THe Lefschetz Theory and Hodge Theory of Smooth Projective Varieties

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Lefschetz Theory - Setting

- X: smooth projective variety over $\mathbb{C}$, $\dim X = n$ (over $\mathbb{C}$).
- Pencil: $G \subseteq \left(\mathbb{CP}^N\right)^*$, $\dim G = 1$. $b \in G \leftrightarrow H_b$.
  $\dim\left(\bigcap_{b\in G} H_b\right) = N - 2$. $\quad\rightarrow$ axis
- X^*: $\{\text{hyperlanes tangent to } X\}$. $\quad\rightarrow$ class
- For $b \in G$: we look at $H_k(X_b)$, $X_b \stackrel{\text{def}}{=} H_b \cap X$.
- Blowing up X along the axis: $Y = \{(x,t) \in X \times G \mid x \in H_t\}$.
  $\rightarrow$ smooth projective (irreducible) variety.

The Lefschetz Theorem

Theorem (Lefschetz)

$H_q(X, X_b) = 0$ *for all* $q \leq \dim X - 1$

Decomposing G

- $G \simeq S^2$.
- $G := D^+ \cup D^-$, crit $\subset \operatorname{int} D^+$.
- $Y^+ \stackrel{\text{def}}{=} f^{-1}(D^+)$.
- For $b \in G$, $Y_b \stackrel{\text{def}}{=} f^-1(b)$.

Theorem (Main Lemma)

$H_q(Y_+, Y_b) = 0$ *if* $q \neq \dim X$, *and* $H_n(Y_+, Y_b)$ *is free of rank* $r = \operatorname{class} X$.

Hard Lefschetz I.

- We look at the $n-1$th homology, with choose field coefficients.

Theorem (Hard Lefschetz I.)

$H_{n-1}(X_b) = I \oplus V.$

Hard Lefschetz II.

- $X' \stackrel{\text{def}}{=} X \cap$ axis. $X' \subset X_b \subset X \quad \longrightarrow$

  $$0 = X_{n+1} \subset X_n \subset \ldots \subset X_3 \subset X_2 = X' \subset X_1 = X_b \subset X_0 = X$$

  $\Longrightarrow \dim X_i = n - i$
- $u \in H^2(X)$: Poincaré dual of the fundamental class $[X_b] \in H_{2n-2}(X)$.

Theorem (Hard Lefschetz II.)

For all $q = 1, \ldots n$, we have

$$H_{n+q}(X) \simeq H_{n-q}(X), \; x \mapsto u^q \cap x$$

Hard Lefschetz III.

Theorem (Hard Lefschetz III. - Primitive Decomposition)

$\forall x \in H_{n+q}$ $\exists! x_0, x_1, \ldots$ s.t. $x = x_0 + u \cap x_1 + u^2 \cap x_2 + \ldots$, and $\forall x \in H_{n-q}$ $\exists! x_0, x_1, \ldots$ s.t. $x = u^q \cap x_0 + u^{q-1} \cap x_1 + \ldots$, where the above x_i are all primitive elements, i.e $u^{q+1} \cap x = 0$ (note that $q+1$ is the smallest such index j for which a nonzero x can have the property that $u^j \cap x = 0$).

Hard Lefchetz IV. - sl_2-module

The last form of the Hard Lefschetz theorem considers the Lie-algebra sl_2 of 2×2 matrices with trace 0. sl_2 is 3-dimensional, with the following basis elements:

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

satisfying the Lie-bracket relations:

$$[eh] = -2e, \quad [fh] = 2f, \quad [ef] = h.$$

Theorem (Hard Lefschetz 4 - sl_2-module)

$H_*(X)$ *is an* sl_2*-module.*

Here

$$f : H_j(X) \to H_{j-2}(X), \quad x \mapsto u \cap x$$
$$h : H_j(X) \to H_j(X), \quad x \mapsto (j-n)x.$$

e can be defined using the primitive decomposition.

Lefschetz - Monodromy

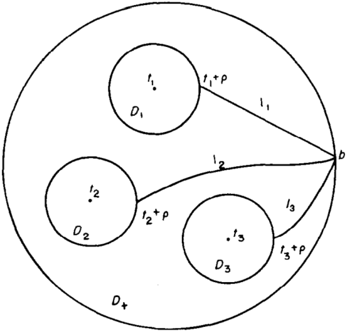
The Setting of the Monodromy

Theorem

1. *$f : Y \to G$ has class X critical values.*
2. *For a generic G, f is Morse.*

- $G^* \stackrel{\text{def}}{=} G \setminus \{$ critical values$\}$.
- $Y^* \stackrel{\text{def}}{=} f^{-}1(G^*)$ is a fibre bundle (locally trivial), with fibres Y_b.
- $\implies \pi_1(G^*, b)$ acts on the homology of Y_b $\to$ monodromy.

D^+



Lefschetz - Monodromy

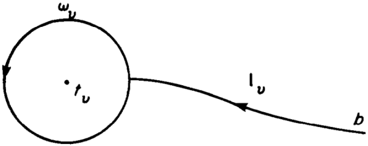
Mondoromy Action

- l_i can be contracted, $\pi_1(G^*, b)$ is generated by $[w_1], [w_2], \ldots [w_r]$.

