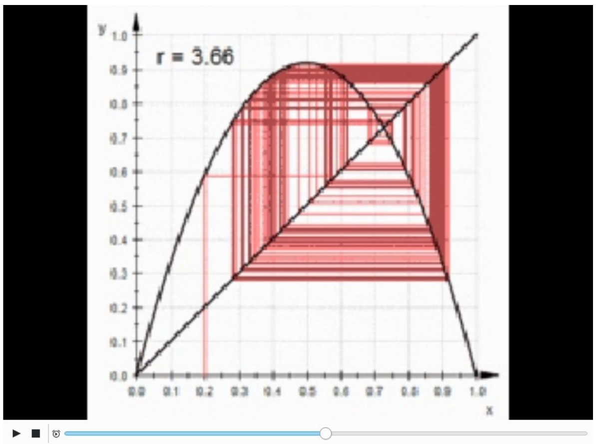
The effect of the change of c on f_c^4. Bifurcations. First the birth of an attractiong period 2, then of an attracting period 4 orbit.

37



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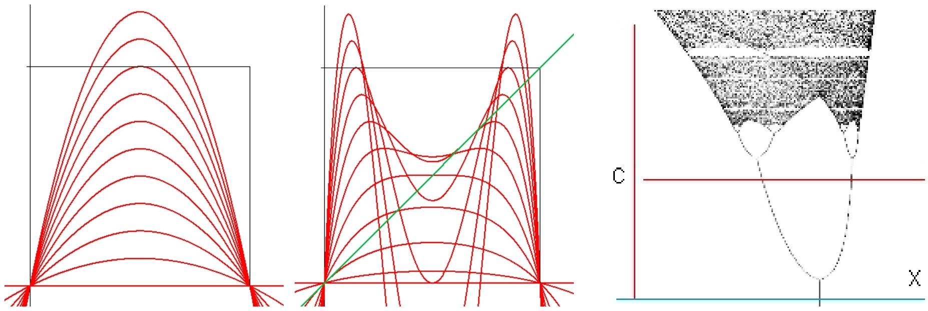
Animation from Wikipedia: the orbit of $x = 0.2$ for different parameter values.

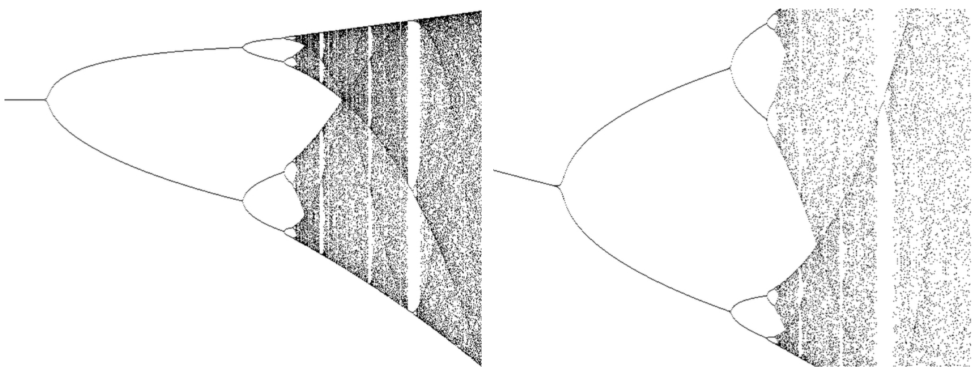
Connection with the Mandelbrot set: Take the restriction of $P_C(z) = z^2 + C$ to the real axis and the logistic family $f_c(x) = cx(1-x)$. The intersection of the Mandelbrot M with $\mathbb{R}$ equals $[-2, 0.25]$. $C = \frac{1-(c-1)^2}{4}$ gives a one-to-one parameter correspondence with the parameters of the logistic family.

38



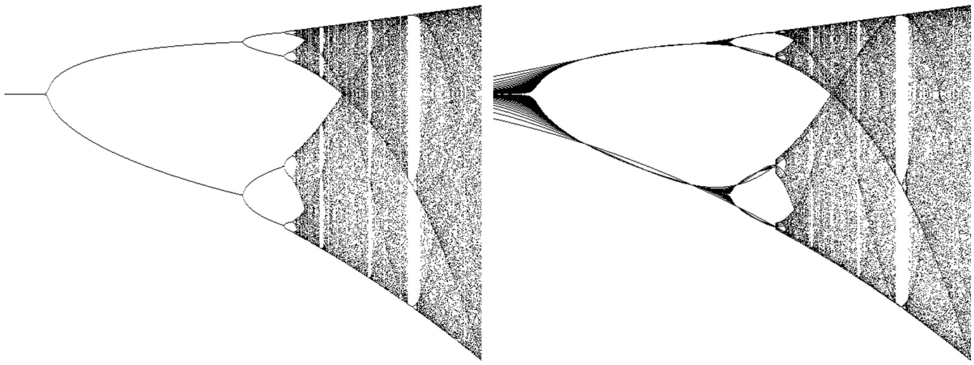
The bifurcation diagram of $f_c(x) = cx(1-x)$.
This is a fractal. Dimension= 0.4498??
The above figure is a rotated version of the usual image.
Each horizontal section corresponds to a parameter value c.
Starting with an almost arbitrary initial x the first few thousand terms of $f_c^{1001}(x), f_c^{1002}(x), \dots$ are plotted. (If there is an attracting fixed point or periodic orbit then these iterates are almost on it. Otherwise, a smeared image corresponds to more chaotic behavior.

39



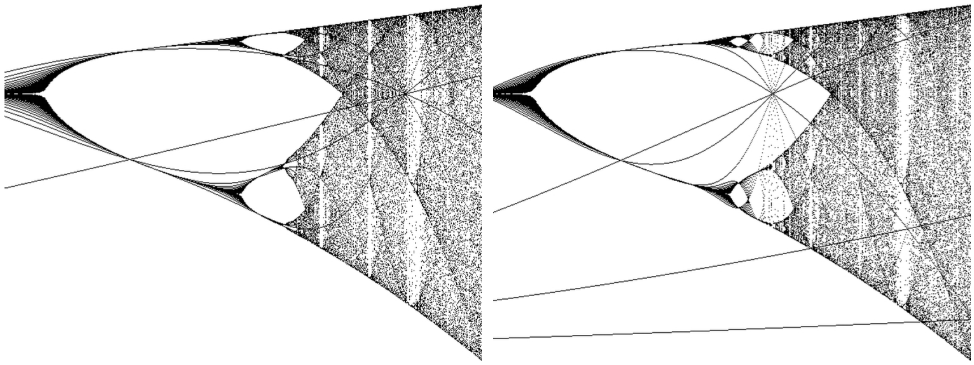
The bifurcation diagram of $f_c(x) = cx(1-x)$.
This is the usual view the x-axis is vertical and the
c parameter-axis is horizontal.
On the right there is a blow-up part of the diagram it is non-linearly similar to the original.

40



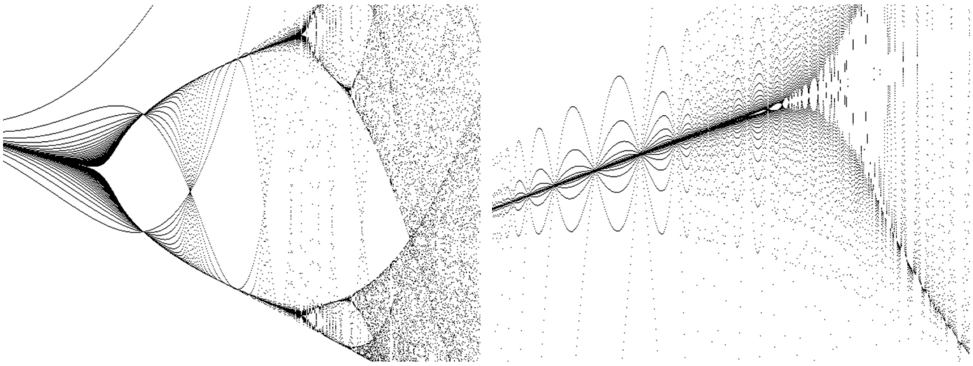
The bifurcation diagram of $f_c(x) = cx(1-x)$.
On the right we do not omit the first 1000 iterates
we plot $f_c^1(x), f_c^2(x), \ldots.$

41



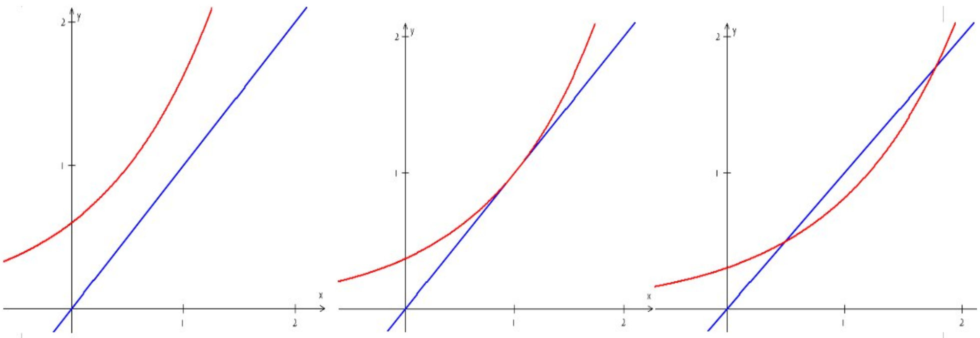
The bifurcation diagram of $f_c(x) = cx(1-x)$.
We do not omit the first 1000 iterates
we plot $f_c^1(x), f_c^2(x), \ldots.$
We vary the initial value.

42



The bifurcation diagram of $f_c(x) = cx(1-x)$.
We do not omit the first 1000 iterates
we plot $f_c^1(x), f_c^2(x), \ldots$.
We vary the initial value.
Blow-ups of parts of the diagrams.

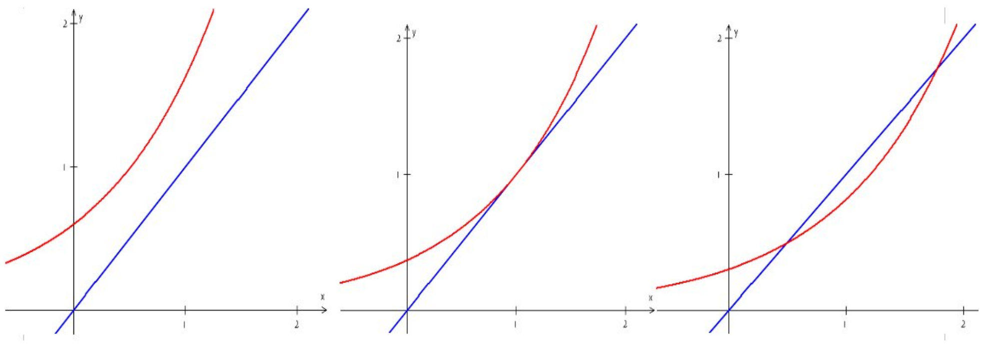
43



Bifurcation types in 1D. Saddle-node (tangent) bifurcation

Example: $E_c(x) = ce^x$, $c > 0$. We have a bifurcation at $c_0 = \frac{1}{e}$.
On the figures we have the graphs corresponding to
$c = 0.6$, $c = c_0$, $c = 0.3$.
If $c > c_0 = \frac{1}{e}$ then $E_c(x) > x$ for $\forall x \in \mathbb{R} \Rightarrow E_c^n(x)$ is monotone increasing.
We show that $E_c^n(x) \to \infty$.
Proof.: If not then $E_c^n(x)$ is bounded and has a finite limit say x_∞.
Since E_c is continuous $E_c(x_\infty) = E_c(\lim_{n\to\infty} E_c^n(x)) = \lim_{n\to\infty} E_c(E_c^n(x)) = \lim_{n\to\infty} E_c^{n+1}(x) = x_\infty$, but E_c does not have any fixed points. ■

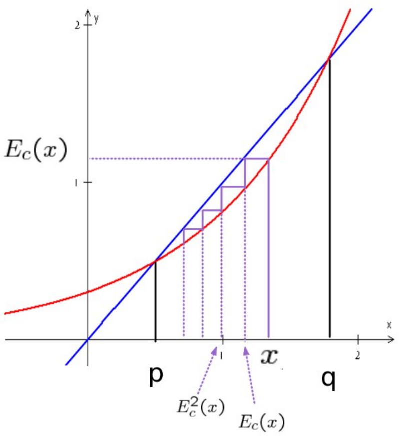
44



Example: $E_c(x) = ce^x$, $c > 0$. We have a bifurcation at $c_0 = \frac{1}{e}$.

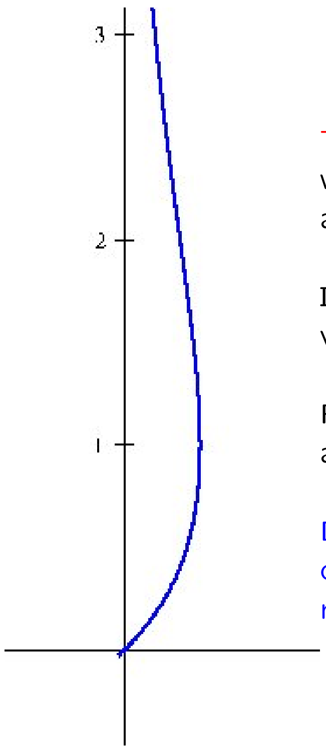
If $c = c_0 = \frac{1}{e}$ then $E_c(1) = 1$.
If $x < 1$ then by strict monotonicity $E_c(x) < E_c(1) = 1$. $E_c(x) > x$ implies that $E_c^n(x)$ is str. monotone incr. and bded $\Rightarrow$converges to an $x_\infty \leq 1$.
Arguing as before x_∞ is the only fixed point of E_c in $(-\infty, 1]$ $\Rightarrow$$x_\infty = 1$.
If $x > 1$ then $E_c(x) > x$ implies that $E_c^n(x)$ is str. monotone incr., it could not converge to a finite $x_\infty > 1$ since it would be a fixed point of E_c.
$\Rightarrow$$E_c^n(x) \to \infty$.
The fixed point 1 is attracting from the left, and repelling from the right.

45



Example: $E_c(x) = ce^x$. If $0 < c < \frac{1}{e}$ then there exist two fixed points $p < q$.
$|E_c'(p)| < 1$ (attracting) and
$|E_c'(q)| > 1$ (repelling).
If $x < p$ then by strict monotonicity $E_c(x) < E_c(p) = p$. $E_c(x) > x$ implies that $E_c^n(x)$ is str. monotone incr. and bded
$\Rightarrow$converges to $x_\infty = p$.
If $p < x < q$ then by strict monotonicity $E_c(p) = p < E_c(x) < E_c(q) = q$.
$E_c(x) < x$ implies that $E_c^n(x)$ is str. monotone decr. and bded
$\Rightarrow$converges to $x_\infty = p$.
If $x > q$ then $E_c(x) > x$ implies that $E_c^n(x)$ is str. monotone incr., it could not converge to a finite $x_\infty > 1$ since it would be a fixed point of E_c. $\Rightarrow$$E_c^n(x) \to \infty$.

46



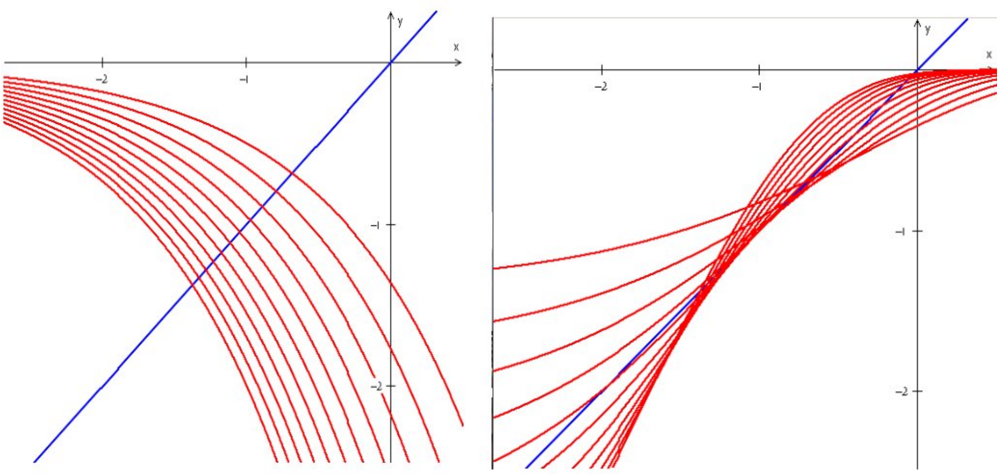
The bifurcation diagram
we plot the location of the fixed (or periodic points agains the parameter).

In our case c is on the horizontal, x is on the vertical axis.

For $E_c(x) = ce^x$, $c > 0$ we have a bifurcation at $c_0 = \frac{1}{e}$.

During this bifurcation as the parameter c decreases “out of nowhere” an attracting and a repelling fixed point is “born”.

