An extremal problem in the cyclic permutation

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1 Introduction

Let n, k be positive integers and $[n] = \{1, 2, \ldots, n\}$ denote the n-element set. Let $2^{[n]}$ be the power set of $[n]$ and a subset of $2^{[n]}$ is called a *family* of $[n]$. We denote the family of all k-elements subset of $[n]$ by $\binom{[n]}{k}$. A family $\mathcal{F}$ is called *inclusion-free* if for any $F_1, F_2 \in \mathcal{F}$, $F_1 \subsetneq F_2$. As the first theorem in extremal finite set theory, Sperner determined the upper bound of $|\mathcal{F}|$ for inclusion-free families $\mathcal{F}$.

Theorem 1.1 (Sperner theorem)**.** *[3] max$|\mathcal{F}| = \binom{n}{\lfloor \frac{n}{2} \rfloor}$ where the max is taken over all inclusion-free families.*

For a poset P, we say that a subposet Q' of Q is a (weak) copy of P, if there exists a bijection $f : P \to Q$, such that for any $p, p' \in P$, the relation $p \prec_P p'$ implies $f(p) \prec_Q f(p')$. If a poset Q does not contain a weak copy of P, then it is P*-free*. Specially, a inclusion-free family is a P_2-free poset, where P_2 is the total order on 2 elements.

A family $\mathcal{F}$ is called *intersecting* if for any $F_1, F_2 \in \mathcal{F}$ the intersection $F_1 \cap F_2 \neq \emptyset$. In 1961, Erdős, Ko and Rado gave the upper bound of $|\mathcal{F}|$ for any intersecting family $\mathcal{F}$.

Theorem 1.2. *[4]* $\max |\mathcal{F}| = 2^{n-1}$*, where the* max *is taken over all intersecting families* $\mathcal{F}$*.*

Theorem 1.3. *[4] Let $\mathcal{F}$ be an intersecting family. There is another intersecting family $\mathcal{G}$ such that $\mathcal{F} \subset \mathcal{G}$ and $|\mathcal{G}| = 2^{n-1}$.*

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Theorem 1.4 (Erdős-Ko-Rado Theorem)**.** *[4] Let k ($1\leq k \leq \frac{n}{2}$) be a fixed integer. Then* $\max|\mathcal{F}| = \binom{n-1}{k-1}$ *over all intersecting families* $\mathcal{F} \subset \binom{[n]}{k}$

A *cyclic permutation* π of the elements of $[n]$ is an ordering of the elements along a cycle. A subset A of $[n]$ is called an *interval* (along π) if its element are consecutive along π. The following statements are well-known.

Theorem 1.5. *[2] An inclusion-free family $\mathcal{A}$ of intervals along π has at most n elements. If $\mathcal{A}$ has n elements, then all of its elements have same size.*

Theorem 1.6. *[2] For some $1 \leq k \leq \frac{n}{2}$, if $\mathcal{A}$ is an intersecting family of k-element intervals along π, then $|\mathcal{A}| \leq k$.*

In this report, we determine the largest size of a intersecting V-free family of intervals along a fixed permutation π, where the poset $V = \{x, y, z\}$ such that $x \prec y$ and $x \prec z$. It is clear that $|\mathcal{A}| \leq n$ if $n \in \{1, 2\}$ and $|\mathcal{A}| \leq n+1$ if $n \in \{3, 4\}$, where the upper bound is tight. Thus, we only consider the case $n \geq 5$. Besides, we can assume that π is actually the identity permutation without loss of generality. For general permutaion π, the proof is similar.

Theorem 1.7. *For a intersecting V-free family $\mathcal{A}$ of intervals along a fixed permutation π, if $n \geq 5$, then $|\mathcal{A}| \leq \left\lfloor \frac{3}{2}n \right\rfloor$. In particular, if $|\mathcal{A}| = \left\lfloor \frac{3}{2}n \right\rfloor$, then $|A| > \frac{n}{2}$ for any $A \in \mathcal{A}$.*

Further, let $\mathcal{A}$ be the union of a set of all intervals with size $\left\lfloor \frac{n}{2} \right\rfloor + 1$ and a set of all intervals with size $\left\lfloor \frac{n}{2} \right\rfloor + 2$ and starting point $2i$ ($1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$). Then we have $|\mathcal{A}| = \left\lfloor \frac{3}{2}n \right\rfloor$. Thus, the upper bound of Theorem 1.7 is tight.

