Pricing Portfolios Of Financial Products

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Abstract

This report is written to fulfill the requirements for the subject "Directed Studies 1". During this semester I concentrated on basic option pricing of European and American options, but the goal and the next step of the general project work in the next 2 semesters will be pricing baskets which can contain different options and bonds.

1 Introduction

The objective of this report is to establish the foundations of the theory of pricing financial products on a basic conceptual level, as well as presenting two derivative pricing-models with their implementation on Python.

We can classify financial market instruments into two different classes; the underlying stocks: shares, bonds, foreign currencies and their respective derivatives. Derivatives are contracts that guarantee some payment or delivery in the future contingent on the behavior of the underlying asset, they can be used to reduce risk by allowing a party to fix a price for a future transaction now.

The basic intuition that underlies valuation is the absence of arbitrage. An arbitrage opportunity is an investment strategy that exploits different prices for the same asset and generates riskless profit by simultaneously entering into transactions in these markets. Such a disparity between the prices cannot persist for a long time.

Throughout this report, we are only going to consider options, which is a contract that gives to its owner, the right, but not the obligation, to buy or sell a specific quantity of an underlying asset at a specified strike price. There are two types of option;

- **Call** options: they give their holder the right to *buy* the underlying asset by a certain expiry date T for a certain price.
- **Put** options: they give their holder the right to *sell* the underlying asset by a certain date for a certain price K, which is referred to as the strike price.

The payoff for the options is $(\gamma(S_T - K))^+$, where $\gamma = 1$ for call options and $\gamma = -1$ for put options. *American* options can be exercised at any time up to the expiration date in contrast to *European* options that can be exercised only on the expiration date itself. The main goal of pricing theory is to determine the price of a derivative at an arbitrary time $t < T$.
Under the assumptions of the Black-Scholes model, European options have a closed price formula, while there does not exist a closed form for pricing an American option. The two most popular pricing methods for American options are the binomial pricing method and the LSM method, which we will discuss in this report.

2 Black-Scholes Model

One of the essential assumptions of the Black-Scholes model is that the underlying asset, most commonly a stock, is modelled as a geometric Brownian motion, and the LSM method is based on the same modelling framework.

A stochastic process $\{S_t\}_{t \in \mathbb{R}^+}$ is said to follow a Geometric Brownian motion if it satisfies the following stochastic differential equation (SDE):

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

where W_t is a Wiener process, $\mu \in \mathbb{R}$ is the drift and $\sigma \in \mathbb{R}$ is the volatility.

It can be seen that given any initial value S_0, the SDE has the following solution:

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right).$$

Black Scholes Formula

Within the Black-Scholes framework, the option prices have an explicit formula which is the following for call options:

$$V(s,T) = s\Phi\left(\frac{log\frac{s}{k} + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)$$

$$-ke^{-rT}\Phi\left(\frac{log\frac{s}{k} + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)$$

While that of a Put option can be priced similarly

3 Binomial options pricing model

The simplest technique for pricing an option involves constructing a binomial tree. A binomial tree is a diagram representing different possible paths that might be followed by the stock price over the life of an option. The underlying assumption is that the stock price follows a random walk. In each time step δt, it has a certain probability p_u of moving up by a certain percentage amount u and a certain probability $p_d = 1 - p_u$ of moving down by a certain percentage d. The condition $u < 1 + R < d$ has to be satisfied for the model to be arbitrage-free, where R is the risk free rate. In the limit, as the time step $\delta t \to 0$ becomes smaller, this model approaches the Black–Scholes model.

Let us consider the one-step binomial model. The values of options at maturity is just their payoff, and to determine their values at time zero, we consider the general portfolio (ϕ, ψ), namely ϕ of the stock S (worth ϕS_0) and ψ of the cash bond B (worth ψB_0). If the portfolio were to be built at time zero it would cost $\phi S_0 + \psi B_0$, one tick later it would be worth one of these possible values:

- $\phi S_u + \psi B_0 exp(r\delta t)$ `after an up move`
- $\phi S_d + \psi B_0 exp(r\delta t)$ `after a down move`

We set up the portfolio such that there is no uncertainty in its value. We then argue that, because the portfolio has no risk, the return it earns must equal the risk-free interest rate. This enables us to work out the cost of setting up the portfolio and therefore the option's price, thus the problem reduces to solving these two equations.

$$\phi S_u + \psi B_0 exp(r\delta t) = f(u)$$

$$\phi S_d + \psi B_0 exp(r\delta t) = f(d)$$

where f is our claim on the stock S.