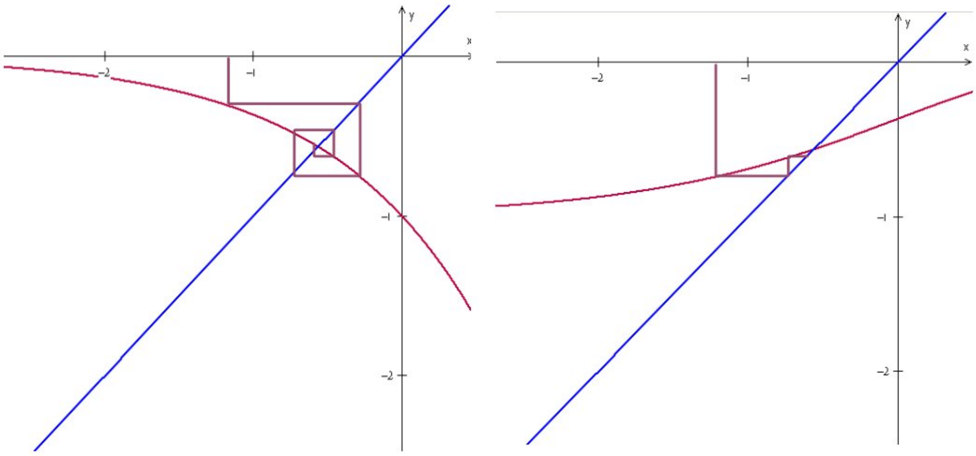
47



Bifurcation types in 1D. Period doubling (flip) bifurcation

Example: $E_c(x) = ce^x$, $c < 0$. We have a bifurcation at $c_1 = -e$.
To the left there is E_c for several parameter values, to the right there is E_c^2.

48

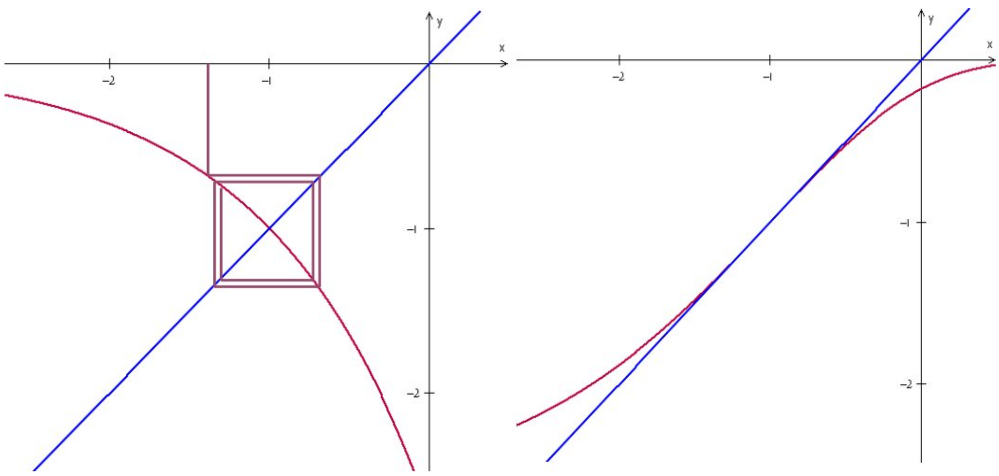


Example: $E_c(x) = ce^x$.

If $0 > c > -e$ then there is one fixed point at p,

$0 > E_c'(p) > -1$, attractive fixed point.

49



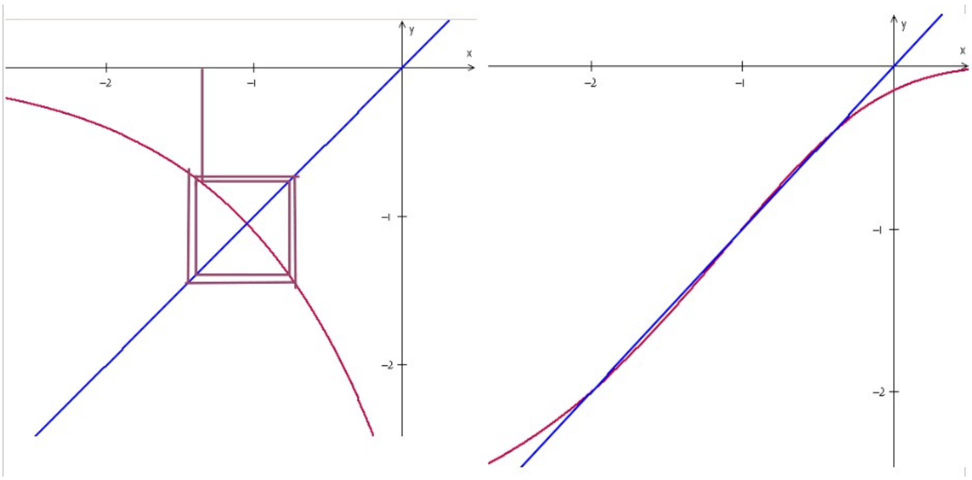
Example: $E_c(x) = ce^x$.

If $c = c_1 = -e$ then $E_{c_1}(-1) = (-e) \cdot e^{-1} = -1$ fixed point.

$E_{c_1}'(-1) = (-e)(e^{-1}) = -1$.

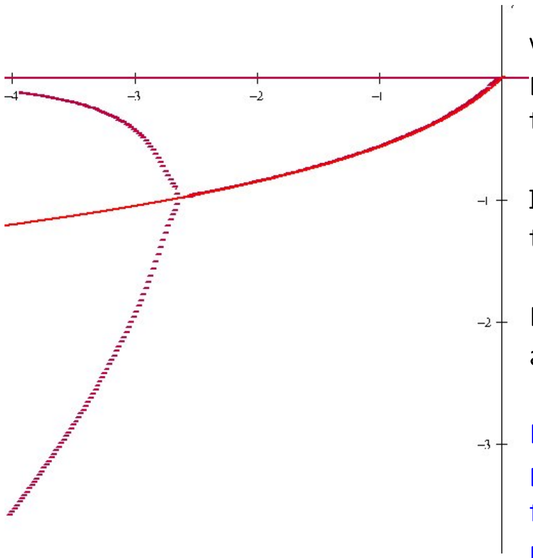
We still have a (weakly) attracting fixed point.

50



Example: $E_c(x) = ce^x$, $c < 0$.
If $c < -e$ then there is a fixed point at p. $E_c'(p) < -1$.
The fixed point becomes repelling.
But, at least for parameter values not much smaller than $-e$,
an attractive period two cycle is "born".

51



The bifurcation diagram
we plot the location of the fixed (or periodic points agains the parameter).

In our case c is on the horizontal, x is on the vertical axis.

For $E_c(x) = ce^x$, $c < 0$ we have a bifurcation at $c_1 = -e$.

During this bifurcation as the parameter c decreases the attractive fixed point becomes repelling and a new attractive period 2 cycle is born.

52

EXERCISES

ZOLTÁN BUCZOLICH:

Introduction to Dynamical Systems, Fractals and Ergodic Theory

Exercise set #1.

1. For the Greek method of computing $\sqrt{2}$ we used $f(x) = \dfrac{x + \frac{2}{x}}{2}$.
   a) Prove that $f(x) \geq \sqrt{2}$ for $\forall x > 0$.
   b) Prove that if $x \geq \sqrt{2}$ then $f(x) \leq x$.
   c) Show that f is not a contraction on $(0, +\infty)$.
   d) Show that if $I = [1, 2]$ then $f(I) \subset I$ and f is a contraction on I.
   e) Prove that $f^n(x) \to \sqrt{2}$ for any $x > 0$.

2. Suppose $(X, \mathcal{B}, \mu, T)$ is a given dynamical system. Suppose $f : X \to \mathbb{R}$ is measurable. Show that $\bar{f} = \limsup_{n\to\infty} \dfrac{1}{n} \sum_{k=0}^{n-1} f(T^k x)$ is T-invariant, that is $\bar{f} \circ T = \bar{f}$.

3. Prove that for Lebesgue almost every $x \in \mathbb{T}$ the orbit $E_2^n(x) = \{2^n x\}$ is dense in $\mathbb{T}$.

Further problems:

4. Suppose $(X, \mathcal{B}, \mu, T)$ is a given dynamical system. Without using Birkhoff's ergodic theorem try to give a proof (as elementary as possible) of the fact, that if $f \in L^1(\mu)$ then $\dfrac{f(T^k x)}{k} \to 0$ for μ a.e. $x \in X$.

5. Suppose that $(X, \mathcal{B}, \mu, T)$ is a given invertible dynamical system. Suppose $A \in \mathcal{B}$ is invariant in the "almost everywhere" sense, that is $\mu(T^{-1}A \triangle A) = 0$. Show that there is $A' \in \mathcal{B}$ such that $\mu(A' \triangle A) = 0$ and A' is invariant in the stricter sense, that is, $T^{-1}A' = A'$.

Exercise set #2.

6. Give a "real analysis" proof (without using Fourier analysis) of the ergodicity of the irrational rotation T_α in $\mathbb{T}$.

7. a) Give an example of a homeomorphism T of a complete metric space X which has a dense orbit ($\exists x \in X$ s.t. $\mathcal{O}_T(x) = \{T^n x : n \in \mathbb{Z}\}$ is dense) but there is no x with a dense positive semiorbit ($\forall x \in X$, $\mathcal{O}_T^+(x) = T^n : n \in \mathbb{Z}_{\geq 0}\}$ is not dense).
   b) Give an example of a homeomrphism T of a compact metric space X which has a dense orbit but there is no x with a positive dense semiorbit.

8. Prove that $(X, \mathcal{B}, \mu, T)$ is ergodic if and only if $\lim\limits_{n\to\infty} \dfrac{1}{n} \sum_{k=0}^{n-1} \mu(T^{-k}A \cap B) \to \mu(A)\mu(B)$ for any $A, B \in \mathcal{B}$.

Further problems:

9. (Koopman–von Neumann) A set $S \subset \mathbb{N}$ is of zero density if $\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \chi_S(k) = 0$, where $\chi_S(k) = 1$ if $k \in S$, otherwise $\chi_S(k) = 0$. Suppose $f : \mathbb{N} \to [0, M]$, with $M \in (0, +\infty)$. Prove that $\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f(k) = 0$ iff $\exists S \subset \mathbb{N}$ of denisity zero such that $\lim_{n\to\infty,\ n\notin S} f(n) = 0$.

10. (Kakutani–Rokhlin lemma) Suppose that $(X, \mathcal{B}, \mu)$ is non-atomic (no sets $A \in \mathcal{B}$ such that for any $B \subset A$, $B \in \mathcal{B}$ we have $\mu(B) = 0$ or $\mu(B) = \mu(A)$). Suppose that $T : X \to X$ is invertible, ergodic and measure preserving, morover $n \in \mathbb{N}$ and $\varepsilon > 0$ are given. Show that $\exists A \subset X$, $A \in \mathcal{B}$ such that $A,\ TA,\ \ldots,\ T^{n-1}A$ are pairwise disjoint and $\mu(X \setminus \bigcup_{k=0}^{n-1} T^k A) < \varepsilon$. (Hint: Take $B \in \mathcal{B}$ with very small measure and consider every n'th level of the Kakutani skyscraper above B.)

MÁRTON ELEKES:
INTRODUCTION TO MEASURE THEORY, GEOMETRIC MEASURE THEORY, GEOMETRIC DECOMPOSITIONS AND DESCRIPTIVE SET THEORY

The goal of this course is to introduce some basic notions, and discuss their basic properties that are needed in the later, more advanced courses of the Summer School. The topics covered are measure theory, geometric measure theory, Hausdorff measures, Hausdorff dimension, box dimension, groups of isometries, geometric decompositions, Borel sets and Baire category.

EXERCISES

MÁRTON ELEKES:

Introduction to measure theory, geometric measure theory, geometric decompositions and descriptive set theory

Core problems

1. a) Prove that $[a, b] \times [c, d]$ is a Borel set.
b) Prove that the set of irrational numbers is a Borel set.

2. Prove that λ^* is an outer measure on $\mathbb{R}^d$.

3. Prove that $\mathcal{H}^s$ is a metric outer measure on $\mathbb{R}^d$ for every s.

4. a) $\dim_s(C \times C) = ?$
b) $\dim_b\left(\left\{\frac{1}{k} : k = 1, 2, \dots\right\}\right) = ?$

5. Prove that $\mathcal{H}^{\frac{\log 2}{\log 3}}(C) < \infty$, and conclude that $\dim_H(C) \le \frac{\log 2}{\log 3}$.

6. Show that $A \subset \mathbb{R}^d$ is nowhere dense iff A^c contains a dense open set.

Extra problems

7. Prove that the Borel sets are exactly the σ-algebra generated by the open sets.

8. Prove that $\overline{\dim}_b(A \times B) \le \overline{\dim}_b(A) + \overline{\dim}_b(B)$.

9. Let $A \subset \mathbb{R}^{d_1}$ and $f : A \to \mathbb{R}^{d_2}$ be a Lipschitz function. Prove that $\dim_H(f(A)) \le \dim_H(A)$.

10. Prove that $A \subset \mathbb{R}^d$ is comeagre iff A contains a dense G_δ set.

TAMÁS KELETI:
THE KAKEYA PROBLEM

How large area is needed to rotate a needle? How small a hedgehog can be? Are lines much bigger than line segments? What do these questions have to do with the Kakeya conjecture, which claims that if a compact set in $\mathbb{R}^n$ has unit line segments in every direction then the set must have Hausdorff / Minkowski dimension n? Why is this conjecture so important to some of the leading mathematicians? What partial results could they prove?

EXERCISES

TAMÁS KELETI:

The Kakeya problem

1. a) Prove that if $B \subset \mathbb{R}^2$ has (2-dimensional) Lebesgue measure zero then $B \times [0,1]$ has (3-dimensional) Lebesgue measure zero.

b) Recall that a *Besicovitch set* in $\mathbb{R}^n$ is a set $B \subset \mathbb{R}^n$ that contains a unit line segment in every direction. Using that there exists a Besicovitch set of Lebesgue measure zero in $\mathbb{R}^2$, show that there exists a Besicovitch set of Lebesgue measure zero in $\mathbb{R}^3$ as well.

2. a) Prove that there exists a sequence $a_0, a_1, a_2, \ldots \in [0,1]$ such that $a_0 = 0$, $\varepsilon_n = |a_{n+1} - a_n| \searrow 0$ and the intervals $[a_n - \varepsilon_n, a_n + \varepsilon_n]$ cover every point of $[0,1]$ infinitely many times.

b) Let $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of x. Let

$$f(t) = \sum_{n=1}^{\infty} \frac{a_{n-1} - a_n}{2^n} \{2^n t\},$$

where (a_n) is the sequence obtained in (a). Check that the above infinite sum converges for any $t \in [0,1]$.

c) Prove that the set

$$K = \{(x, tx + f(t)) \; : \; x, t \in [0,1]\} \subset \mathbb{R}^2$$

contains unit line segments of all slopes in $[0,1]$.

d)* Show that every vertical line intersects K in a set of (1-dimensional Lebesgue) measure zero.

e) Prove that a suitable union of four rotated copies of K is a Besicovitch set of zero measure in $\mathbb{R}^2$.

MIKLÓS LACZKOVICH:
THE BANACH–TARSKI PARADOX

The Banach–Tarski paradox is one of the most surprising results of pure mathematics. It states that a three dimensional ball can be decomposed into a finite number of pieces such that a suitable rearrangement of the pieces constitutes a decomposition of a larger ball, or, more generally, of an arbitrary bounded set with a nonempty interior. In this course we show how the result emerged from the problem of invariant measures, and cover the preliminaries needed for the proof including some geometry (isometries of the Euclidean space), and group theory (free groups). Then we prove the Banach–Tarski paradox, and discuss some improvements and generalizations.

Further reading:

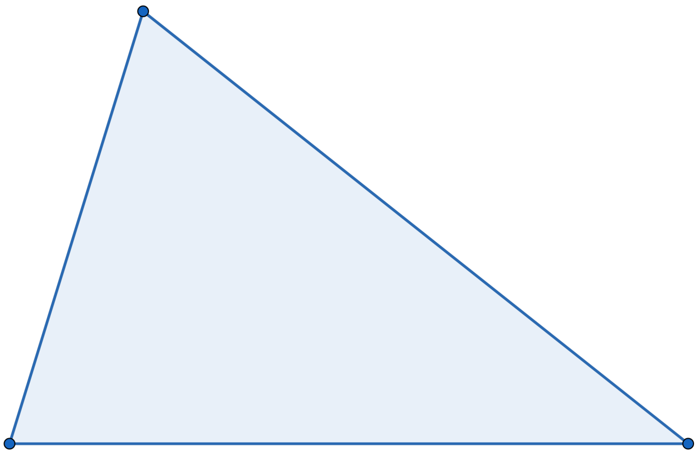
[1] S. Wagon: *The Banach-Tarski paradox.* Cambridge Univ. Press, 1986. First paperback edition, 1993.

