Helly-type theorems and boxes
Directed studies 2

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1 Introduction

Helly-type theorems state that if a property A holds for any **subfamily** of a family of sets $\mathcal{F}$ that is of a **given finite size** h and property, then some property B holds for the whole **family** $\mathcal{F}$ of **arbitrary finite size** n. An equivalent and often useful formulation provided by negations is that if $\mathcal{F}$ doesn't have property B, then some subfamily of size h doesn't have property A. The minimal number h for which a given Helly-type statement holds will be referred to as the **Helly-number**.
The structure of this paper is the following. Section 2 gives an overview of some of the most notable Helly-type theorems that are currently known and provides a more detailed description of a particular type of Helly theorem, namely box-piercing theorems. Section 3 offers some results in an attempt to expand this particular direction along the lines of other notable Helly-type theorems. Finally, Section 4 presents the proofs of the results.

2 Helly-type theorems

Helly's original statement is about the **emptyness** of the **intersection** of a family of **convex sets in Euclidean space**.
Theorem (Helly). *For a finite family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$ if any $(d+1)$-tuple of sets in $\mathcal{F}$ has a non-empty intersection, then all sets in $\mathcal{F}$ have a non-empty intersection.*
Note that here property A and B are the same. This theorem is equivalent to **Radon's theorem** about the convex hull of points in $\mathbb{R}^d$
Lovász and later **Bárány** introduced a property on the **subfamilies**, namely that they be **systems of distinct representatives** of a given substructure that gives a stronger result, the so called **Colorful Helly Theorem**.
Theorem (Colorful Helly Theorem, Lovász, Bárány). *For finite families $\mathcal{F}_1, ..., \mathcal{F}_{d+1}$ of convex sets in $\mathbb{R}^d$ if any colorful selection $C_1 \in \mathcal{F}_1, ..., C_{d+1} \in \mathcal{F}_{d+1}$ has a non-empty intersection, then there is a family $\mathcal{F}_i$ such that all sets in $\mathcal{F}_i$ have a non-empty intersection.*
The **original Helly** theorem is the **subcase** of this statement when all families are the same. The statement follows from Helly's theorem by considering a **lexicographic ordering** on the points of $\mathbb{R}^d$.
More recently **Kalai** and **Meshullum** proved an **extended version** of this theorem which states that not only is there an intersecting family, but it can also be extended by a colorful selection from the other families while still intersecting.
Bárány, **Katschalski** and **Pach** showed a Helly-type theorem about a stronger property B on the family of convex sets. Their **Quantitative Volume Theorem** provides a condition not only for the emptyness of the intersection, but also gives a **lower bound** for the **volume of intersection** of sets.
Theorem 1 (Quantitative Volume Theorem, Bárány, Katschalski, Pach) *For a finite family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$ if any $2d$-tuple has an intersection of volume at least 1, then all sets in $\mathcal{F}$ have an intersection of volume at least $c_d = d^{-2d^2}$.*
Note that here property B is weaker than A although both are lower bounds on the volume of the intersection. This is sometimes the case with quantitative volume theorems. Note also that the Helly number is larger than in the original Helly theorem. The constant c_d was later reduced to d^{-2d} by others.

2.1 Piercing boxes

Another possible variant of Helly's theorem **generalizes** the notion of **intersection** with the notion of **piercing**.

Definition: A set P **pierces** a family of sets $\mathcal{F}$ if for any set $S \in \mathcal{F}$ there is an element $p \in P$ such that $p \in S$. If $|P| = n$, then $\mathcal{F}$ is n**-pierceable**.
Note that if an intersection of sets is non-emty if and only if it is 1-pierceable.
All previously discussed statements were about families of convex sets. However, there are **no Helly-type theorems** about n-piercing **for all families** of **convex sets** if both property A and B is n-piercability for $n > 1$. For example **Chakraborty** et al. showed that for any constant $h > 0$ there exists a family of **circles** in the plane such that any subfamily of size h is 2-pierceable but the whole family is **not 2-pierceable**.
Danzer and **Grünbaum** showed the **Helly-number** for all possible Helly-type theorems for n**-piercing** families of **axis-parallel boxes in Euclidean space** where both property A and B are n-piercing.
Theorem 2 (Danzer, Grünbaum). *If $h = h(d, n)$ is the smallest positive integer such that for any finite family $\mathcal{F}$ of axis-parallel boxes in $\mathbb{R}^d$ every h-tuple from $\mathcal{F}$ is n-pierceable implies that $\mathcal{F}$ is n-pierceable then following are the values of h:*

$$\begin{aligned}
&h(d,1) = 2 \\
&h(1,n) = n+1 \\
&h(d,2) = \begin{cases} 3d & : \quad 2 \mid d \\ 3d-1 : & \quad 2 \nmid d \end{cases} \\
&h(2,3) = 16 \\
&h(d,n) = \aleph_0 \quad n \geq 3, (d,n) \neq (2,3)
\end{aligned}$$