Theorem (Picard-Lefschetz)

If $q \neq n-1$ then $\pi_1(G^*, b)$ acts trivially on $H_q(Y_b)$. For $q = n-1$, the elementary path w_i acts by $(w_i)_*(x) = x + (-1)^{(n-1)/2}(x, \delta_i)\delta_i$.

w_i



Hodge Theory - Riemannian Manifolds

- Oriented $M \quad \rightarrow dvol$
- Hodge-star operator: $\alpha \wedge *\beta = (\alpha, \beta) dvol$. Here $(.\,,\,.)$ is induced by the metric: $(dx_{i_1} \wedge \ldots \wedge dx_{i_k}, dx_{j_1} \wedge \ldots \wedge dx_{j_k}) = \det(g(dx_{i_u}, dx_{j_v})_{uv})$.
- We get inner products: $\langle \alpha, \beta \rangle = \int_X (\alpha, \beta) dvol = \int_X \alpha \wedge *\beta$.
- $\implies$ we have norms.

Theorem (The Hodge Theorem)

Every de Rham cohomology class has a unique representative that minimalizes the norm. This is called the harmonic representative.

If d^* is the adjoint of d with respect to the above defined inner product, then a form is harmonic iff $d^*\alpha = d\alpha = 0$. By defining the Hodge Laplacian $\Delta = d^*d + dd^*$, we get the further characterization: α is harmonic iff $\Delta\alpha = 0$.

Forms on $\mathbb{C}$ manifolds

- M compact $\implies$ holomorphic forms on M are constant.
- $\mathcal{E}_X^k$ is the sheaf of complex-valued $\mathcal{C}^\infty$ k-forms.
- Ω_X^p is the sheaf of holomorphic p-forms.
- $\mathcal{E}^{(p,0)}$ is the $\mathcal{C}^\infty$-submodule of $\mathcal{E}_X^p$ generated by Ω_X^p. $\mathcal{E}^{(0,p)} = \overline{\mathcal{E}^{(p,0)}}$, and $\mathcal{E}^{(p,q)} = \mathcal{E}^{(p,0)} \wedge \mathcal{E}^{(0,q)}$.

Theorem (Dolbeault's theorem)

For any complex manifold X,

1. $0 \to \Omega_X^p \to \mathcal{E}^{(p,0)} \xrightarrow{\overline{\partial}} \mathcal{E}^{(p,1)} \xrightarrow{\overline{\partial}} \dots$ *is a soft resolution (i.e a resolution in which each element is a soft sheaf).*
2. $$H^q(X, \Omega_X^p) \simeq \frac{\ker\left(\mathcal{E}^{(p,q)} \to \mathcal{E}^{(p,q+1)}\right)}{\operatorname{im}\left(\mathcal{E}^{(p,q-1)} \to \mathcal{E}^{(p,q)}\right)}$$

Kähler Manifolds

- Strong results of Hodge theory for $\mathbb{C}$ manifolds only hold for a subclass of manifolds.
- We call a Riemannian metric on a $\mathbb{C}$-manifold Hermitian, if the multiplication by $\sqrt{-1}$ is orthogonal.
- If $z_i = x_i + \sqrt{-1}y_i$ are local analytic coordinates, for a Hermitian metric H we have that $H = \sum h_{ij} dz_i \otimes \overline{dz_j}$, where (h_{ij}) is a positive definite Hermitian matrix.
- A $\mathbb{C}$-manifold is Kähler, if it admits a Hermitian metric that is locally Euclidean up to second order, i. e. if for any point $p \in X$ there exist analytic local coordinates $z_1, \ldots, z_n$ with $z_i = 0$ at p, such that

$$h_{ij} \equiv \delta_{ij} \mod (x_1, y_1, \ldots, x_n, y_n)^2.$$

Kähler Manifolds II.

- All smooth complex varieties are Kähler.
- By the Kodaira embedding theorem and Chow's theorem we in fact have that a compact complex manifold is a nonsingular projective algebraic variety iff it has a Kähler metric with rational Kähler class.
- We can extend the Hodge star to $\mathcal{E}_X^*$, and define $\overline{*}(\alpha) = \overline{*\alpha}$, to get two Hodge star operators on our complex manifold. We define additional operators on X:
  - $\Delta_\partial = \partial^*\partial + \partial\partial^*$ with bidegree $(0,0)$
  - $\Delta_{\overline{\partial}} = \overline{\partial}^*\overline{\partial} + \overline{\partial}\overline{\partial}^*$ with bidegree $(0,0)$
  - $L = \omega\wedge$ with bidegree $(1,1)$
  - $\Lambda = - * L *$, with bidegree $(-1,-1)$.

Hodge Decomposition

Theorem

Suppose that X is a compact Kähler manifold.

- $H^q(X, \Omega_X^p)$ *is isomorphic to the space of harmonic* (p, q)*-forms.*
- *As a corollary,* $H^p(X, \Omega_X^q) \simeq H^{n-p}(X, \Omega_X^{n-q})$

Theorem (Hodge decomposition)

If X is a compact Kähler manifold, then

- *A form a is harmonic iff its* (p, q)*-components are.*
- $H^i(X, \mathbb{C}) \simeq \bigoplus_{p+q=i} H^q(X, \Omega_X^p)$
- *Complex conjugation induces an* $\mathbb{R}$*-linear isomorphisms between* (p, q) *and* (q, p) *forms. Therefore* $H^q(X, \Omega_X^p) \simeq H^p(X, \Omega_X^q)$.

Hard Lefschetz with Hodge Theory

We have defined L, Λ. H: multiplication by $(n-i)$.

Theorem

1. $[\Lambda, L] = H$
2. $[H, L] = -2L$
3. $[H, \Lambda] = 2\Lambda$.