[2] G. Tomkowicz and S. Wagon: *The Banach-Tarski paradox.* Second edition. Encyclopedia Math. Appl., **163**, Cambridge University Press, New York, 2016.

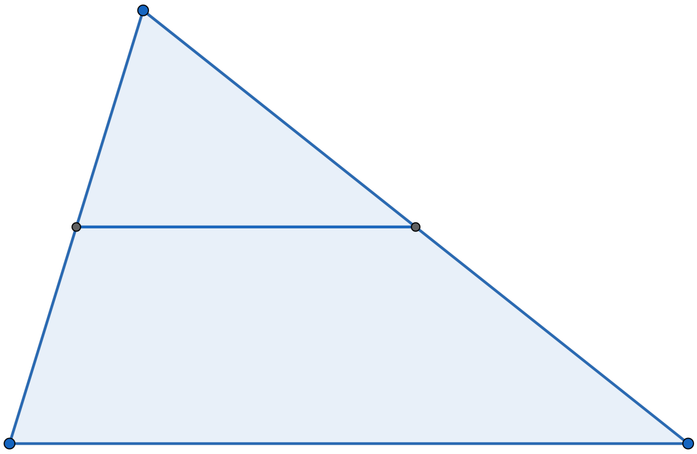
[3] M. Laczkovich: *Conjecture and Proof.* The Mathematical Association of America, 2001.

[4] M. Laczkovich: *Paradoxes in measure theory.* In: Handbook of Measure Theory (editor: E. Pap), Elsevier, 2002. Vol. I, 83-123.

Dissecting polygons



Dissecting polygons



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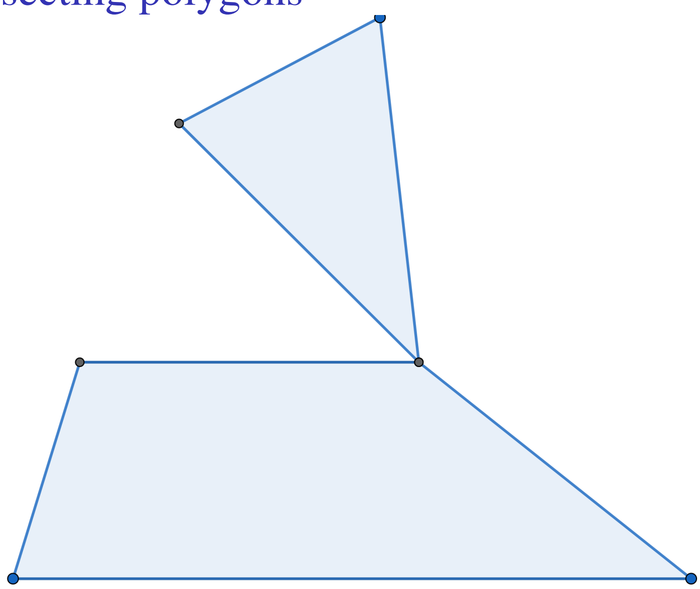

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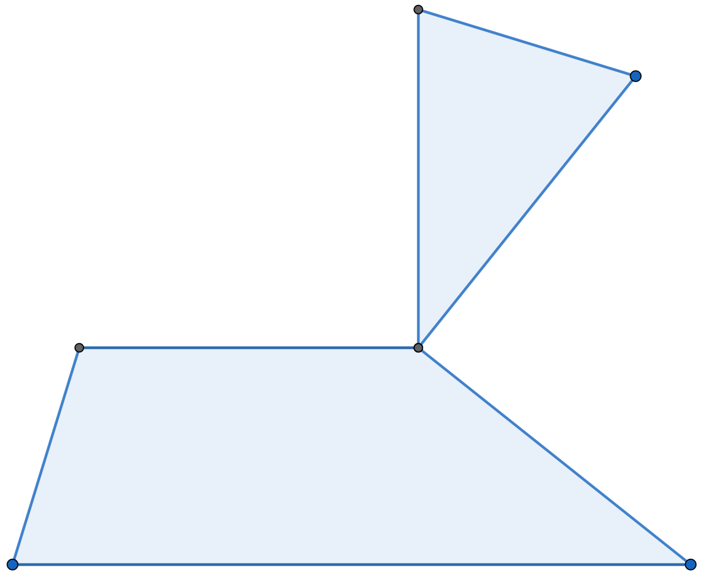

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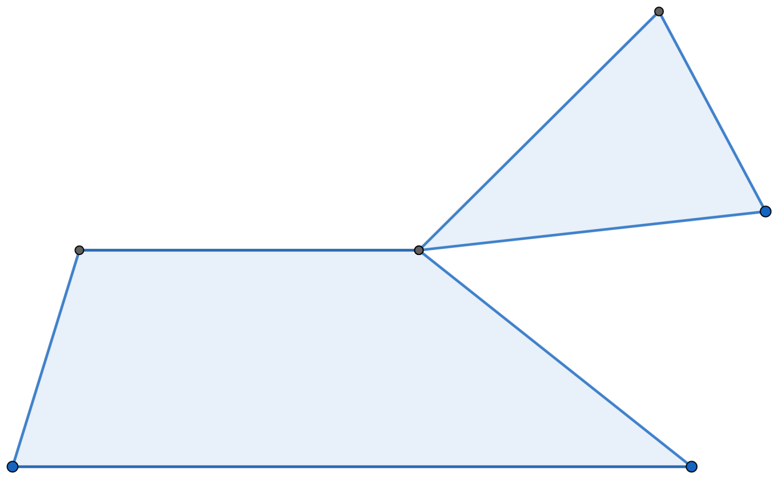

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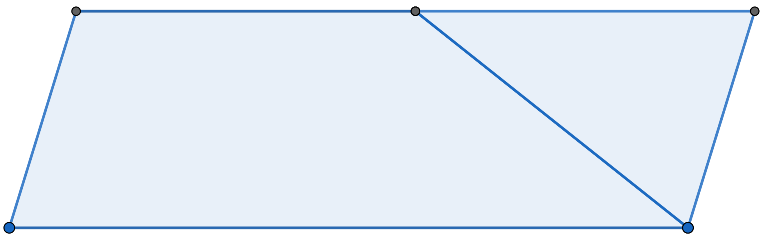

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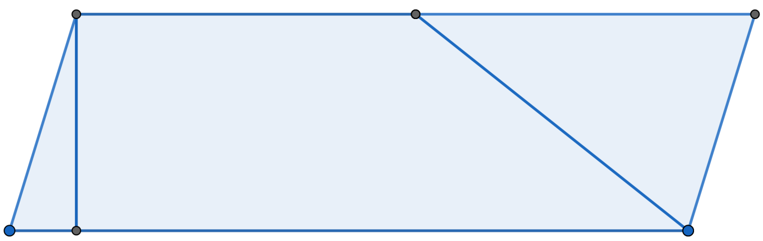

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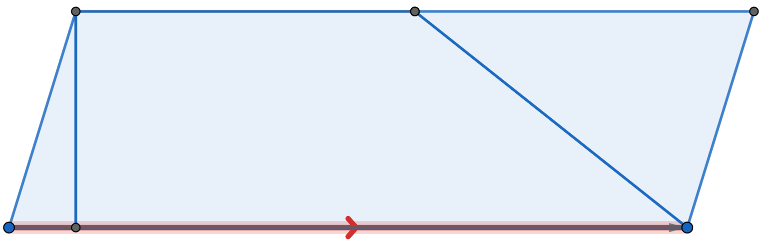

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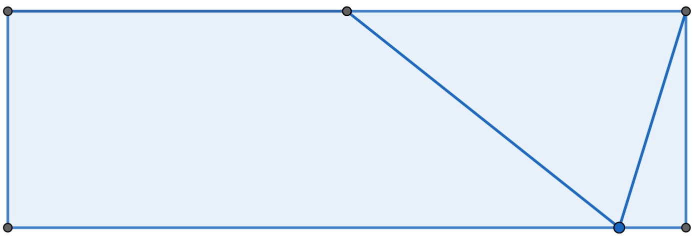

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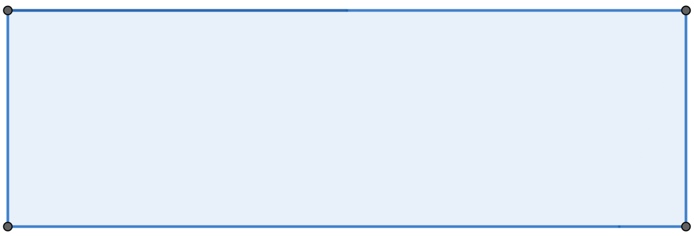
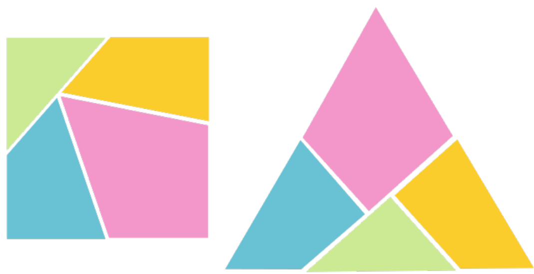


Dissecting polygons and polyhedra

Bolyai–Gerwien–Wallace theorem

Given any two polygons of the same area, it is possible to cut the first into finitely many polygons which can be reassembled to yield the second.



https://en.wikipedia.org/wiki/File:Triangledissection.svg

Hilbert's third problem

Given any two polyhedra of equal volume, is it always possible to cut the first into finitely many polyhedral pieces which can be reassembled to yield the second?

Theorem (Dehn)

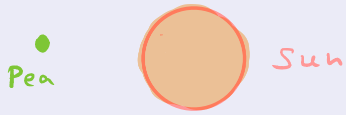
No.
Dehn invariant. For example, cube and regular tetrahedron.

Banach–Tarski paradox (1924)

The unit ball in $\mathbb{R}^3$ can be divided into finitely many pieces that can be rearranged to obtain the union of two disjoint unit balls.

Definition

We say that two sets $A, B \subset \mathbb{R}^d$ are equidecomposable if there exist finite partitions
$A = A_1 \cup^* \ldots \cup^* A_n$
$B = B_1 \cup^* \ldots \cup^* B_n$
where $B_i = \gamma_i(A_i)$ for some isometry γ_i.



Banach–Tarski paradox

Any two bounded sets in $\mathbb{R}^d$, $d \geq 3$, with non-empty interiors are equidecomposable.

Remark

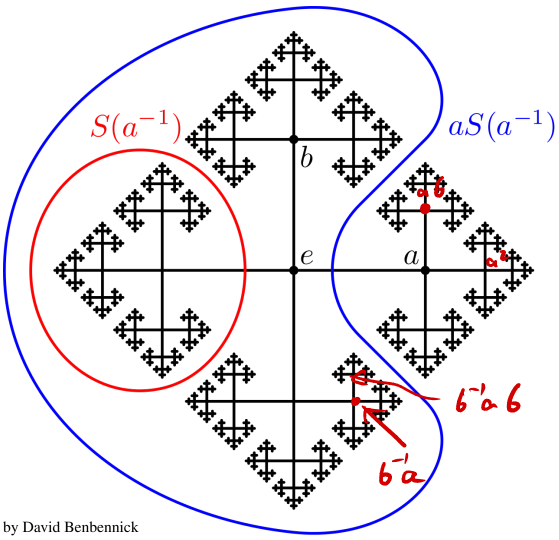
Not true in $\mathbb{R}^2$.

Isometries of $\mathbb{R}^2$ form an *amenable* group.

Hausdorff paradox (1914)

Hausdorff paradox

The unit sphere S^2 is equidecomposable to the disjoint union of two unit spheres modulo countable sets.



by David Benbennick
https://commons.wikimedia.org/wiki/File:Paradoxical_decomposition_F2.png

There exist two rotations in $SO(3)$ generating the free group $\mathbb{F}_2$.

$\mathbb{F}_2 = \{e\} \cup S(a) \cup S(b) \cup S(a^{-1}) \cup S(b^{-1})$

S(a) = all the words starting with a.

$\mathbb{F}_2 = S(a) \cup aS(a^{-1})$

$\mathbb{F}_2 = S(b) \cup bS(b^{-1})$

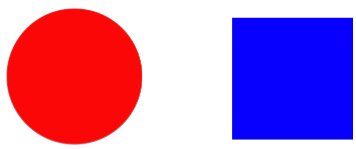
do this in all orbits

No such paradox in $\mathbb{R}^2$.

Tarski's circle squaring problem (1925)

Question

Is the disc equidecomposable to a square?
(Is it possible to cut a disc into finitely many pieces and rearrange them to obtain a square of the same area?)

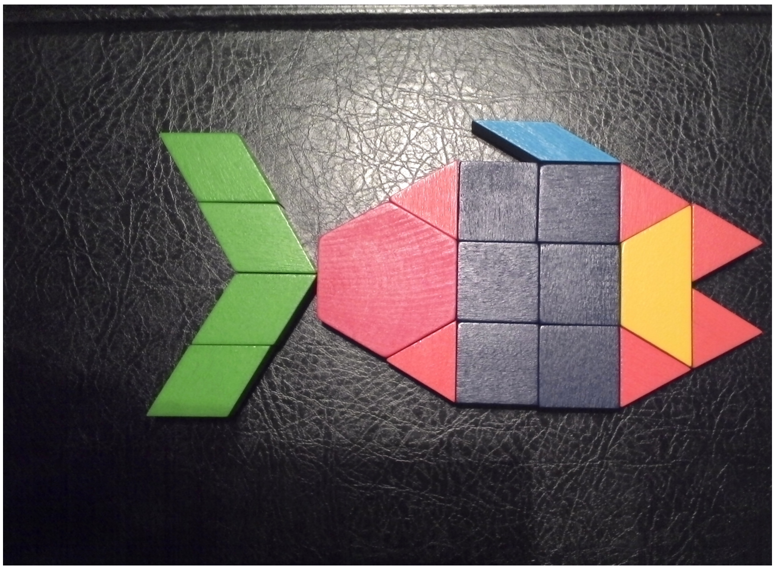


Answer (Laczkovich, 1990)

Yes.
And only translations needed.

How not to look for equidecompositions

Dividing one set into pieces and then trying to reassemble to yield the other usually does not work.



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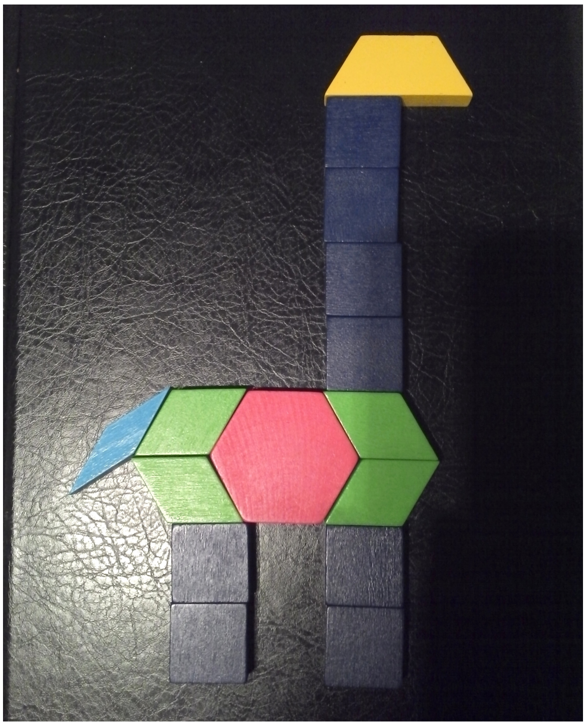
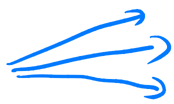


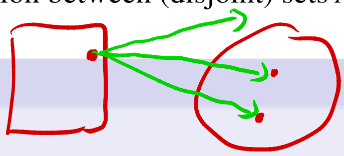
The right way to find equidecompositions

Take a lot of isometries and then find the partitions that work.

In graph theoretic language



Fix isometries/translations $\gamma_1, \ldots, \gamma_n$.
We are trying to find an equidecomposition between (disjoint) sets A, B using these isometries.



Bipartite graph G

- Vertices: $A \cup B$.
- Edges: $\{(a, b) \in A \times B \ : \ \exists i \ \ b = \gamma_i(a)\}$.

Perfect matching

- A set of edges covering every vertex exactly once.
- A bijection $f : A \to B$ such that $\forall x \ \exists i \ f(x) = \gamma_i(x)$.

Claim

There exists a perfect matching in G $\iff$ A is equidecomposable to B using $\gamma_1, \ldots, \gamma_n$.

