Superlinear convergence of the conjugate gradient method for elliptic partial differential equations with unbounded reaction coefficient

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Outline

The problem

Let $N \geq 2$, $p > 2$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain. We consider the elliptic problem

$$\begin{cases} -\mathrm{div}(G\nabla u) + \eta u = g, \\ u_{\partial\Omega} = 0, \end{cases} \tag{1.1}$$

where $\eta = \eta(x)$ is a general variable (i.e. nonconstant) coefficient.

Objectives

- Study the mesh-independent superlinear convergence of the preconditioned conjugate gradient method (CGM) applied to the discretization of (1.1).
- Find an eigenvalue-based estimation of the rate of superlinear convergence.
- Extend the results done in [6] from $\eta \in \mathrm{C}(\overline{\Omega})$ to $\eta \in \mathrm{L}^q(\Omega)$.

Assumptions

(i) The symmetric matrix-valued function $G \in \mathrm{C}^1(\overline{\Omega}, \mathbb{R}^N \times \mathbb{R}^N)$ satisfies

$$G(x)\xi \cdot \xi \geq m|\xi|^2$$

for all $\xi \in \mathbb{R}^N$, with some $m > 0$ independent of ξ.

(ii) There exists $2 < p < \frac{2N}{N-2}$ such that $\eta \in \mathrm{L}^{p/(p-2)}(\Omega)$.

(iii) $\partial\Omega$ is piecewise C^2 and Ω is locally convex at the corners.

(iv) $g \in \mathrm{L}^2(\Omega)$.

Then problem (1.1) has a unique weak solution in $\mathrm{H}^1_0(\Omega)$.

Construction of the discretization

Let $V_h \subset H_0^1(\Omega)$ be a given FEM subspace. We look for the numerical solution u_h of (1.1) in V_h:

$$\int_\Omega (G\nabla u_h \cdot \nabla v + \eta u_h v) = \int_\Omega gv, \quad v \in V_h. \tag{1.2}$$

The corresponding linear algebraic system has the form

$$(\mathbf{G}_h + \mathbf{D}_h)\mathbf{c} = \mathbf{g}_h.$$

We apply the matrix $\mathbf{G}_h$ as preconditioner,

$$(\mathbf{I}_h + \mathbf{G}_h^{-1}\mathbf{D}_h)\mathbf{c} = \tilde{\mathbf{g}}_h \tag{1.3}$$

with $\tilde{\mathbf{g}}_h = \mathbf{G}_h^{-1}\mathbf{g}_h$. Now, we apply the CGM for the system (1.3) and study the error vectors $e_k = c - c_k$.

Main result

Theorem 1

Let $2 < p < \frac{2N}{N-2}$. *Then there exists* $\boldsymbol{C} > 0$ *such that for all* $k \in \mathbb{N}$

$$\left(\frac{\|e_k\|_A}{\|e_0\|_A}\right)^{\frac{1}{k}} \leq \frac{\boldsymbol{C}}{k^{\frac{1}{s}}} \to 0, \ as\ k \to \infty \tag{2.4}$$

where $\alpha = \frac{1}{N} - \frac{1}{2} + \frac{1}{p}$ *and* $s > \frac{1}{\alpha}$.

Sketch of the proof

Let $D = \mathrm{H}_0^1(\Omega) \cap \mathrm{H}^2(\Omega)$. We define the operators

$$Su \equiv -\mathrm{div}(G\nabla u), \quad u \in D \quad \text{and} \quad Qu \equiv \eta u, \quad u \in \mathrm{H}_0^1(\Omega)$$

and since $p < 2^* = \frac{2N}{N-2}$, the embedding $\mathcal{I} : \mathrm{H}_0^1(\Omega) \to \mathrm{L}^p(\Omega)$ is compact and by assumption (iii) the symmetric operator S maps onto $\mathrm{L}^2(\Omega)$.