If we bought this (ϕ, ψ) portfolio and held it, the equations guarantee that we achieve our goal and we have synthesized the derivative. When we rewrite the formula for the value V of the (ϕ, ψ) portfolio, we get:

$$V = exp(-r\delta t)\left((1-q)f(d) + qf(u)\right)$$

$$\text{where} \quad q = \frac{exp(r\delta t) - d}{u - d}$$

This is the discounted expected value of the claim under the risk-neutral measure.

We can generalize the case of the one-step model to an arbitrary n-step model. During each time step, the stock either moves up to u times its initial value or moves down to d times its initial value. After calculating the value of the option at the final layer of the tree, we traverse backwards through the layers of the tree, calculating the value of the claim at each node similar to what we did in the one-step model, until we reach the root of the tree.

American options on the other hand have more optionality than just choosing between two alternatives at the maturity date. The buyer of the option then has to make decisions from moment to moment to decide when and if to call the option. The buyer of an American call has the choice when to stop, and that choice can only use price information up to the present moment. The payoff of the option at that (stopping) time $\tau$is:

$$(S_\tau - K)^+$$

and the value of the option is (maximised over all possible stopping strategies):

$$\sup_\tau \mathbb{E}_\mathbb{Q}\left(e^{-r\tau}(S_\tau - K)^+\right)$$

We now move on to consider how American options can be valued using a binomial tree such as that in. The procedure is to work back through the tree from the end to the beginning, testing at each node to see whether early exercise is optimal. The value of the option at the final nodes is the same as for the European option. At earlier nodes the value of the option is the greater of the payoff from early exercise and the discounted expected value (under the risk free rate) of future payoffs.

4 Longstaff-Schwartz Method (LSM)

We present here the LSM algorithm for the special case of pricing American options. This is done by approximating the optimal path-wise stopping rule that maximizes the value of the American option.

The basic idea of the Longstaff-Schwartz algorithm, is to use least-squares regression on a finite set of functions to approximate conditional expectation values.

First of all, the time axis has to be discretized —i.e., if the American option is alive within the time horizon $[0, T]$, early exercise is only allowed at discrete times $0 < t_1 < t_2 < \cdots < t_N = T$.

For a given time t_j, decision to perform early exercise is carried out if the payoff from immediate exercise exceeds the continuation value —i.e. the value of the option in case it was not exercised at t_j. This continuation value can be expressed as conditional expectation of the option payoff with respect to the risk-neutral pricing measure $\mathcal{Q}$. We start the algorithm by generating multiple paths representing the behaviour of the stock over the time horizon.

- The first step of the actual algorithm is to determine the cashflow vector C_{t_N} at the last time step T_N. These cashflows are the payoff of the option in the terminal value of each simulation path i.
$$C_{i,t_N} = (S_{i,t_N} - K)^+$$
- Second, we consider the spot prices at time-step t_{N-1}, and estimate the exercise value, selecting only the in-the-money paths, i.e:
$$\left(S_{i,t_{N-1}} - K\right)^+ > 0$$
- In order to obtain the continuation values, LSM regress the discounted future cash-flows onto a finite set of basis functions of our values for the spot price. The regression is done by using the values from all of the paths. In my implementation, the relationship between the variables is modelled as a second degree polynomial. If S is the spot price, a_j are coefficients and B_j is the set of basis functions, then the continuation value for path i with values S_{i,t_n} at time t_n is:
$$Cont_{i,t_{n-1}} = \sum_j a_j(t_n)B_j(S)$$
- Once both continuation and exercise values are ready, early exercise is performed if:
$$C_{i,t_{n-1}} > Cont_{i,t_{n-1}}$$
- Once we finish our backward iteration process and reach the initial point, we can build a cashflow or value matrix, and the option value is given by the arithmetic average of the values.

I would like to note that I implemented both of these methods via Python. .

References

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