Proof. If $f : A \to B$ is a bijection, let
$A_i = \{x \in A \ : \ f(x) = \gamma_i(x)$ and there is no smaller i with the same property$\}$.
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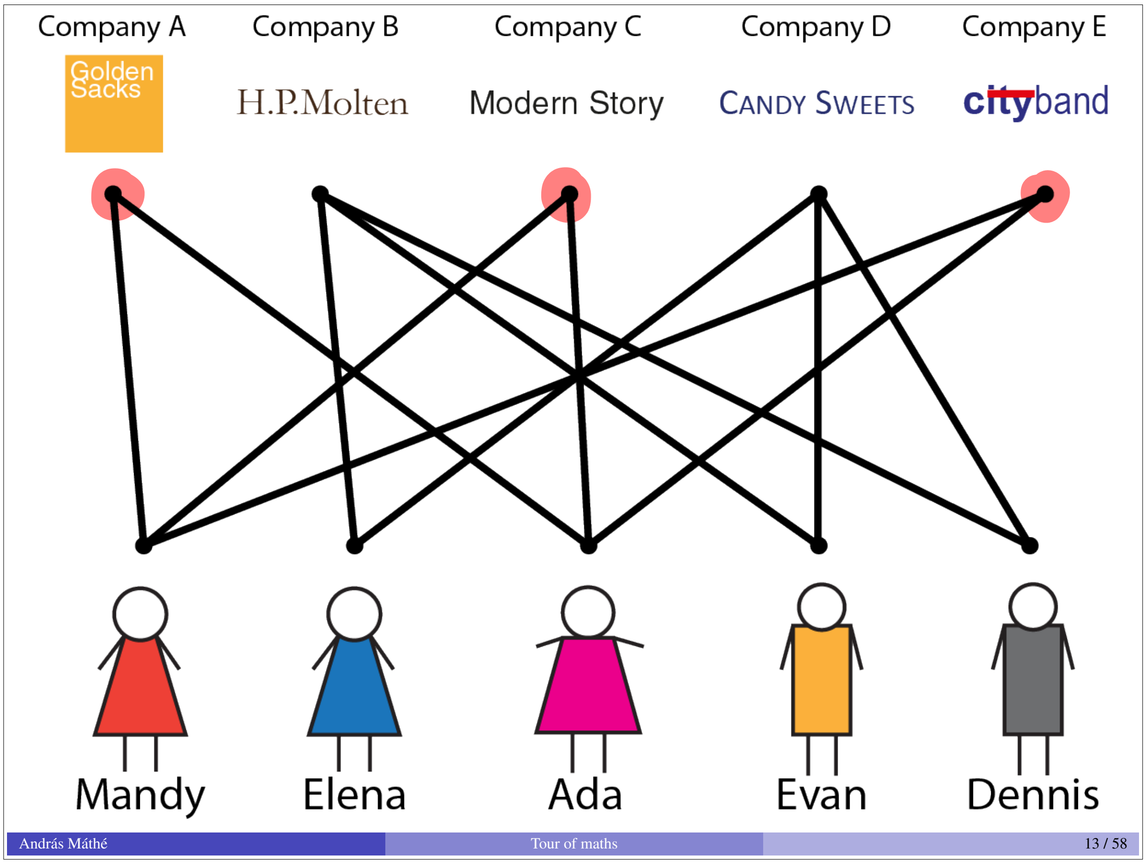
Company A
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Golden Sacks
H.P.Molten
Modern Story
CANDY SWEETS
cityband
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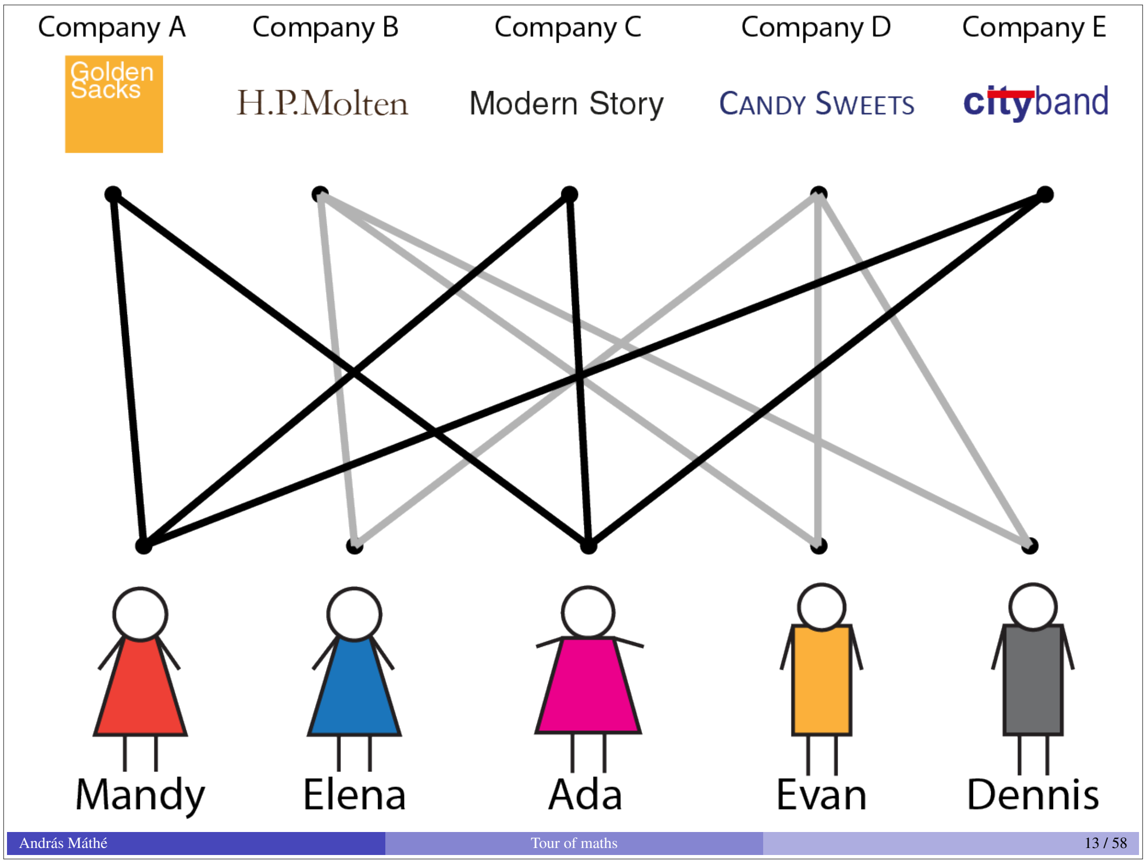
Elena
Ada
Evan
Dennis
András Máthé
Tour of maths
13 / 58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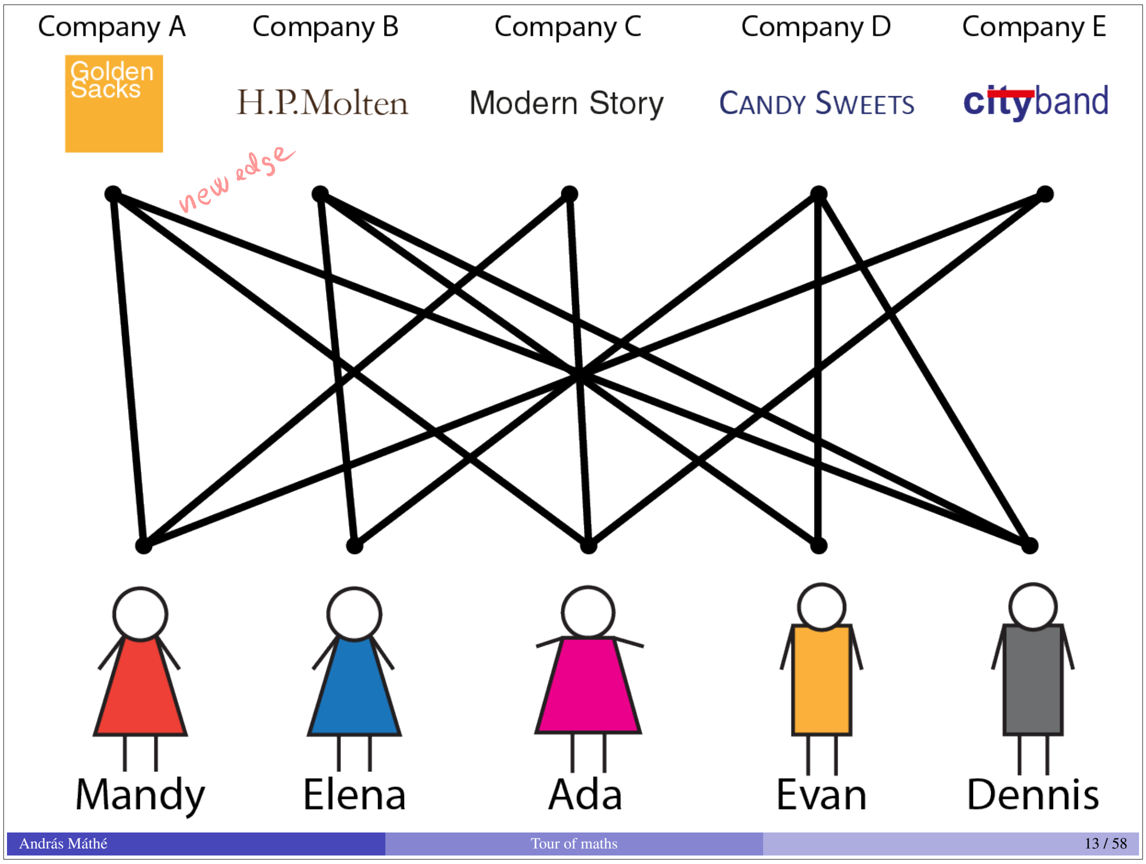
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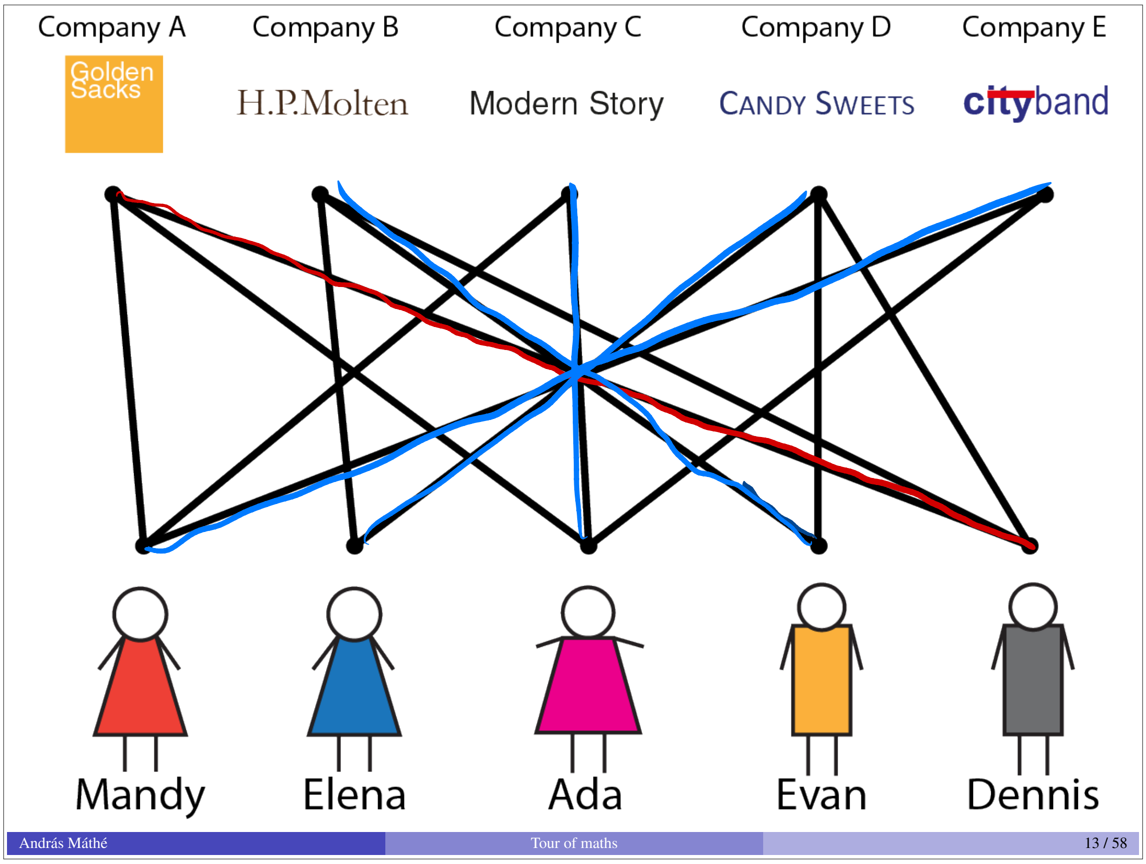
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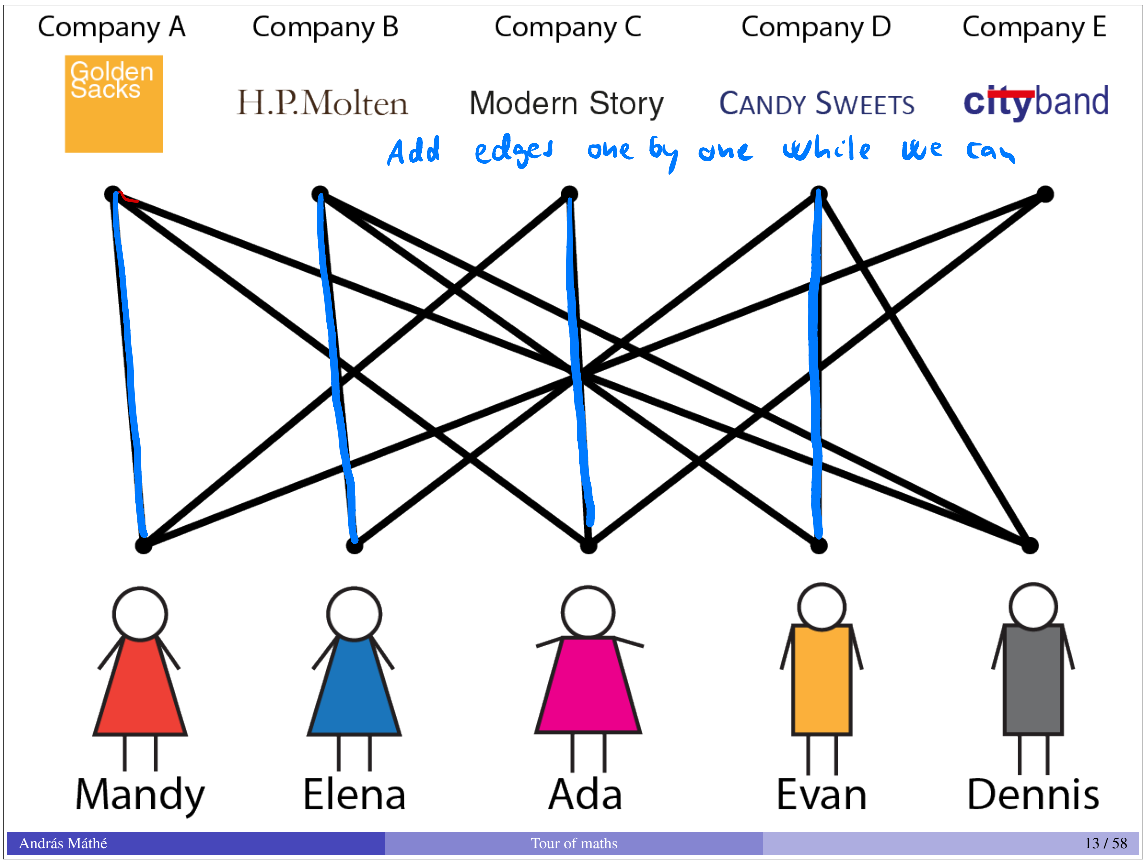
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Add edges one by one while we can
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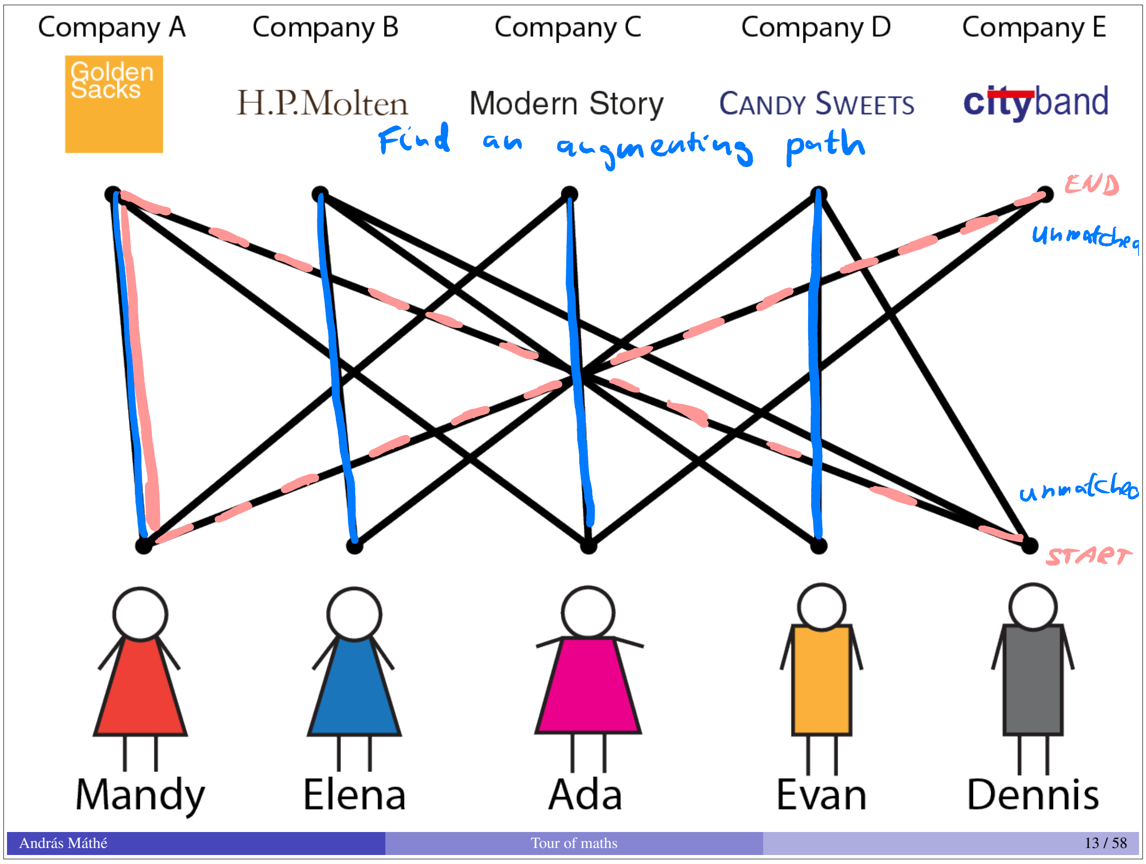
Tour of maths
13 / 58


