An extremal problem in the cyclic permutation

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Notations

Let n, k be positive integers and $[n] = \{1, 2, \ldots, n\}$ denote the n-element set. Let $2^{[n]}$ be the power set of $[n]$ and a subset of $2^{[n]}$ is called a *family* of $[n]$. We denote the family of all k-elements subset of $[n]$ by $\binom{[n]}{k}$.

Inclusion-free families

A family $\mathcal{F}$ is called *inclusion-free* if for any $F_1, F_2 \in \mathcal{F}$, $F_1 \subsetneq F_2$. As the first theorem in extremal finite set theory, Sperner determined the upper bound of $|\mathcal{F}|$ for inclusion-free families $\mathcal{F}$.

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Theorem (Sperner theorem)

$max|\mathcal{F}| = \binom{n}{\lfloor \frac{n}{2} \rfloor}$ *where the max is taken over all inclusion-free families.*

P-free families

For a poset P, we say that a subposet Q' of Q is a (weak) copy of P, if there exists a bijection $f : P \to Q$, such that for any $p, p' \in P$, the relation $p \prec_P p'$ implies $f(p) \prec_Q f(p')$. If a poset Q does not contain a weak copy of P, then it is *P-free*.

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Specially, a inclusion-free family is a P_2-free poset, where P_2 is the total order on 2 elements.

Intersecting families

A family $\mathcal{F}$ is called *intersecting* if for any $F_1, F_2 \in \mathcal{F}$ the intersection $F_1 \cap F_2 \neq \emptyset$. In 1961, Erdős, Ko and Rado gave the upper bound of $|\mathcal{F}|$ for any intersecting family $\mathcal{F}$.

Theorem (Erdős-Ko-Rado, 1961)

$\max |\mathcal{F}| = 2^{n-1}$, *where the* max *is taken over all intersecting families* $\mathcal{F}$.

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Theorem (Erdős-Ko-Rado Theorem)

Let k ($1 \leq k \leq \frac{n}{2}$) be a fixed integer. Then $\max |\mathcal{F}| = \binom{n-1}{k-1}$ over all intersecting families $\mathcal{F} \subset \binom{[n]}{k}$

Cyclic permutation

A *cyclic permutation* π of the elements of $[n]$ is an ordering of the elements along a cycle. A subset A of $[n]$ is called an *interval* (along π) if its element are consecutive along π. The following statements are well-known.

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Theorem

An inclusion-free family $\mathcal{A}$ of intervals along π has at most n elements. If $\mathcal{A}$ has n elements, then all of its elements have same size.

Theorem

For some $1 \leq k \leq \frac{n}{2}$, if $\mathcal{A}$ is an intersecting family of k-element intervals along π, then $|\mathcal{A}| \leq k$.

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An extremal problem in cyclic permutation

In this report, we determine the largest size of a intersecting V-free family of intervals along a fixed permutation π, where the poset $V = \{x, y, z\}$ such that $x \prec y$ and $x \prec z$.

It is clear that $|\mathcal{A}| \leq n$ if $n \in \{1, 2\}$ and $|\mathcal{A}| \leq n + 1$ if $n \in \{3, 4\}$, where the upper bound is tight. Thus, we only consider the case $n \geq 5$.

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Besides, we can assume that π is actually the identity permutation without loss of generality. For general permutaion π, the proof is similar.

Main result

Theorem

For a intersecting V-free family $\mathcal{A}$ of intervals along a fixed permutation π, if $n \geq 5$, then $|\mathcal{A}| \leq \left\lfloor \frac{3}{2} n \right\rfloor$. In particular, if $|\mathcal{A}| = \left\lfloor \frac{3}{2} n \right\rfloor$, then $|A| > \frac{n}{2}$ for any $A \in \mathcal{A}$.

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For a intersecting V-free family $\mathcal{A}$ of intervals along a fixed permutation π, if $n \geq 5$, then $|\mathcal{A}| \leq \left\lfloor \frac{3}{2}n \right\rfloor$. In particular, if $|\mathcal{A}| = \left\lfloor \frac{3}{2}n \right\rfloor$, then $|A| > \frac{n}{2}$ for any $A \in \mathcal{A}$.