Chakraborty, **Ghosh** and **Nandi** combined previous statements and showed an **extended colorful Helly-type theorem for n-piercing intervals and 2-piercing axis-parallel boxes**.
Note that the cases $n = 3, d = 2$ and $n \geq 3, d \geq 3$ are not yet known.
The simple colorful version of this theorem is a trivial consequence of the extended version. Furthermore, the proof of the extended version is not an essential part of the proof as it only follows by adding a last step to the proof after already showing the colorful version.

3 Results

This section presents an attempt at introducing a same kind of variant for **Theorem 2** as **Theorem 1** is for the original Helly-theorem. Thus, it introduces frameworks which allows for statements about volume that generalize box-piercing. This is achieved by the notion of punching holes into boxes.

3.1 Punching holes into boxes

Definition: For volume set $\mathcal{V} \subset \mathbb{R}_{>0}$ and enumeration $\nu : \mathcal{V} \to \mathbb{Z}_{>0}$ a family a of d-dimensional boxes $\mathcal{F} = \{\prod_{j=1}^{d}[a_{ij}, b_{ij}] : i \in \mathcal{I}\}$ for some index set $\mathcal{I}$ is $\mathcal{V}, \nu$**-punchable** if there is a family of d-dimensional boxes $\mathcal{H}$ such that

$$\sum_{v \in \mathcal{V}} v = |\mathcal{H}| \tag{1}$$

$$\forall v \in \mathcal{V} \quad \nu(v) = |\{H \in \mathcal{H} : \mathrm{Vol}(H) = v\}| \tag{2}$$

$$\forall B \in \mathcal{F} \quad \exists H \in \mathcal{H} \quad H \subset B \tag{3}$$

If (3) holds for some families of boxes $\mathcal{F}, \mathcal{H}$ then $\mathcal{H}$ **punches** $\mathcal{F}$. If the volume set has 1 element $\mathcal{V} = \{v\}$ and $\nu(v) = n$ and there is a family $\mathcal{H}$ for which (1),(2),(3) hold, then $\mathcal{F}$ is n**-punchable**.

3.2 Statements

Proposition 1: *For a family of intervals $\mathcal{F} = \{I_i = [a_i, b_i] \subset \mathbb{R} : i \in \mathcal{I}\}$ if any subfamily of $n+1$-elements is n-punchable, then $\mathcal{F}$ is n-punchable.*
Proposition 1.1: *If any translates of a set of d-dimensional boxes $\mathcal{H} = \{A, B\}$ punches any subfamily of $3d$ elements of the family $\mathcal{F}$ then $\mathcal{H}$ punches $\mathcal{F}$.*
Statement 1: *In a Helly-type theorem about 2-punching boxes, the Helly-number has to be at least $3d$ for $2 \mid d$ and $3d-1$ for $2 \nmid d$.*
Proposition 2: *For any dimension d there is a family $\mathcal{F}$ of d-dimensional boxes such that any $(4d-2)$-tuple is 2-punchable, but F is only $\{\varepsilon\}, 2$-punchable for any $\varepsilon > 0$.*

Corollary 2.1: *In a Helly-type theorem about 2-punching boxes, the Helly-number has to be at least* $4d-1$.
Statement 2: *If any* $2d$ *element subfamily of a family of d-dimensional boxes is* 1*-punchable, then* $\mathcal{F}$ *is* 1*-punchable.*
Conjecture: *For a family of d-dimensional boxes* $\mathcal{F} = \{\prod_{i=1}^{d}[a_i, b_i] \subset \mathbb{R}^d : i \in \mathcal{I}\}$ *if any subfamily of* $4d$*-elements is* 2*-punchable, then* $\mathcal{F}$ *is* 2*-punchable.*

4 Proofs

Proof of Proposition 1:
Observation: *If* $A, B \subset \mathbb{R}^d$ *are convex sets, then their Minkowski-difference* $A - B$ *is also a convex set.*
Observation: *If* $A, B, C \subset \mathbb{R}^d$ *then* $C + t \subset A \cap B$ *for some* $t \in \mathbb{R}^d$ *if and only if* $A - C \cap B - C \neq \emptyset$ *where* $S - T$ *denotes the Minkowski-difference.*
Proof: $A - C$ equals the set of vectors v such that $C + v \subset A$.
Observation: *All intervals of volume 1 are translates of each other.*
Let $I = [0, 1]$, then any tuple $I_1, ..., I_{n+1}$ is n-punchable if and only if $I_i - I, ..., I_{n+1} - I$ is n-pierceable. The proposition follows thus from the theorem of Danzer and Grünbaum about n-piercing intervals (**Theorem 1**). ■
Proposition 1.1 also follows from **Theorem 1** by the same argument, considering the part about 2-piercing boxes.
Statement 1 also immediately follows from **Theorem 1** by noting that if a set of boxes is 2-punchable, then it is also 2-pierceable.