Company A
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Golden Sacks
H.P.Molten
Modern Story
CANDY SWEETS
cityband
Find an augmenting path
END
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unmatched
START
Mandy
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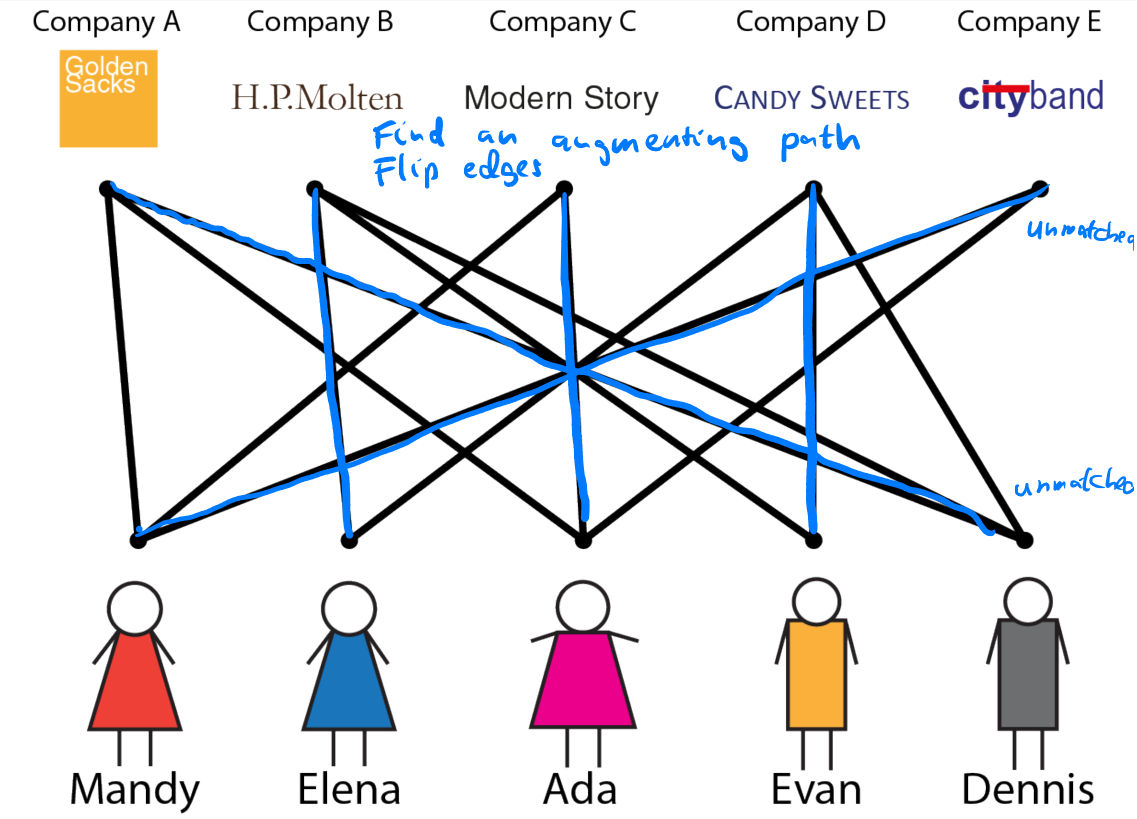
Evan
Dennis
András Máthé
Tour of maths
13 / 58



In graph theoretic language

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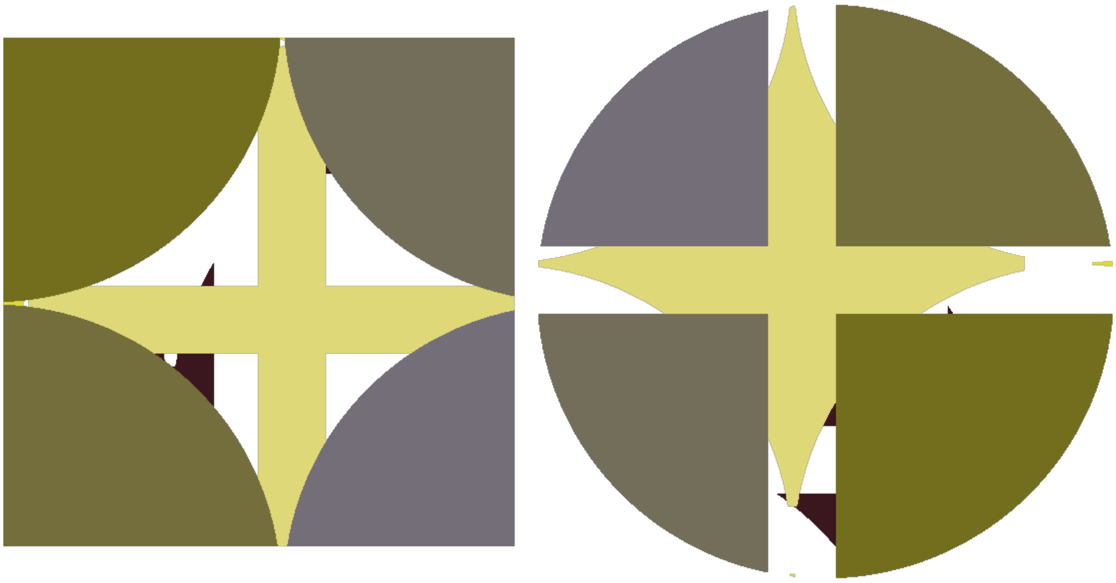
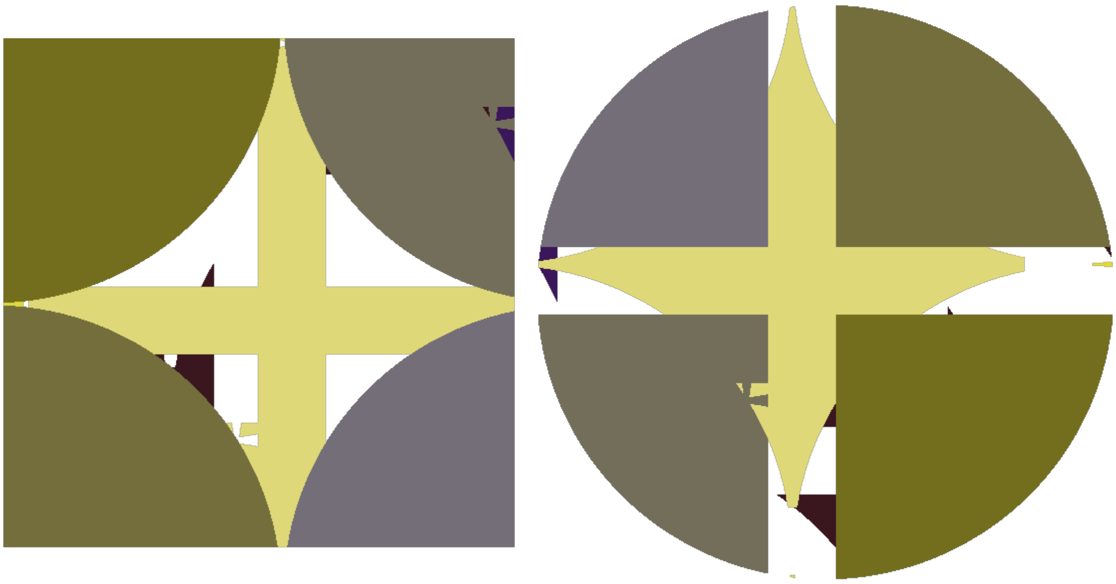
⋮

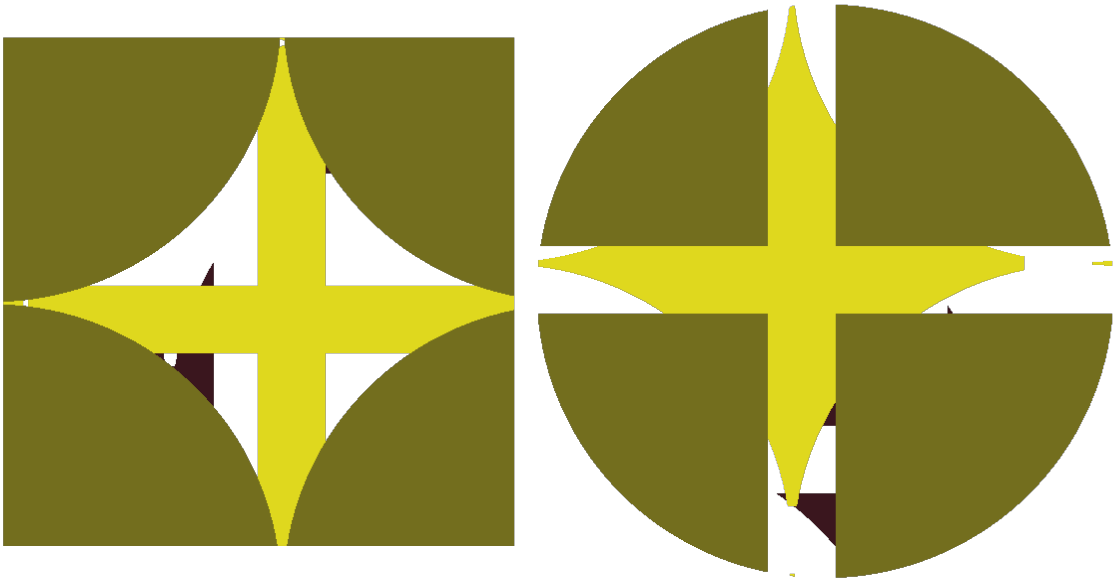
Building up (partial) equidecompositions 1



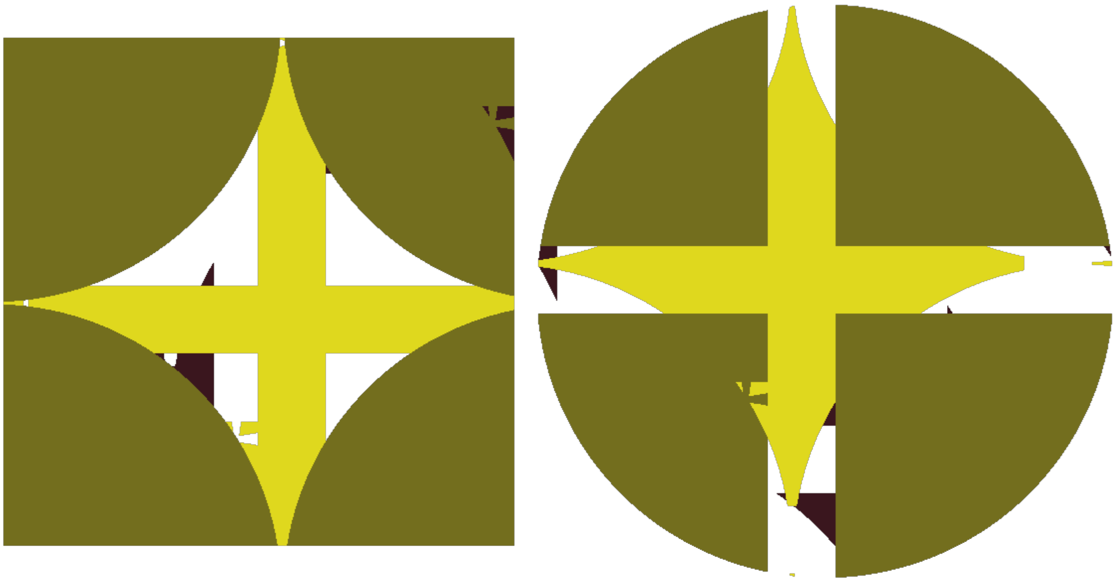
Building up (partial) equidecompositions 2



Building up (partial) equidecompositions 1 v2



Building up (partial) equidecompositions 2 v2



In graph theoretic language 2

Fix isometries/translations $\gamma_1, \dots, \gamma_n$.
We are trying to find an equidecomposition between (disjoint) sets A, B using these isometries.

Bipartite graph G

- Vertices: $A \cup B$.
- Edges: $\{(a,b) \in A \times B \,:\, \exists i \;\; b = \gamma_i(a)\}$.

Perfect matching

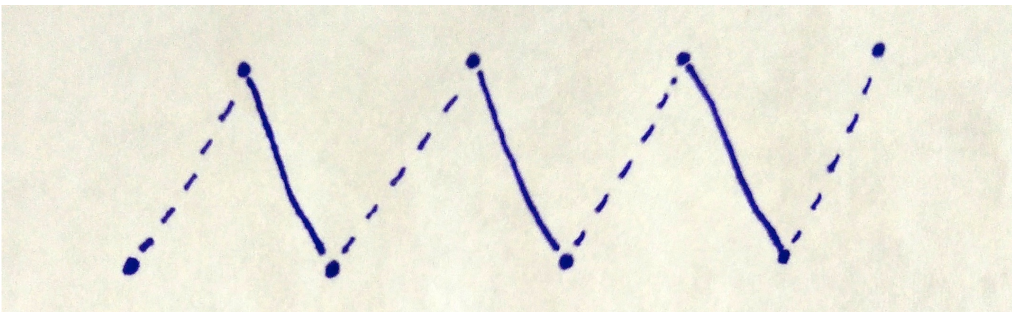
- A set of edges covering every vertex exactly once.
- A bijection $f : A \to B$ such that $\forall x \;\; \exists i \;\; f(x) = \gamma_i(x)$.

Claim

There exists a perfect matching in G $\iff$ A is equidecomposable to B using $\gamma_1, \dots, \gamma_n$.

Proof. If $f : A \to B$ is a bijection, let
$A_i = \{x \in A \,:\, f(x) = \gamma_i(x)$ and there is no smaller i with the same property$\}$.
⋮

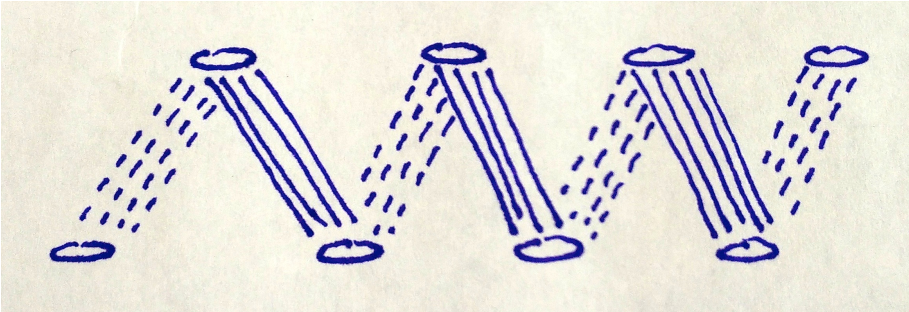
Finding maximum matchings in finite bipartite graphs



Maximum matching algorithm

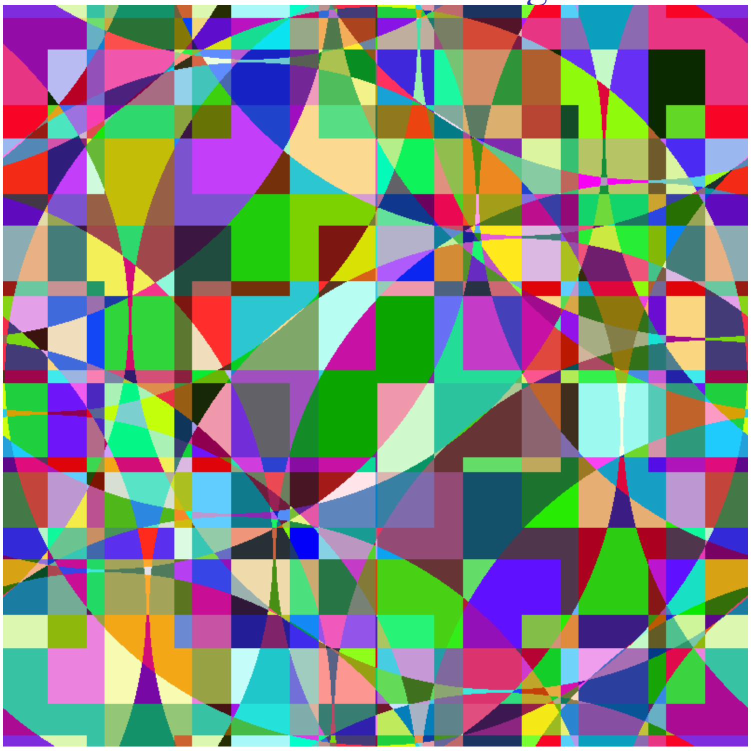
1. Start with empty matching.
2. Find an augmenting path.
3. Increase the size of the matching by flipping edges along the augmenting path.
4. Iterate if we can still find augmenting paths.
5. The algorithm finishes in finite time: we obtain a maximum matching.

