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Directed Studies 1

Numerical solution of a nonlinear plate equation

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PURPOSE OF PROJECT

- ➔ Understand the weak form of the nonlinear plate equation (a 4th order PDE) and prove that it has a unique weak solution.

- ➔ Theory for the numerical solution: construction and proof of convergence. It consists of finite element discretization, Newton linearization and conjugate gradient method.

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Elasto-plastic bending of clamped plates

Weak formulation :

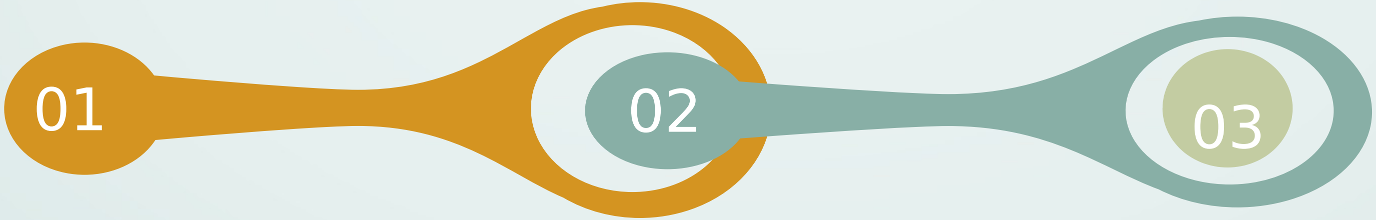
Prove existence and uniqueness

Theory for the numerical solution ;

1.Apply a finite element discretization

2. Apply a Newton iteration (so-called "outer iteration")

3.Apply the PCG method("inner iteration"),



The PDE :

Elasto-plastic bending of a clamped thin plane plate Ω is described by a fourth order nonlinear Dirichlet boundary value problem .

The formulation of the problem is the following :

$$
\begin{cases}
\frac{\partial^2}{\partial x^2}\left(\bar{g}(E(D^2u))\left(\frac{\partial^2 u}{\partial x^2}+\frac{1}{2}\frac{\partial^2 u}{\partial y^2}\right)\right)+\frac{\partial^2}{\partial x\partial y}\left(\bar{g}(E(D^2u))\left(\frac{\partial^2 u}{\partial x\partial y}\right)\right) \\
\qquad +\frac{\partial^2}{\partial y^2}\left(\bar{g}(E(D^2u))\left(\frac{\partial^2 u}{\partial y^2}+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}\right)\right)=\alpha(x) \\
u_{|\partial\Omega}=\frac{\partial u}{\partial \nu}\Big|_{\partial\Omega}=0
\end{cases}
$$

This problem is written briefly as

where the scalar function satisfies the condition

with suitable constants independent of the variable .

where

The weak formulation of

The weak formulation of the problem : find such that

For regular functions , the weak formulation is obtained via multiplying our problem by , integration and the divergence theorem. In this way we have

Instead of (1) .The latter can be obtained form here by defining

And , which yields that

Prove the existence and uniqueness of the weak solution :

Theorem 1:

Let H be a real Hilbert space and let the operator have the following properties:

(i) F has a bihemicontinuous symmetric Gateaux derivative

(ii) there exists a constant such that

Then for any the equation has a unique solution

Prove the existence and uniqueness of the weak solution :

Remark 1:

Let F have the form :

where the scalar function satisfies the condition

Then (2) defines an operator which has a bihemicontinuous symmetric Gateaux derivative satisfying

Proposition 1 :The elasto-plastic bending problem of a clamped plate has a unique weak solution

Proof :

For any matrices let us introduce the following notations:

,

we verify directly via Remark 1 for the operator

where

Using the weak form and previously notations , we have

=

=

The obtained form of F is a special case of (2) with

hence Theorem 1 can be applied.

Finite element discretization:

Galerkin's method for nonlinear operator equations :

Let be a real Hilbert space and a given operator, which is uniformly monotone and Lipschitz continuous.
Consider the operator equation (1)
where . The equation (1) admits a unique solution .
Let us write the equation (1) in its equivalent variational forms involving test functions :

Let The approximate solution is defined by the subspace equation
(2)

The equation (2) admits a unique solution .
The coefficients of the expansion can be obtained as follows. We set

Let us introduce the real functions (
and let , Now put these functions together in so the coefficients of can be obtained by solving the nonlinear algebraic system of equations
(c) =

("Nonlinear Céa's lemma") :

For the Galerkin solution ,the quasi-optimality relation

}

holds true

Example:

Consider the problem :

where and due to the assumptions on there exists a unique weak solution , that is,

Let be some finite element subspace ,Then the approximate solution satisfies the subspace equation :

and the coefficients can be obtained by solving the nonlinear system of algebraic equations :

$$\mathcal{A}(c) = \mathcal{B}$$

where and

Here, $\mathcal{A}$ inherits the uniform monotonicity and Lipschitz continuity of f, so unique solvability of this system follows from the theorem on the Galerkin method. Further, the nonlinear Céa’s lemma holds true

Newton linearization :

Theorem 1:

Let X and Y be Banach spaces and let be Fréchet differentiable, then assume that ,

and is Lipschitz continuous with constant L. If ,then the iteration

satisfies

(1)

(2) If is such that

then

m

Theorem 2: *Assume conditions of Theorem1 and let be the solution of the equation let be arbitrary*

Then :

Decreases monotonically and locally qudratically i.e after a suitable index

and

where

Example :

Consider the finite element solution :

and define the operator as

The Newton iteration requires more differentiability, so we need more conditions on f.

Assumption (N) :

  and the Jacobians are symmetric and

there are such that :

Further,assume that   is Lipschitz continuous

It can be proved that the operator introduced in (2) inherits the properties of : is Gateaux differentiable and

from which it follows that

Using the assumption ,we also have that is Lipschitz continuous with a suitable constant .Therefore,Theorem 2 holds true for our problem

Inner-outer iterations

1.Assumptions:

Let Assumptions (N) hold. Let be a finite dimensional subspace with the inner product and let

and

Denote by the solution of

The operator F is Gâteaux differentiable and its derivative is given by

The operator F' inherits the Lipschitz continuity of , Let L denote the Lipschitz constant of F'.

2.Construction:

Let and define the sequence as follows :

(a) The outer iteration defines the sequence

(2.1)

where is the numerical solution of :

Further,

is constant satisfying

(2.3)

(b) To determine   in (2.2) , the inner iteration defines a sequence

using a preconditioned conjugate gradient method .Here we have :

Let

Then we consider the preconditioned form of (2.2) :

Finally,for which

with and

Theorem 3:

Let Assumptions be satisfied. Then construction yields the following convergence results :

(1) The outer iteration () satisfies

monotonically

with speed depending on the sequence () up to locally quadratic order. Namely, if , then the convergence is linear.

Further, if

with some constant , then the convergence is locally of order

yielding also the convergence estimate of weak order

with suitable constants

(2) there holds

cond(

And, accordingly, the inner iteration satisfies

Therefore, the number of inner iterations for the nth outer step is at most

with

Let be a real function that is in the variable , and there exists such that

If these conditions hold for “ ", then the conditions of Theorem 3 are satisfied for the plate problem

Hence the inner-outer method works for our model problem.

THANK YOU