Further, let $\mathcal{A}$ be the union of a set of all intervals with size $\left\lfloor \frac{n}{2} \right\rfloor + 1$ and a set of all intervals with size $\left\lfloor \frac{n}{2} \right\rfloor + 2$ and starting point $2i$ $(1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor)$. Then we have $|\mathcal{A}| = \left\lfloor \frac{3}{2}n \right\rfloor$. Thus, the upper bound of the theorem is tight.

Defination of $\mathcal{M}_{\mathcal{A}}$

Since $\mathcal{A}$ is V-free, each chain of A contain at most two elemnts. Let $\mathcal{M}_{\mathcal{A}}$ be the set such that $\mathcal{A} \backslash \mathcal{M}_{\mathcal{A}}$ is the set of all maximal elements in each chain with length two in the poset $\mathcal{A}$ with inclusion order.

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It should be noted that for each $A \in \mathcal{A} \backslash \mathcal{M}_{\mathcal{A}}$, there eixsts $M \in \mathcal{M}_{\mathcal{A}}$ such that $M \subset A$. Also, $\mathcal{M}_{\mathcal{A}}$ is an inclusion-free and intersecting family. Since $\mathcal{A}$ is V-free, for any $M \in \mathcal{M}_{\mathcal{A}}$, there exists at most one element $A \in \mathcal{A} \backslash \mathcal{M}_{\mathcal{A}}$ such that $M \subset A$.

Lemmas

First, we claim that $\max|\mathcal{A}| \geq n+1$. For example, a set of all intervals with length at least $n-1$ is an intersecting V-free family with $|\mathcal{A}| = n+1$. So we only need to consider the case $|\mathcal{A}| \geq n+1$.

Lemma

If $|M| \leq \frac{n}{2}$ for any $M \in \mathcal{M}_{\mathcal{A}}$, then $|\mathcal{A}| \leq n$.

Corollary

If $|\mathcal{A}| \geq n+1$, then there exists $A \in \mathcal{A}$ such that $|A| > \frac{n}{2}$.

Lemmas

Since each pair of two distinct elements of $\mathcal{M}_{\mathcal{A}}$ must have different starting points, $|\mathcal{M}_{\mathcal{A}}| \leq n$. In the following we consider two cases, $|\mathcal{M}_{\mathcal{A}}| = n$ and $|\mathcal{M}_{\mathcal{A}}| \leq n-1$, to prove $|\mathcal{A}| \leq \left\lfloor \frac{3}{2}n \right\rfloor$ provided $|\mathcal{A}| \geq n+1$.

The case of $|\mathcal{M}_{\mathcal{A}}| = n$

Lemma

If $|\mathcal{A}| \geq n+1$ *and* $|\mathcal{M}_{\mathcal{A}}| = n$*, then* $|\mathcal{A}| \leq \left\lfloor \frac{3}{2} n \right\rfloor$*.*

Proof. Let $\mathcal{M}_{\mathcal{A}} = \{M_1, M_2, \ldots, M_n\}$ and M_i has starting point i and endpoint b_i. Then we can find that $|M_1| = |M_2| = \cdots = |M_n| = |M_1|$. For each $A \in \mathcal{A} \backslash \mathcal{M}_{\mathcal{A}}$, there eixsts at least one i such that one of the following statements holds:

- $A \to M_i, M_{i-1}$;
- $A \to M_i, M_{i+1}$.

Besides, there exists at most one element $A' \in \mathcal{A} \backslash \mathcal{M}_{\mathcal{A}}$ such that $M' \subset A'$ for any $M' \in \mathcal{M}_{\mathcal{A}}$. Thus, $|\mathcal{A}| = |\mathcal{M}_{\mathcal{A}}| + |\mathcal{A} \backslash \mathcal{M}_{\mathcal{A}}| \leq \frac{3}{2} |\mathcal{M}_{\mathcal{A}}| = \left\lfloor \frac{3}{2} n \right\rfloor$. □

The case of $|\mathcal{M}_{\mathcal{A}}| \leq n-1$

Lemma

If $|\mathcal{A}| \geq n+1$ and $|\mathcal{M}_{\mathcal{A}}| \leq n-1$, then $|\mathcal{A}| \leq \left\lfloor \frac{3}{2}n \right\rfloor$ with equality holds if and only if $|\mathcal{M}_{\mathcal{A}}| = n-1$ and n is odd.