Finding measurable maximum matchings in infinite bipartite graphs?



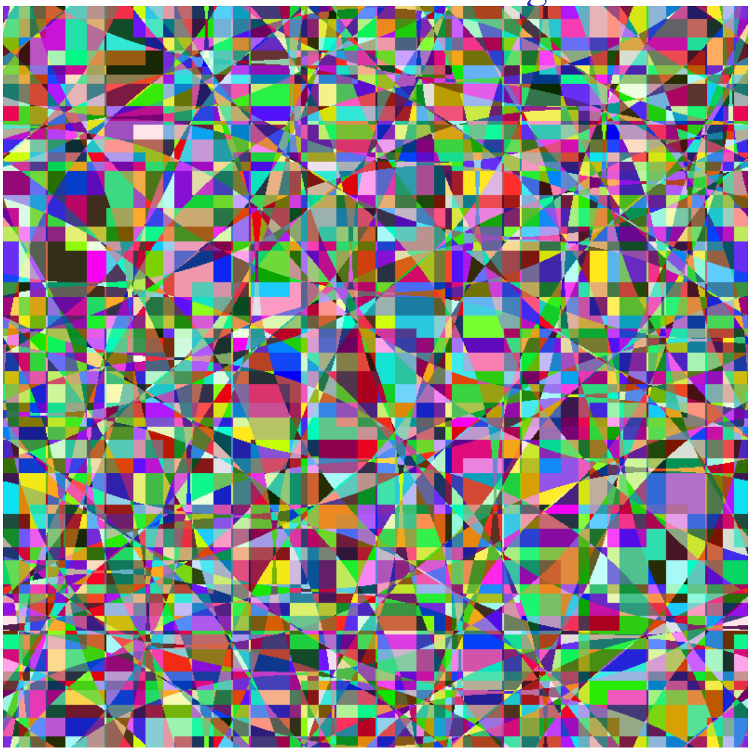
1. Start with empty matching.
2. Find a large family of disjoint augmenting paths. (Elek–Lippner)
3. Increase the size of the matching by flipping edges along the augmenting path.
4. Iterate.
5. The algorithm does not finish in finite time. The matchings might or might not converge.

What a local rule sees – 1 neighbourhood



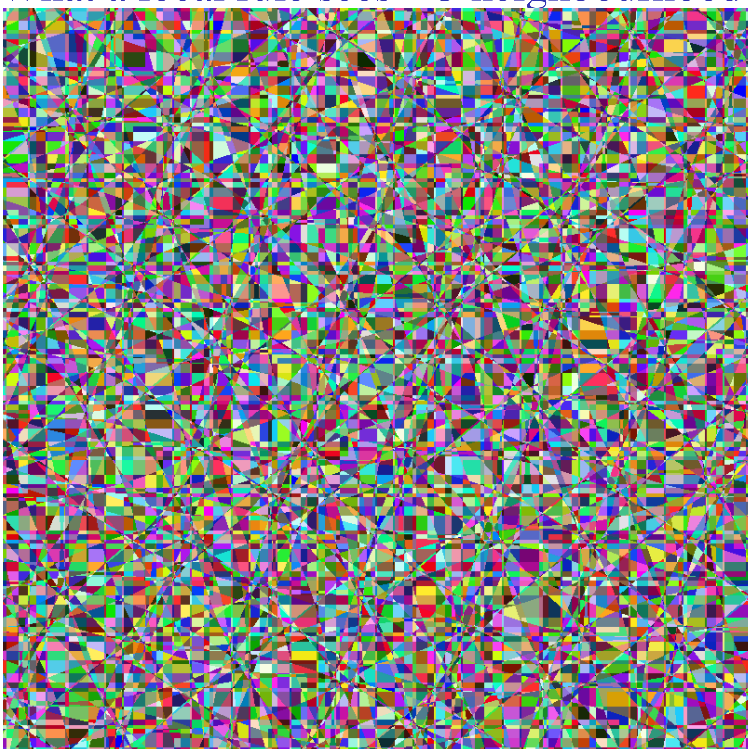
$d = 2;\ v_1,\ v_2;\ $ 9 transl.

What a local rule sees – 2 neighbourhood



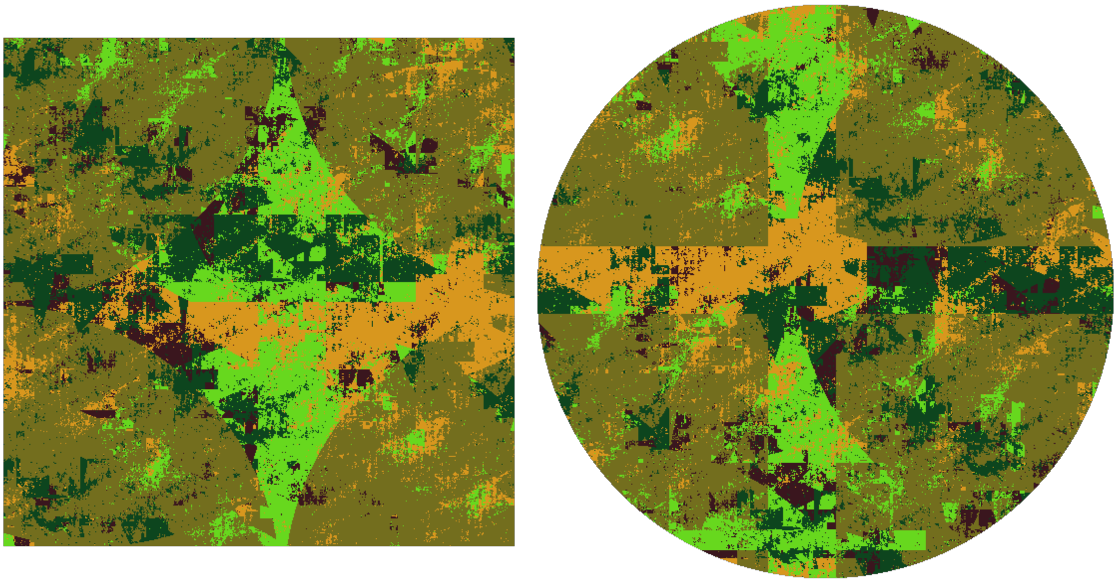
$d = 2;\ v_1,\ v_2;$ 25 transl.

What a local rule sees – 3 neighbourhood



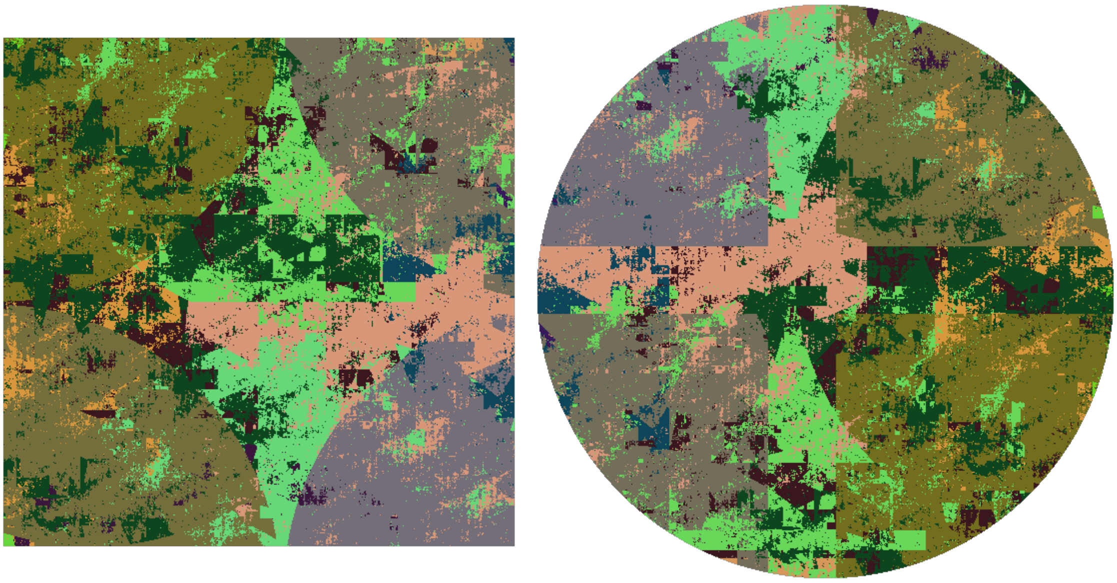
$d = 2;\ v_1,\ v_2;$ 49 transl.

Measurable circle squaring



Experiments suggest: 5 translations on the torus, 16 on the plane, may be enough.
(torus size: 580)

Measurable circle squaring



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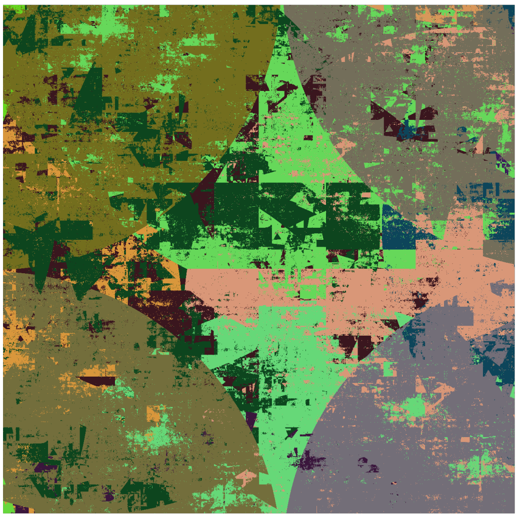
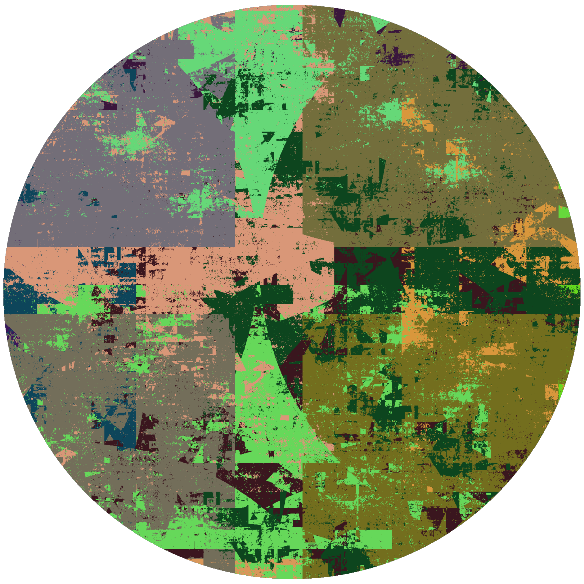
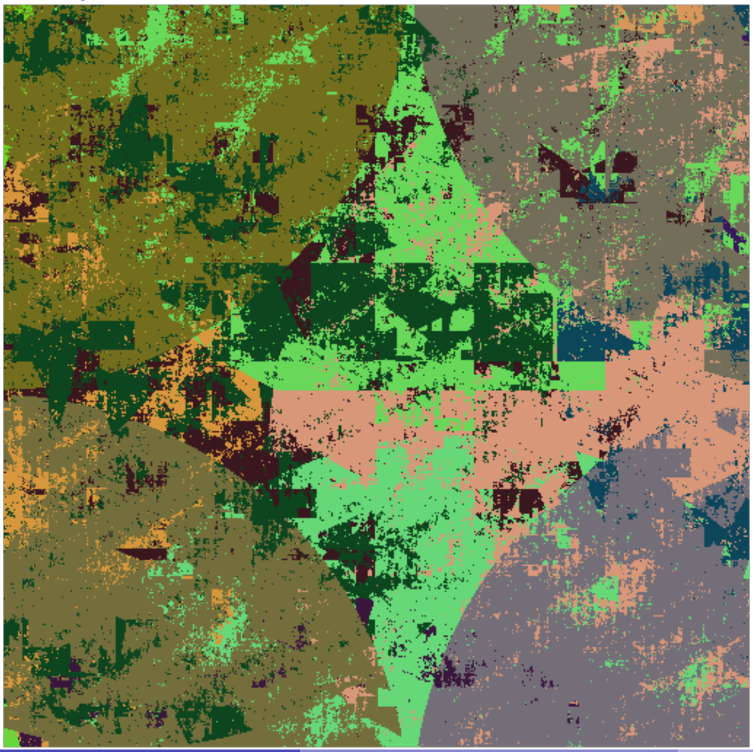
Measurable circle squaring



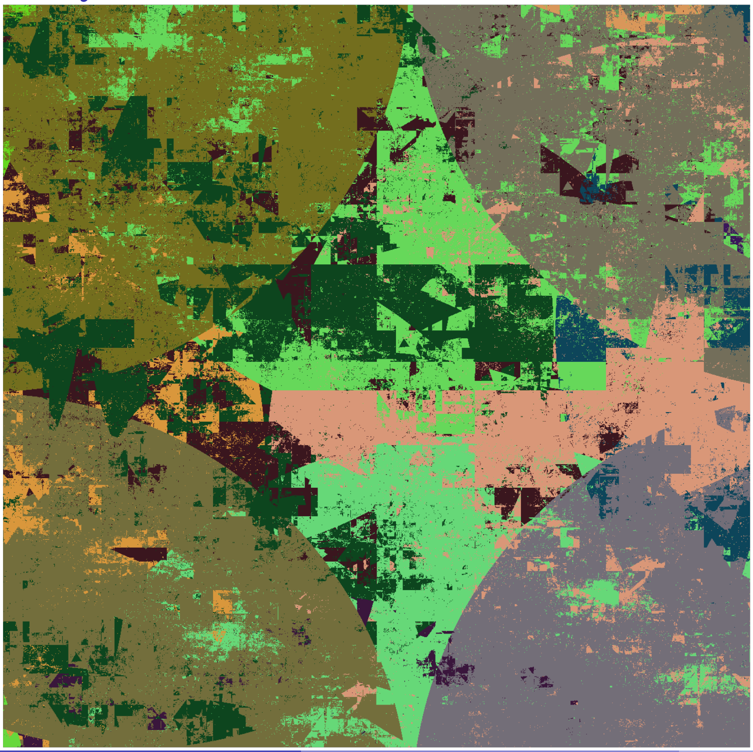


Experiments suggest: 5 translations on the torus, 16 on the plane, may be enough.
(torus size: 1531)