Proof of Proposition 2:
The following families of boxes of size $4d$ have the given property.
For d dimensions let $B'_{ij} = \prod_{k=1}^{d} I_k$ for $1 \leq i \leq d$, $1 \leq j \leq 4$, where $I_k = [-2, 2] = I$ for $k \neq i$ and

$$I_i = \begin{cases} [-2 + \varepsilon/2, -1 + \varepsilon/2] : j = 1 \\ [-1 - \varepsilon/2, -\varepsilon/2] : j = 2 \\ [\varepsilon/2, 1 + \varepsilon/2] : j = 3 \\ [1 - \varepsilon/2, 2 - \varepsilon/2] : j = 4 \end{cases}$$

Then for all i and j let $B_{ij} = cB'_{ij}$ where $c = \varepsilon^{-\frac{d-1}{d}}$. Finally let the family of boxes be $\mathcal{F} = \{cB_{ij} : 1 \leq i \leq d, 1 \leq j \leq 4\}$.
We will refer to the index i for box B_{ij} as the box's narrow dimension and call boxes $B_{ij}, j \in \{1, 2, 3, 4\}$ the i-narrow boxes.
Observation: For every dimension i there are 4 i-narrow boxes in $\mathcal{F}$: two pairs, B_{i1},B_{i2} and B_{i3}, B_{i4} which are intersecting and have no intersections between the different pairs.
Observation: Consider the box A with vertices $V(A) = \{\underline{v} \in \{-c, c\}^d\}$. In the family of axis-parallel hypercubes $\mathcal{A} = \{B_{\underline{v}} : \underline{v} \in V(A)\}$ with edge-lenghts $c\varepsilon$ having the vertices of A at their center, for any diagonally opposite pair of vertices $\underline{v}, -\underline{v}$ the pair of boxes $B_{\underline{v}}, B_{-\underline{v}}$ $\{\varepsilon\}$, 2-punches the family $\mathcal{F}$. These punching pairs have maximal volume and there are no other punching pairs of maximal volume.
Observation: For any subfamily $\mathcal{F}' \subset \mathcal{F}$ there is a pair of hypercubes in $\mathcal{A}$ that punches $\mathcal{F}'$.
Claim: *Any subfamily* $\mathcal{F}' \subset \mathcal{F}$ *of size* $4d-2$ *is 2-punchable.*
Proof: Since $|\mathcal{F}| = 4d$ and $|\mathcal{F}'| = 4d - 2$ there is either *(A)* an index $1 \geq i \geq d$ for which there are only 2 i-narrow boxes in $\mathcal{F}'$ or *(B)* there are two indices - $1 \geq i_1, i_2 \geq d, i_1 \neq i_2$ - for which there is an i_1- and i_2-narrow box missing from $\mathcal{F}'$.
In case *(A)* let $B_{\underline{v}}$ and $B_{\underline{w}}$ be a pair of boxes punching the subfamily $\mathcal{F}'$ and let i be the dimension missing boxes belonging to it, while $B_{ij_1} \supset B_{\underline{v}}$ and $B_{ij_2} \supset B_{\underline{w}}$ are the boxes in $\mathcal{F}'$ with narrow dimension i. If $B_{\underline{v}} = \prod_{k=1}^{d}[v_k - c\varepsilon/2, v_k + c\varepsilon/2]$ and $B_{\underline{w}} = \prod_{k=1}^{d}[w_k - c\varepsilon/2, w_k + c\varepsilon/2]$, then boxes $B_1 = \left(\prod_{k=1}^{i-1}[v_k - c\varepsilon/2, v_k + c\varepsilon/2]\right) \times \pi_i(B_{ij_1}) \times \left(\prod_{k=i+1}^{d}[v_k - c\varepsilon/2, v_k + c\varepsilon/2]\right)$ and $B_2 = \left(\prod_{k=1}^{i-1}[w_k - c\varepsilon/2, w_k + c\varepsilon/2]\right) \times \pi_i(B_{ij_2}) \times \left(\prod_{k=i+1}^{d}[w_k - c\varepsilon/2, w_k + c\varepsilon/2]\right)$ will also punch $\mathcal{F}'$ where $\pi_i : \mathbb{R}^d \to \mathbb{R}$ denotes the $\underline{x} \mapsto x_i$ projection onto dimension i. Note, that $B_{\underline{v}} \subset B_1$ and $B_{\underline{w}} \subset B_2$ and one can think of B_1 and B_2 as a sort of extension of $B_{\underline{v}}$ and $B_{\underline{w}}$ along the dimension i such that they fill B_{ij_1} and B_{ij_2} along the narrow dimension. Since $\pi_i(B_1), \pi_i(B_2) \subset \pi_1(B_{kl})$ for any