2 Proof of Theorem 1.7

Since $\mathcal{A}$ is V-free, each chain of A contain at most two elemnts. Let $\mathcal{M}_{\mathcal{A}}$ be the set such that $\mathcal{A}\backslash\mathcal{M}_{\mathcal{A}}$ is the set of all maximal elements in each chain with length one in the poset $\mathcal{A}$ with inclusion order. It should be noted that for each $A \in \mathcal{A}\backslash\mathcal{M}_{\mathcal{A}}$, there eixsts $M \in \mathcal{M}_{\mathcal{A}}$ such that $M \subset A$. Also, $\mathcal{M}_{\mathcal{A}}$ is an inclusion-free and intersecting family. Since $\mathcal{A}$ is V-free, for any $M \in \mathcal{M}_{\mathcal{A}}$, there exists at most one element $A \in \mathcal{A}\backslash\mathcal{M}_{\mathcal{A}}$ such that $M \subset A$.

First, we claim that $\max|\mathcal{A}| \geq n+1$. For example, a set of all intervals with length at least $n-1$ is an intersecting V-free family with $|\mathcal{A}| = n+1$. So we only need to consider the case $|\mathcal{A}| \geq n+1$.

Lemma 2.1. *[2] If $|M| \leq \frac{n}{2}$ for any $M \in \mathcal{M}_{\mathcal{A}}$, then $|\mathcal{A}| \leq n$.*

Proof. Denote $|\mathcal{M}_{\mathcal{A}}| = m$ and $\mathcal{M}_{\mathcal{A}} = \{M_1, M_2, \dots, M_m\}$. Let a_i and b_i be the starting point and endpoint of M_i for $1 \le i \le m$, respectively. Since each two distinct elements of $\mathcal{M}_{\mathcal{A}}$ must have different starting points and endpoints, $a_i \ne a_j$ and $b_i \ne b_j$ for different i and j. Without lose of generality, we assume M_1 is a shortest interval of $\mathcal{M}_{\mathcal{A}}$, which implies exactly one of $\{a_i, b_i\}$ belongs to M_1 for any $i \ne 1$.

Let I be the index of i with $a_i \notin M_1$ and $i \ne 1$. For $i \in I$, we denote the interval with starting point $b_i + 1$ and length $|M_i|$ by M'_i. Since $\mathcal{M}_{\mathcal{A}}$ is intersecting, $M'_i \notin \mathcal{M}_{\mathcal{A}}$. Then we consider the family $\mathcal{M}'_{\mathcal{A}}$ obtained from $\mathcal{M}_{\mathcal{A}}$ by removing all M_i and adding M'_i for any $i \in I$. It is clear that $\mathcal{M}'_{\mathcal{A}}$ is still an inclusion-free intersecting family and $|\mathcal{M}'_{\mathcal{A}}| = |\mathcal{M}_{\mathcal{A}}|$. Also, M_1 is still a shortest interval in $\mathcal{M}'_{\mathcal{A}}$ and the staring points of other intervals in $\mathcal{M}'_{\mathcal{A}}$ must be in M_1. Thus, $|\mathcal{M}_{\mathcal{A}}| = |\mathcal{M}'_{\mathcal{A}}| \le |M_1| \le \frac{n}{2}$.

Recall that for any $M \in \mathcal{M}_{\mathcal{A}}$, there exists at most one element $A \in \mathcal{A} \backslash \mathcal{M}_{\mathcal{A}}$ such that $M \subset A$. So $|\mathcal{A} \backslash \mathcal{M}_{\mathcal{A}}| \le |\mathcal{M}_{\mathcal{A}}|$ and we can know that $|\mathcal{A}| \le 2|\mathcal{M}_{\mathcal{A}}| \le n$. □

Corollary 2.2. *If $|\mathcal{A}| \ge n + 1$, then there exists $A \in \mathcal{A}$ such that $|A| > \frac{n}{2}$.*

Since each pair of two distinct elements of $\mathcal{M}_{\mathcal{A}}$ must have different starting points, $|\mathcal{M}_{\mathcal{A}}| \le n$. In the following we consider two cases, $|\mathcal{M}_{\mathcal{A}}| = n$ and $|\mathcal{M}_{\mathcal{A}}| \le n-1$, to prove $|\mathcal{A}| \le \left\lfloor \frac{3}{2}n \right\rfloor$ provided $|\mathcal{A}| \ge n + 1$.