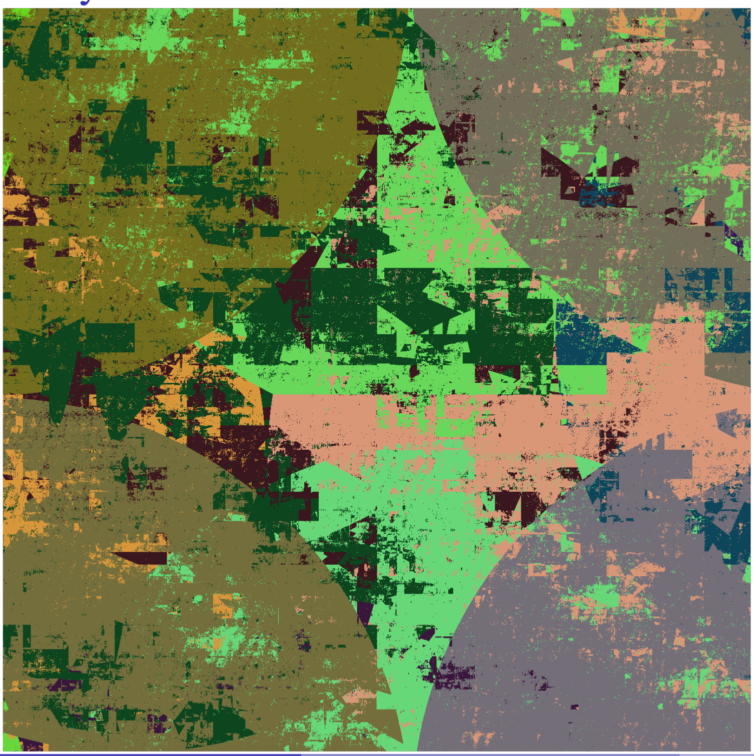
Stability – torus size: 580



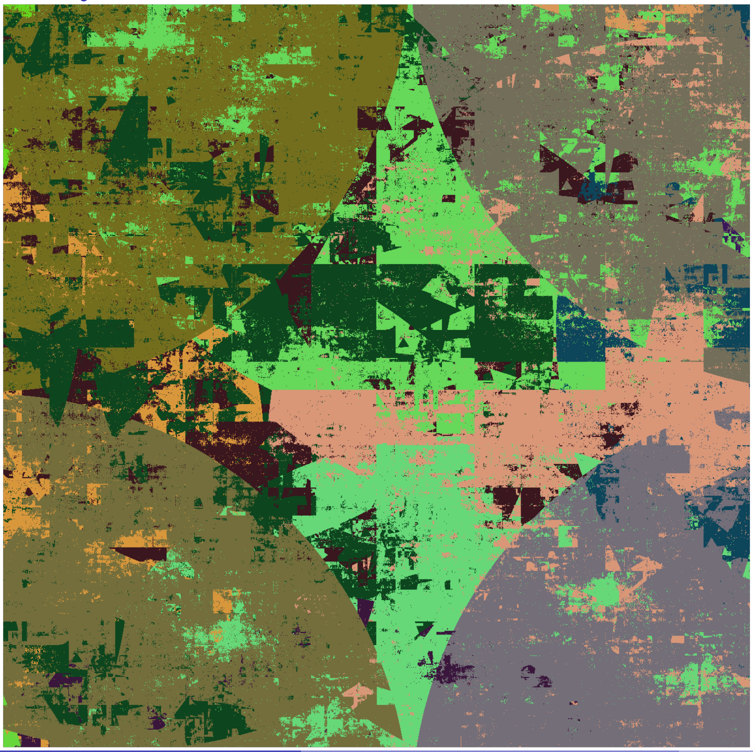
Stability – torus size: 1501



Stability – torus size: 1521



Stability – torus size: 1531



EXERCISES

ANDRÁS MÁTHÉ:

Tarski's circle squaring problem

1. (Required background: the union of countable many sets of Lebesgue measure zero is still of Lebesgue measure zero.)

Show that if there is a measurable equidecomposition modulo nullsets, and there is an equidecomposition, then there is a measurable equidecomposition. That is:

Let $A, B \subset \mathbb{R}^d$ be measurable sets, and let λ denote Lebesgue measure (or let $A, B \subset S^{d-1}$, and let λ be the surface area on the sphere).

- Assume that there are isometries $\gamma_1, \ldots, \gamma_n$, disjoint measurable sets $A_1, \ldots, A_n, N_A$, disjoint measurable sets $B_1, \ldots, B_n, N_B$ such that $\lambda(N_A) = 0$, $\lambda(N_B) = 0$, and

$$A = A_1 \cup \ldots \cup A_n \cup N_A$$
$$B = B_1 \cup \ldots \cup B_n \cup N_B$$

where $B_i = \gamma_i(A_i)$ for each $i = 1, \ldots, n$.

- Also assume that there are isometries $\gamma'_1, \ldots, \gamma'_m$, disjoint (arbitrary) sets $A'_1, \ldots, A'_m$, disjoint (arbitrary) sets $B'_1, \ldots, B'_m$ such that

$$A = A'_1 \cup \ldots \cup A'_m$$
$$B = B'_1 \cup \ldots \cup B'_m$$

where $B'_i = \gamma'_i(A'_i)$ for each $i = 1, \ldots, m$.

Show that there are isometries $\gamma''_1, \ldots, \gamma''_k$ and disjoint measurable sets $A''_1, \ldots, A''_k$, and disjoint measurable sets $B''_1, \ldots, B''_k$, such that

$$A = A''_1 \cup \ldots \cup A''_m$$
$$B = B''_1 \cup \ldots \cup B''_m$$

where $B''_i = \gamma''(A''_i)$ for each $i = 1, \ldots, k$.

Hint. Consider the group Γ generated by the isometries γ_i, γ'_i. This group has countable many elements. Let $N = \cup_{\gamma \in \Gamma} \gamma(N_A \cup N_B)$. Then $\lambda(N) = 0$. Use the measurable equidecomposition on the complement of N, and use the (non-measurable) equidecomposition on N . (Check that they can be 'glued' together.)

Corollary. If $d \geq 3$ and $A, B \subset \mathbb{R}^d$ are measurable, bounded, and have non-empty interior and equal measure $\lambda(A) = \lambda(B)$, then assumption 1 was covered in the lectures; assumption 2 is the Banach–Tarski paradox, so the conclusion holds as well: there is a measurable equidecomposition.

2. (Required background: familiarity with the L^2 norm.)

This exercise explains what it means for an averaging operator to have a spectral gap and asks you to prove that spectral gap implies the 'expansion property' that we used in lectures.

Let $\gamma_1, \dots, \gamma_n$ be rotations of the sphere S^2, let μ denote the normalized surface area, so $\mu(S^2) = 1$. Consider the space $L^2(S^2, \mu)$, this is the space of square-integrable measurable functions, so $f \in L^2(S^2, \mu)$ if f is measurable and

$$\int |f|^2 d\mu < \infty.$$

This is a Hilbert space with norm $\|f\|_2 = \left(\int |f|^2\right)^{1/2}$. Let $T : L^2(S^2, \mu) \to L^2(S^2, \mu)$ be the associated averaging operator defined by

$$(Tf)(x) = \frac{1}{n} \sum_{i=1}^{n} f(\gamma_i^{-1}(x))$$

for $x \in S^2$ and $f \in L^2(S^2, \mu)$. We say that T has a spectral gap if there is a constant $c > 0$ such that

$$\|Tf\|_2 \leq (1 - c)\|f\|_2 \text{ whenever } \int f = 0.$$

(Why is it called spectral gap? Clearly, $Tf = f$ for constant functions, so 1 is an eigenvalue and it is not hard to see that it is the largest eigenvalue. The subspace in L^2 that is orthogonal to the constant functions is exactly the space of functions with integral zero. So the inequality means that "all other eigenvalues are at most $1 - c$". We could assume that T is self-adjoint by insisting that if a rotation is in the list, its inverse is also in the list.)

Drinfeld showed that one can find rotations such that T has this spectral gap. Show that this implies that for every $\varepsilon > 0$, there is $c_\varepsilon > 0$ such that for every measurable set $X \subset S^2$ we have

$$\mu(\cup_i \gamma_i(X)) \geq \min\big((1 + c_\varepsilon)\mu(X), 1 - \varepsilon\big).$$

Then verify (as indicated in the lectures) that by choosing more rotations we can improve this to the following statement: For every $\varepsilon > 0$ and for every $C > 0$, there are finitely many rotations γ_i , such that for every measurable set $X \subset S^2$ we have

$$\mu(\cup_i \gamma_i(X)) \geq \min\big((C\mu(X), 1 - \varepsilon\big).$$

Hint (for the first statement). Given a measurable set X, consider

$$f(x) = 1_X(x) - \mu(X).$$

Then $\int f = 0$. The rest is calculations.

BOREL COMBINATORICS AND DISTRIBUTED ALGORITHMS

ZOLTÁN VIDNYÁNSZKY

These are notes of the summer school lectures at ELTE, 2023.

1. Borel Combinatorics

As we have seen, the most straightforward generalizations of finite combinatorial objects often have counter-intuitive behavior: for example, the Banach-Tarski paradox relies on the existence of a perfect matching in the appropriate graph. To eliminate this kind of behavior, one can investigate instead definable (i.e., Borel/measurable/Baire measurable) generalizations of combinatorial objects. This is the main idea behind the field of Borel combinatorics.

A *graph* G *on a set* X, is a symmetric subset of X^2. In this case $X = V(G)$ is called the vertex set and G is called the edge set. We will call x and y *adjacent/connected/neighbors* if $(x, y) \in G$.

If G is a graph, the *chromatic number of* G, $\chi(G)$ is the minimal n, such that G admits an n*-coloring*, that is a map $c : V(G) \to n$ with

$$\forall x, y \in V(G)\ ((x, y) \in G \implies c(x) \neq c(y)).$$

If $V(G)$ is a Borel space[1], we can define the *Borel chromatic number of* G, $\chi_B(G)$ to be the minimal n, such that G admits an *Borel* n*-coloring*, that is a Borel map $c : V(G) \to n$ with

$$\forall x, y \in V(G)\ ((x, y) \in G \implies c(x) \neq c(y)),$$

here n is endowed with the trivial Borel structure.[2]

If G is a graph, a set $S \subseteq V(G)$ is G*-independent*, if it contains no edges, or formally, if $S^2 \cap G = \emptyset$.

Claim 1.1. *G admits a Borel n-coloring iff $V(G)$ can be covered with n-many G-independent sets.*

Recall that a *connected component* of a vertex v of a graph G is the collection of vertices w, such that there is a path from v to w in G, i.e., a sequence of vertices $v_0, \dots, v_n$ with $v_0 = v$, $w = v_n$ and $(v_i, v_{i+1}) \in G$. A *cycle* is an injective sequence of vertices $v_0, \dots, v_n$ with $n > 1$, such that $(v_n, v_0), (v_i, v_{i+1}) \in G$ for all i. A graph is *acylcic* if it contains no cycles. A graph is d*-regular*, if every vertex has exactly d neighbors.

1.1. Examples.

(1) (The Example) Let $\alpha \in [0, \pi]$ be such that $\frac{\alpha}{\pi}$ is irrational. Denote by T_α the irrational rotation of the circle, S^1 by α. For $x, y \in S^1$ define

$$xGy \iff T_\alpha(x) = y \vee T_\alpha(y) = x.$$

[1] for the sake of this note, Borel space will be identified with Borel subsets of $[0, 1]$

[2] the reader, not familiar with Borel measurability should take the below claim as a definition.

1

Clearly, G is an acyclic 2-regular graph.

Proposition 1.2. $2 = \chi(G) < \chi_B(G) = 3$.

Proof. To show $\chi(G) = 2$ just notice that connected components of G are bi-infinite lines, hence they admit a 2-coloring.

To see $\chi_B(G) \leq 3$ fix some interval I on S^1 with diameter less than α. Clearly I is G-independent. Since α/π is irrational, for every $x \notin I$, there is some $n > 0$ with $T^n(x) \in I$. Then let $c(x) = 2 \iff x \in I$ and $c(x)$ be the parity of the minimal n with $T^n(x) \in I$.

Now, for $\chi_B(G) > 2$ assume that $B_0 \cup B_1 = S^1$ is a Borel 2-coloring. Then, there is an i and a nonempty open interval U with the property that $U \setminus B_i$ is meager. But then (as $2\alpha/\pi$ is also irrational), there is an odd n with $T^n_\alpha(U) \cap U \neq \emptyset$. Now, $T^n_\alpha(U) \cap B_i$ is not meager, as it contains $T^n_\alpha(U) \cap U \cap B_i$. On the other hand $T^n_\alpha(U) \cap B_i$ must be meager, as we started with a coloring and T_α is category preserving. □

(2) (Group actions) Let Γ be a countable group and $S \subseteq \Gamma$ be a generating set. Assume that $\Gamma \curvearrowright X$ is an action of Γ on the set X. As there is no danger of confusion we always denote the action with the symbol $\cdot$. The *Schreier graph* $Sch(\Gamma, S, X)$ of such an action is a graph on the set X such that $x \neq x'$ are adjacent iff for some $\gamma \in S \cup S^{-1}$ we have that $\gamma \cdot x = x'$.

Probably the most important example of a Schreier graph is the (right) *Cayley graph,* $Cay(\Gamma, S)$ that comes from the right multiplication action of Γ on itself. That is, $g, h \in \Gamma$ form an edge in $Cay(\Gamma, S)$ if there is $\sigma \in S$ such that $g \cdot \sigma = h$. Another example is the graph of the left-shift action of Γ on the space 2^Γ: the left-shift action is defined by

$$\gamma \cdot x(\delta) = x(\gamma^{-1} \cdot \delta)$$

for $\gamma \in \Gamma$ and $x \in 2^\Gamma$. Observe that the Schreier graph of this actions is a Borel graph, where we endow the space A^Γ with the product topology.

Let $Free(Sch(\Gamma, S, X)) = \{x : \forall \gamma \in \Gamma(\gamma \cdot x = x \implies \gamma = 1)\}$, the *free part* of the action.

Claim 1.3.
- *For $x \in Free(Sch(\Gamma, S, 2^\Gamma))$ the connected component of x is isomorphic to $Cay(\Gamma, S)$.*

  Proof. The first statement is obvious, while the second is HW. □

  Thus, the typical connected component looks like the Cayley graph of the graph.

Proof. The first statement is obvious, while the second is HW. □

Thus, the typical connected component looks like the Cayley graph of the graph.

(3) Let $[\mathbb{N}]^\mathbb{N}$ denote the collection of the infinite subsets of the natural numbers. The *shift-graph*, $\mathcal{G}_\mathcal{S}$ on $[\mathbb{N}]^\mathbb{N}$ is defined as the symmetrization of the graph of the *shift-map* $\mathcal{S}$, that is,

$$\mathcal{S}(x) = x \setminus \{\min x\}.$$

Clearly $\mathcal{G}_\mathcal{S}$ is acyclic, and *locally finite*, that is, every vertex has finitely many neighbors.

2. The LOCAL model

Now we turn to the investigation of a model of distributed computing by Linial.

Definition 2.1. A *t-round local algorithm* is defined as follows. Given a finite graph/digraph G with $|V(G)| = N$, the vertices of which are imagined to be computers. At the beginning, the computers have no information about the graph, except for knowing their own unique label/identifier, that is, a number $\in \{1, \dots, N\}$. The computation is divided into rounds; in each round, a node can perform a computation and send some information to its neighbors. The nodes must run the same algorithm. After t rounds, each computer must output the solution to a graph-theoretic problem (e.g., vertex or edge coloring, perfect matching, etc.).

We encode the solution of such a problem by a map $f : V(G) \to k$. In this note we will only talk about coloring problems.

Since no constraints are imposed on the length of the computation or messages sent, it is easy to see the following.

Claim 2.2. *A t-round LOCAL algorithm gives rise to a map from labelled t neighborhoods of points to k. Conversely, every such a map corresponds to a t-round local algorithm.*

Thus, the only objective becomes to minimize the number of rounds required to perform the given task. Observe that any coloring problem can be solved on an N-sized graph can be solved by an N-round local algorithm (in fact, by a d-round algorithm, where d is the diameter of the graph). Hence, we are interested in algorithms, which work in significantly less rounds than the diameter of the graph.

Now we consider a very concrete example.

Claim 2.3. *There is no local algorithm to 2-color an N-long path in $\frac{N}{5}$-many rounds.*

Proof. Otherwise, if A was such an algorithm, there were sequences of labels such that $A(q_0, \dots, q_k) = A(q'_0, \dots, q'_k)$ and $\{q_i : i \leq k\} \cap \{q'_i : i \leq k\} = \emptyset$. But then there is a labeling of the path such that the middle vertices corresponding to the sequences above have odd distance. □

The situation with 3-coloring is dramatically different. Define the $\log^*$ function by recursion as follows: let $\log^*(x) = 0$ if $x \leq 1$, and $1 + \log^*(\log_2 x)$ if $x > 1$.

Theorem 2.4. *There is a $\log^* N + C$ round algorithm to 3-color the path.*

Proof. For the sake of simplicity, we will assume that the path is directed towards one end, the general case will follow from Proposition ???.

The observation is that the labels already give an N-coloring, and we will step-by-step improve this coloring.

We first need a combinatorial object, which is interesting on its own.

Lemma 2.5. *(Sperner families) Let $k \geq C_0$ be even. There is a family $\mathcal{F}$ of subsets $\{0, 1, \dots, k-1\}$ such that*

- $|\mathcal{F}| \geq \frac{2^k}{k}$.
- *every $A, B \in \mathcal{F}$ distinct, we have $A \setminus B \neq \emptyset$.*

Proof. Take $\mathcal{F}$ to be the $k/2$ sized subsets of k. □

Now, in one round we reduce the number of colors exponentially.

Lemma 2.6. *There exists some constant C, such that given a k-coloring of the path with $k \geq C$, in one round we can output a $k'-$ coloring, where $k' \leq \log_2 k + \log_2 \log_2 k + 2$.*

Proof. Take the minimal k' even with $2^{k'}/k' \geq k$, then k' satisfies the inequality above, fix a family $\mathcal{F}$ as in the above lemma.