We define the energy space H_S as the completion of D under the *energy inner product*

$$\langle u, v\rangle_S = \langle Su, v\rangle = \int_\Omega G\nabla u \cdot \nabla v,$$

and the corresponding norm is denoted by $\|\cdot\|_S$. Then $H_S = \mathrm{H}_0^1(\Omega)$.

Lemma 1

The operator $Q_S = S^{-1}Q$ is well defined and there exists $C > 0$ such that

$$\|Q_S v\|_{H_S} \le C\|v\|_{\mathrm{L}^p(\Omega)}, \quad v \in H_S. \tag{2.5}$$

Hence Q_S is compact and self-adjoint in H_S.

Let us consider (1.3):

$$(\mathbf{I}_h + \mathbf{G}_h^{-1}\mathbf{D}_h)\mathbf{c} = \tilde{\mathbf{g}}_h$$

Let $\mathbf{A} = (\mathbf{I}_h + \mathbf{G}_h^{-1}\mathbf{D}_h)$. It is known [2] that

$$\left(\frac{\|e_k\|_A}{\|e_0\|_A}\right)^{1/k} \leq \frac{2\|\mathbf{A}^{-1}\|}{k}\sum_{j=1}^{k}|\lambda_j(\mathbf{G}_h^{-1}\mathbf{D}_h)|, \quad k = 1, 2, \ldots, n. \quad (2.6)$$

We want to find a mesh independent bound for (2.6).

We have the following result

Proposition 1

For any $k = 1, 2, \ldots, n$

$$\sum_{j=1}^{k} |\lambda_j(\mathbf{G}_h^{-1}\mathbf{D}_h)| \leq \sum_{j=1}^{k} \lambda_j(S^{-1}Q). \tag{2.7}$$

Moreover,

$$\left(\frac{\|e_k\|_A}{\|e_0\|_A}\right)^{1/k} \leq \frac{2\|A^{-1}\|}{k} \sum_{j=1}^{k} \lambda_j(S^{-1}Q). \tag{2.8}$$

Now we wish to get a rate estimation from (2.8).

- Let $\lambda_n = \lambda_n(S^{-1}Q)$. Since $Q_S = S^{-1}Q$ is a compact self-adjoint operator in H_S, by [5, Ch.6, Th.1.5] we have the following characterization of the eigenvalues of Q_S:

  $$\lambda_n(Q_S) = \min\{\|Q_S - L_{n-1}\| \,/\, L_{n-1} \in \mathcal{L}(H_S), \operatorname{rank}(L_{n-1}) \leq n-1\},$$

  for all $n \in \mathbb{N}$.
- Let $a_n(\mathcal{I})$ denote the approximation numbers of the compact embedding $\mathcal{I}\colon \mathrm{H}_0^1(\Omega) \mapsto \mathrm{L}^p(\Omega)$, defined as

  $$a_n(\mathcal{I}) = \min\{\|\mathcal{I} - L_{n-1}\| / \, L_{n-1} \in \mathcal{L}(\mathrm{H}_0^1(\Omega), \mathrm{L}^p(\Omega)), \operatorname{rank}(L_{n-1}) \leq n-1\}.$$

As a consequence of Lemma 1, we obtain the following result

Lemma 2

For all $n \in \mathbb{N}$,

$$\lambda_n(Q_S) \leq \frac{C}{\sqrt{m}} a_n(\mathcal{I}). \tag{2.9}$$

Furthermore, from [4] we have the estimation

$$a_n(\mathcal{I}) \leq \hat{C} n^{-\alpha}, \quad \alpha = \frac{1}{N} - \frac{1}{2} + \frac{1}{p},$$

for some constant $\hat{C} > 0$. Hence

$$\frac{1}{k} \sum_{n=1}^{k} \lambda_n(Q_S) \leq \frac{C\hat{C}}{\sqrt{m}} \left(\sum_{n=1}^{\infty} \frac{1}{n^{s\alpha}} \right)^{\frac{1}{s}} \frac{1}{k^{\frac{1}{s}}} = \frac{\boldsymbol{C}}{k^{\frac{1}{s}}},$$

for any $s\alpha > 1$. Then, by (2.8), the theorem is proved.

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