$B_{kl} \in \mathcal{F}' \setminus \{B_{ij_1}, B_{ij_2}\}$ we have that $B_{\underline{v}} \subset B_{kl}$ implies $B_1 \subset B_{kl}$ and $B_{\underline{w}} \subset B_{kl}$ implies $B_2 \subset B_{kl}$, so B_1 and B_2 truly punches $\mathcal{F}'$.
In case *(B)* let i_1 and i_2 be the dimensions with missing boxes. In this case, any pair of diagonally opposite hypercubes $B_{\underline{v}'}, B_{-\underline{v}'}$ from $\mathcal{A}$ punches $\mathcal{F}'$. Since 3 narrow boxes belong to both dimensions i_1 and i_2 in $\mathcal{F}'$ respectively, there is a pair $B_{\underline{v}}, B_{-\underline{v}}$ which is contained in 2 i_1- and i_2-narrow boxes respectively. Let j_1 and j_2 be the indices of the i_1- and i_2-narrow boxes in $\mathcal{F}'$ which don't intersect any other narrow boxes of their respective dimensions. We can extend $B_{\underline{v}}$ and $B_{-\underline{v}}$ similarly as in case *(A)* so that we get a punching pair of boxes with bigger volume: $B_1 = \left(\prod_{k=1}^{i_1-1}[v_k - c\varepsilon/2, v_k + c\varepsilon/2]\right) \times \pi_{i_1}(B_{i_1j_1}) \times \left(\prod_{k=i_1+1}^{d}[v_k - c\varepsilon/2, v_k + c\varepsilon/2]\right)$ and $B_2 = \left(\prod_{k=1}^{i_2-1}[-v_k - c\varepsilon/2, -v_k + c\varepsilon/2]\right) \times \pi_{i_2}(B_{i_2j_2}) \times \left(\prod_{k=i_2+1}^{d}[-v_k - c\varepsilon/2, -v_k + c\varepsilon/2]\right)$. By the same argument as in case *(A)*, $\pi_{i_i}(B_{i_1j_1}) \subset \pi_{i_1}(B_{kl})$ and $\pi_{i_2}(B_{i_2j_2}) \subset \pi_{i_2}(B_{kl})$ for any $B_{kl} \in \mathcal{F}' \setminus \{B_{i_1j_1}, B_{i_2j_2}\}$, so (B_1, B_2) really punches $\mathcal{F}'$.

We can see that the volume of B_1 and B_2 is $(c\varepsilon)^{d-1} \cdot c = c^d\varepsilon^{d-1} = \left(\varepsilon^{-\frac{d-1}{d}}\right)^d \varepsilon^{d-1} = \varepsilon^{-(d-1)}\varepsilon^{d-1} = 1$.
Thus we have shown a 2-punching for any subfamily $\mathcal{F}' \subset \mathcal{F}$ of size $|\mathcal{F}'| = 4d - 2$, while the whole family $\mathcal{F}$ is only $\{\varepsilon\}$, 2-punchable. ■
Corrolary 2.1 immediately follows from **Proposition 2**, since if property A is 2-punchability and property B is $\{\varepsilon'\}$, 2-punchability, there is a counterexample for $0 < \varepsilon < \varepsilon'$ where every subfamily of size $h < 4d - 1$ has property A but the whole family is only $\{\varepsilon\}$, 2-punchable, thus missing property B.

Statement 2 follows from the fact that if some box B maximally punches a family of d-dimensional boxes $\mathcal{F}$, then each of its $2d$ facets is contained by some facet of a box $B' \in \mathcal{F}'$, where $|\mathcal{F}'| \leq 2d$. Since B is bordered by facets of boxes in $\mathcal{F}'$, $B = \bigcap \mathcal{F}'$, thus, B also maximally punches any subfamily $\mathcal{F}'' \supset \mathcal{F}'$ of size $2d$.
The proposed **Conjecture**, which gives an upper bound for the Helly-number, is thus motivated by the fact that 2 d-dimesional boxes have $4d$ facets in total.

References

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