Proof. Denote $|\mathcal{M}_{\mathcal{A}}| = m$ and $\mathcal{M}_{\mathcal{A}} = \{M_1, M_2, \ldots, M_m\}$. Let a_i and b_i be the starting point and endpoint of M_i for $1 \leq i \leq m$, respectively. When $m < \frac{3}{4}n$, $|\mathcal{A}| \leq 2|\mathcal{M}_{\mathcal{A}}| < \frac{3}{2}n$, the result is trivial. We only need to consider $m \geq \frac{3}{4}n$.

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If $a_1a_2a_3\dots a_{m-1}a_m$ are consecutive, then we assume that $a_i = i$ without loss of generality. Then for any $A \in \mathcal{A}\backslash\mathcal{M}_{\mathcal{A}}$, exactly one element $M_i \in \mathcal{M}_{\mathcal{A}}$ such that $M_i \subset A$ is possible only if one of following statements holds:

- $i = 1$;
- $2 \leq i \leq m-1$ and $b_{i+1} > b_i + 1$;
- $i = m$ and $b_1 > b_m + 1$.

Let t be the largest number of such A among $\mathcal{A}\backslash\mathcal{M}_{\mathcal{A}}$ and $|\mathcal{A}\backslash\mathcal{M}_{\mathcal{A}}| = l$. Since there exists at most one element $A' \in \mathcal{A}\backslash\mathcal{M}_{\mathcal{A}}$ such that $M' \subset A'$ for any $M' \in \mathcal{M}_{\mathcal{A}}$, we have $m = |\mathcal{M}_{\mathcal{A}}| \leq 2(l-t) + t = 2l - t$ and $l \leq \frac{m+t}{2}$.
Also, up to modulo n, $b_1 = n + |M_1| > b_m \geq m - 1 + |M_1| + t - 1$, which implies $t \leq n - m + 1$.
Thus, $|\mathcal{A}| = m + l \leq m + \frac{n+1}{2} \leq \frac{3}{2}n - \frac{1}{2} \leq \left\lfloor \frac{3n}{2} \right\rfloor$, where $|\mathcal{A}| = \left\lfloor \frac{3n}{2} \right\rfloor$ holds if and only if $m = n-1$ and n is odd.

Similarly, consider the permutation is constructed by d segments: $a^j_1 a^j_2 \dots a^j_{m_j}$ $j = 1, 2, \dots, d$. Without loss of generality, assume that $a^1_1 = 1$ and $a^d_{m_d} < n$.

Similarly, consider the permutation is constructed by d segments: $a_1^j a_2^j \dots a_{m_j}^j$ $j = 1, 2, \dots, d$. Without loss of generality, assume that $a_1^1 = 1$ and $a_{m_d}^d < n$.

For each $j = 1, 2, \dots, d$ and $1 \leq i \leq m_j$, let M_i^j be the interval with starting point a_i^j. Let k_i^j be the length of M_i^j. Then

$$t_j + a_1^j + k_1^j + m_j - 1 \leq a_1^{j+1} + k_1^{j+1}. \tag{1}$$

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$$t_j + a^j_1 + k^j_1 + m_j - 1 \leq a^{j+1}_1 + k^{j+1}_1. \tag{1}$$

Sum (1) over all j, up to modulo n, we have

$$\sum_{j=1}^{d} \left(t_j + a^j_1 + k^j_1 + m_j - 1 \right) \leq \sum_{j=1}^{d} \left(a^{j+1}_1 + k^{j+1}_1 \right).$$

Therefore, $t + m - d \leq n$ and similarly, $|\mathcal{A} \backslash \mathcal{M}_\mathcal{A}| \leq \frac{m+t}{2} \leq \frac{n+d}{2}$. Then $|\mathcal{A}| = |\mathcal{M}_\mathcal{A}| + |\mathcal{A} \backslash \mathcal{M}_\mathcal{A}| \leq (n-d) + \frac{n+d}{2} = \frac{3}{2}n - \frac{d}{2} < \lfloor \frac{3}{2} n \rfloor$. $\square$

Observation

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If $|\mathcal{A}| = \left\lfloor \frac{3}{2}n \right\rfloor$, *then* $|A| > \frac{n}{2}$ *for any* $A \in \mathcal{A}$.

Thank you for listening!