Now, there is an injection $b : k \to \mathcal{F}$. Let c be the k-coloring, and for a vertex v define its new color to be any element of $b(c(v)) \setminus b(c(w))$, where the (v, w) edge is directed towards w. Note that this is possible, by the choice of the family $\mathcal{F}$, and this is going to be a k'-coloring, as the new color of a vertex v comes from the set $b(c(v))$, which is avoided by the color of its neighbor. □

Thus, applying this improvement $\log^* N$-many times, we can get to a coloring with k-colors, where k is already too small to use the reduction again (observe that this threshold does not depend on N). To deal with this, we use another technique.

Lemma 2.7. *Let $k \geq 4$. Assume that we are given a k-coloring of the path. Then there is a 1-round algorithm for a $k-1$-coloring.*

Proof. Let c be the k-coloring. Now, for every vertex that has color $k-1$, choose a color not used by its neighbors. □

Hence, the overall algorithm goes as follows: we apply the logarithmic color reduction $\log^* N$-many times, and once we are stuck, we apply the above lemma a constant number of times, until we get a 3-coloring. □

Theorem 2.8. *The bound $\log^* N$ is asymptotically optimal.*

Proof. HW. □

A *rooted directed tree* is a directed, connected acyclic graph with a distinguished vertex, the root, so that every edge is directed towards it.

Theorem 2.9. *There is a $\log^* N + C$ local algorithm to 3-color an N sized rooted directed tree.*

Proof. HW. □

With more sophisticated versions of Sperner families, one can dramatically generalize Theorem 2.4. Let $\Delta(G)$ be the maximal degree (i.e., number of neighbors) in the graph G.

Theorem 2.10. *There exists a $C_{\Delta(G)} \log^* N$ round local algorithm to $\Delta(G)+1$-color a graph G of size N.*

A family of sets $\mathcal{F}$ is Δ-*cover free*, if for all $A_1, \dots, A_{\Delta+1} \in \mathcal{F}$ distinct, we have

$$A_1 \setminus \cup_{1<i\leq\Delta+1} A_i \neq \emptyset.$$

Theorem 2.11. *For any large enough k, there exists a k sized Δ-cover free family of subsets of the set $C_\Delta 2^k$, where C_Δ is an explicit constant depending on Δ.*

Using this statement, it is not hard to give a proof of Theorem 2.10 similarly to the proof of Theorem 2.4.

3. Back to Borel

Now we turn our attention to the Borel realm. Recall Brooks' theorem from finite combinatorics: if a graph has degrees $\leq \Delta$ then its chromatic number is at most $\Delta + 1$. This theorem has an analogue in the Borel context.

Remark 3.1. In what follows, we will not check that the objects defined are Borel. In same cases it is a straightforward calculation in other cases it follows from the Luzin-Novikov theorem, see [6].

Theorem 3.2. *Assume that G is a Borel graph. Then $\chi_B(G) \leq \Delta(G) + 1$.*

Lemma 3.3. *Assume that G is a Borel graph with finite degrees. Then $\chi_B(G) \leq \aleph_0$.*

Proof. Fix a basis $(U_n)_{n\in\mathbb{N}}$ of the underlying space. Color each x by the minimal n such that for any $y \in X$ with $(x, y) \in G$ we have $y \notin U_n$ (such an n exists as all $x \in X$ has only finitely many neighbors). □

Proof of Theorem 3.2. Fix a Borel coloring c of G with countably many colors. Color the elements of $c^{-1}(n)$ by induction on n, producing a coloring $c' : X \to n$. If $\bigcup_{i<n} c^{-1}(i)$ has been already colored and $c(x) = n$ let $c'(x)$ be the minimal $j < d+1$ such that x has no neighbors already colored by j. Since the sets $c^{-1}(n)$ are G-independent and the degrees of G are bounded by d, c' is a Borel $d+1$-coloring. □

Theorem 3.4 $(1, 2, 3, \infty)$**.** *Let G_f be an acyclic Borel graph arising from a symmetrization of a function f. Then $\chi_B(G_f) \in \{1, 2, 3, \aleph_0\}$. Moreover, all these chromatic numbers can be realized.*

Proof. By Lemma 3.3 we have $\chi_B(G_f) \leq \aleph_0$.

Lemma 3.5. *Assume that G_f admits a finite Borel coloring $c : V(G) \to k$ with $k \geq 4$. Then there G_f admits a Borel $k - 1$-coloring.*

Proof. Define a new coloring $c'_0(x)$ by $c'_0(x) = c(f(x))$. Note that for any x the color of all preimages of x is the same. Clearly c'_0 is also a Borel k-coloring. Now, define $c'(x)$ by letting $c'(x) = c'_0(x)$ in case this value is $\leq k - 2$, and otherwise choose a color not used by the neighbors of x (this is possible, as there are at most two colors used). □

Iterating this lemma yields that if $\chi_B(G_f)$ is finite, then $\chi_B(G_f) \leq 3$.

In order to see the second statement, note that we have seen that $\chi_B(G_{T_\alpha}) = 3$, restricting G_{T_α} to any connected component gives an example of a graph with Borel chromatic number 2.

Finally, we claim that $\chi_B(\mathcal{G}_S) = \aleph_0$. This relies on the following generalization of the infinite Ramsey theorem.

Theorem 3.6 (Galvin-Prikry)**.** *Let $k, l \in \mathbb{N}$ and $c : [\mathbb{N}]^{\mathbb{N}} \to l$ be a Borel coloring. There exists a set $A \in [\mathbb{N}]^{\aleph_0}$ such that $c \restriction [A]^{\mathbb{N}}$ is constant.*

To see our claim, towards contradiction, assume that there is Borel l-coloring c of G_S. Then, by the Galvin-Prikry Theorem there is a set A such that all subsets of A are homogeneous. In particular, $c(A) = c(S(A))$, a contradiction. □

Problem 3.7. *It is not known, what are the possible values of Borel chromatic numbers of Borel graphs generated by k functions. Is it the case that they belong to $\{1, 2, \ldots, 2k + 1, \aleph_0\}$?*

4. A transfer

We finish with a transfer theorem from the distributed world to the Borel one. For the sake of simplicity we will first work with paths, for which the below statement is vacuous, as we already now for which l such algorithms do exist. Nevertheless, the idea presented can be transferred to more meaningful contexts.

Proposition 4.1. *Assume that there is some C such that on every path of length N there is a local $C \log^* N$ round l-coloring algorithm. Then every Borel graph such that all connected components are bi-infinite paths can be l-colored in a Borel manner.*

Proof sketch. Let G be a Borel graph whose all connected components are bi-infinite lines.

We can choose an N large enough so that there is a t-round local algorithm for l-coloring of N long paths, where $t \ll N$ (to be specified later).

The idea is that we want to apply this algorithm on the Borel graph G, however, we lack the input labels there. Now, consider the graph G^{4t} on $V(G)$, where x and y are connected if their distance in G is $\leq 4t$. Then $\Delta(G^{4t}) \leq 9t$. In particular, by Theorem 3.2, there is a Borel $9t + 1 < N$ coloring c_0 of G^{4t}.

Claim 4.2. *c_0 assigns labels $\in \{1, 2, \ldots, 9t+1\} \subset \{1, 2, \ldots, N\}$ such that in every $2t + 2$ neighborhood in G, the labels are pairwise distinct.*

Now we run the local algorithm on the Borel graph G in the following way: at every vertex, take its $2t + 1$ neighborhood, the coloring c_0 yields a labeling of this neighborhood by labels $\{1, 2, \ldots, N\}$. Observe also that by the choice of the graph G^{4t}, there can be no two vertices in this neighborhood with the same c_0 label. Hence we can apply the local algorithm in the $2t + 1$-neighborhood.

Finally, observe that since the local algorithm outputs an l-coloring, this must be a (proper) l-coloring of the graph G. Otherwise, two neighboring vertices would get the same color. But this could have happened in the N-long path, contradicting the correctness of the algorithm.

□

Using the same trick, one can show the following. Call a family of finite graphs $\mathcal{F}$ *nice*, if it is closed under taking subgraphs and every graph in $\mathcal{F}$ has degree bounded by d.

Theorem 4.3. *Let $\mathcal{F}$ be a nice family. Assume that there is some C such that on every element of $\mathcal{F}$ of size N there is a local $C \log^* N$ round l-coloring algorithm. Then every Borel graph G such that all the finite neighborhoods of vertices of G are in $\mathcal{F}$ admits a Borel l-coloring.*

The reader interested in the rich theory of Borel combinatorics and its connections to the LOCAL model should consult [9, 8, 1, 2, 4, 3, 5, 10, 7].

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EXERCISES

ZOLTÁN VIDNNYÁNSZKY:

Finite and infinite: connections between distributed computing and Borel combinatorics

Exercise set #1.

1. A set $S \subset \mathbb{R}$ is *nowhere dense* if for every open I there is a $J \subset I$ open with $J \cap S = \emptyset$. Show that the ternary Cantor set is nowhere dense in $\mathbb{R}$.

2. Prove the Baire Category Theorem on $\mathbb{R}$: if $(A_n)_{n\in\mathbb{N}}$ is a sequence of nowhere dense sets, then $\cup A_n \neq \mathbb{R}$.

3. A set is called *meager*, if it can be covered by the union of countable many nowhere dense sets.

a) Show that the countable union of meager sets is meager.
b) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is a homeomorphism and M is meager, then so is $f^{-1}(M)$.
c) Show that for any closed set F we have that $F \setminus int(F)$ is meager.

4. A set S is called *Baire measurable* if there exists an open set U such that $S \setminus U \cup U \setminus S$ is meager.

a) Show that Baire measurable sets are closed under countable unions.
b) Show that Baire measurable sets are closed under complements using the last statement in Problem 3.
c) Conclude that Borel sets are Baire measurable.

5. Complete the proof of the statement from the lecture: show that $\chi_B(G_{T_\alpha}) = 3$.

Exercise set #2.

6. Assume that G is a rooted directed tree, $k \geq 4$, and c is a k-coloring of G. Show that using c, there is a 2-round local algorithm to $k-1$-color G.

7. Show that there is a $C \log^* N$-round local algorithm to 3-color an N sized rooted directed tree. **Hint:** Use problem 6.

8. Using the following statement as a black box, show that the bound $\log N$ is asymptotically optimal for the 3-coloring of paths.

Theorem. *There is some $C > 0$ such that for any large enough N and $k < C\log^* N$ the following holds: assume that the k-tuples of the set $\{0, 1, \ldots, N-1\}$ are colored by 3-colors. Then there is a $k+1$-tuple such that all of its k-tuples have the same color.*

9. Assuming the existence of Δ-cover free families of exponential size show that there exists a $C_{\Delta(G)} \log^* N$ round local algorithm to $\Delta(G)+1$-color a graph G of size N.

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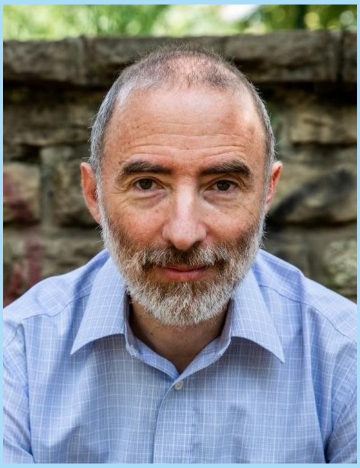

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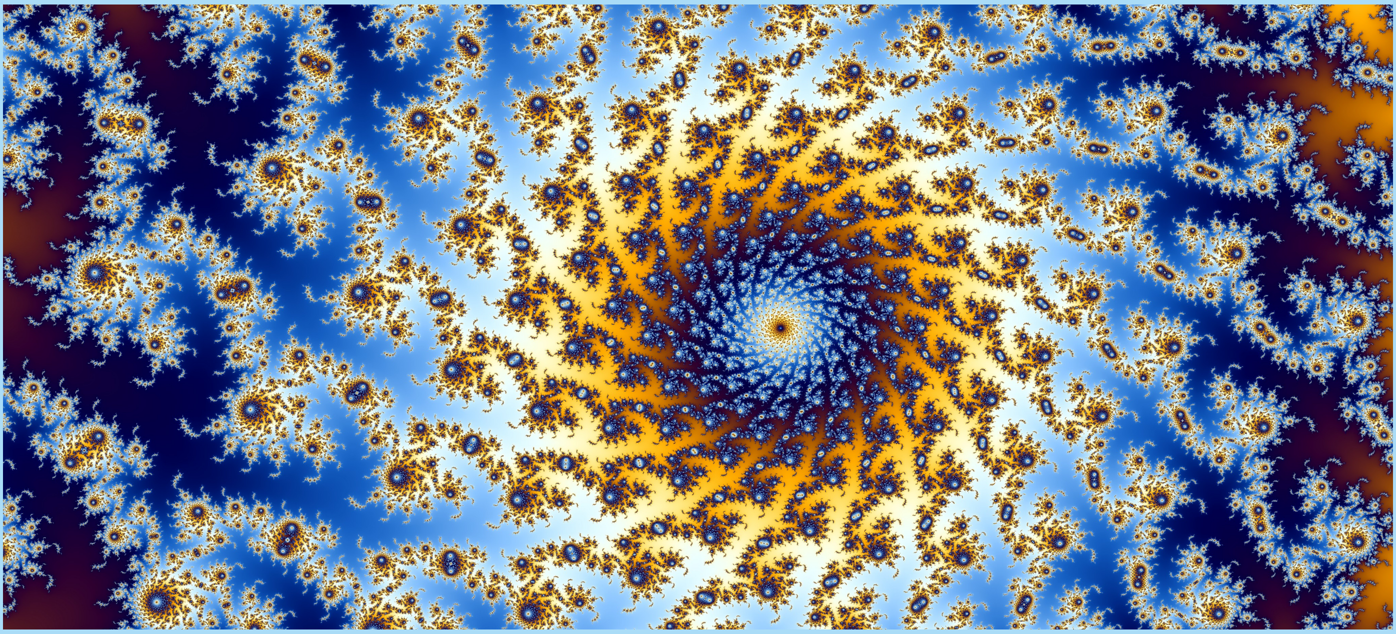

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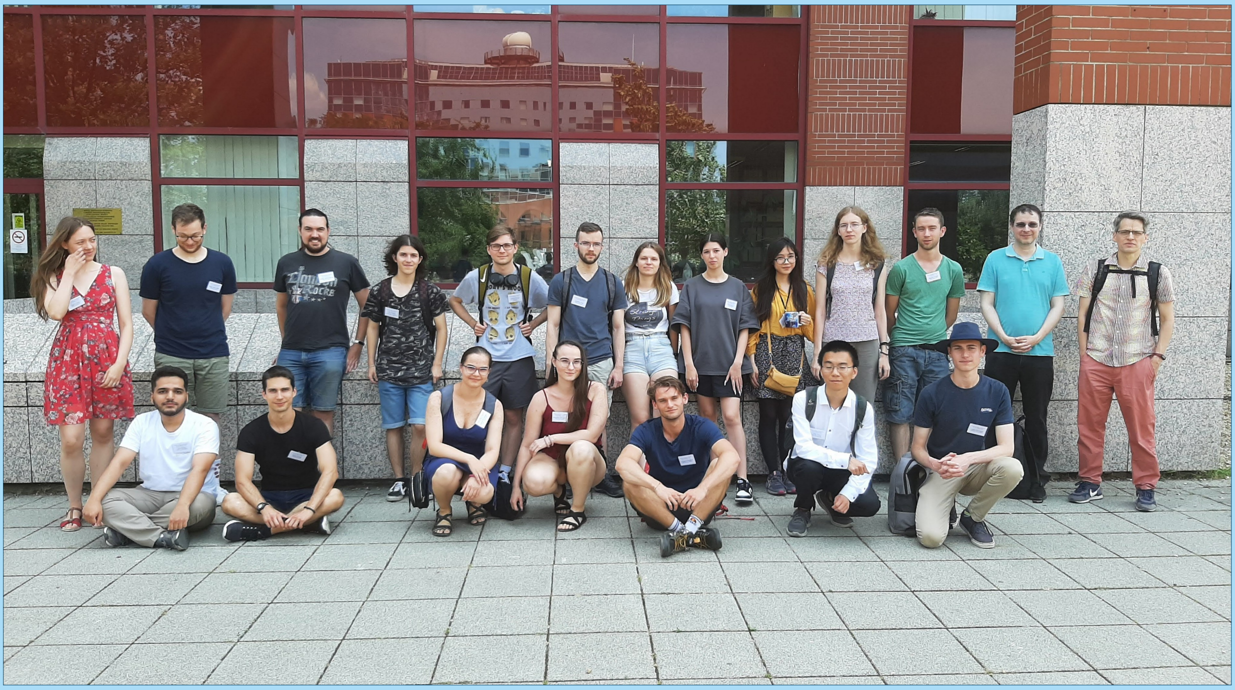

Paradoxical decompositions, fractals and dynamics

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