Lemma 2.3. *If $|\mathcal{A}| \ge n + 1$ and $|\mathcal{M}_{\mathcal{A}}| = n$, then $|\mathcal{A}| \le \left\lfloor \frac{3}{2}n \right\rfloor$.*

Proof. Let $\mathcal{M}_{\mathcal{A}} = \{M_1, M_2, \dots, M_n\}$ and M_i has starting point i and endpoint b_i. Since $\mathcal{M}_{\mathcal{A}}$ is inclusion-free, $b_i < b_{i+1}$ for any $1 \le i \le n - 1$, where the inequality is considered modulo n. Then $|M_i| = b_i - i + 1 \le b_{i+1} - (i+1) + 1 = |M_{i+1}|$. In particular, $|M_n| \le |M_1|$. So $|M_1| = |M_2| = \cdots = |M_n| = |M_1| = k$. Combining with Corollary 2.2, $k > \frac{n}{2}$. When $k = n$, $\mathcal{A} = [n]$ and then $|\mathcal{A}| = 1$, which contradicts with $|\mathcal{A}| \ge n+1$. When $k = n-1$, $\mathcal{A}$ is either $\mathcal{M}_{\mathcal{A}} \cup [n]$ or $\mathcal{M}_{\mathcal{A}}$, and then $|\mathcal{A}| \le n + 1 \le \left\lfloor \frac{3}{2}n \right\rfloor$, where the last inequality holds because $n \ge 5$.

Recall that for each $A \in \mathcal{A} \backslash \mathcal{M}_{\mathcal{A}}$, there eixsts $M_i \in \mathcal{M}_{\mathcal{A}}$ such that $M_i \subset A$. Also, since $|M_i \cap M_{i+1}| = |M_{i-1} \cap M_i| = k-1$, at least one of $\{M_{i-1}, M_{i+1}\}$ is included by A. Besides, there exists at most one element $A' \in \mathcal{A} \backslash \mathcal{M}_{\mathcal{A}}$ such that $M' \subset A'$ for any $M' \in \mathcal{M}_{\mathcal{A}}$. This implies that $|\mathcal{A} \backslash \mathcal{M}_{\mathcal{A}}| \le \frac{1}{2}|\mathcal{M}_{\mathcal{A}}| = \frac{n}{2}$. Thus, $|\mathcal{A}| = |\mathcal{M}_{\mathcal{A}}| + |\mathcal{A} \backslash \mathcal{M}_{\mathcal{A}}| \le \left\lfloor \frac{3}{2}n \right\rfloor$, as desired. □

Lemma 2.4. *If $|\mathcal{A}| \ge n + 1$ and $|\mathcal{M}_{\mathcal{A}}| \le n - 1$, then $|\mathcal{A}| \le \left\lfloor \frac{3}{2}n \right\rfloor$ with equality holds if and only if $|\mathcal{M}_{\mathcal{A}}| = n - 1$ and n is odd.*

Proof. Denote $|\mathcal{M}_{\mathcal{A}}| = m$ and $\mathcal{M}_{\mathcal{A}} = \{M_1, M_2, \dots, M_m\}$. Let a_i and b_i be the starting point and endpoint of M_i for $1 \le i \le m$, respectively. When $m < \frac{3}{4}n$, $|\mathcal{A}| \le 2|\mathcal{M}_{\mathcal{A}}| < \frac{3}{2}n$, the result is trivial. We only need to consider $m \ge \frac{3}{4}n$.

First we discuss the case $a_1a_2a_3\dots a_{m-1}a_m$ are consecutive and then extend the result to other cases.

If $a_1a_2a_3\dots a_{m-1}a_m$ are consecutive, then we assume that $a_i = i$ without loss of generality. Then for any $A \in \mathcal{A}\backslash\mathcal{M}_{\mathcal{A}}$, exactly one element $M_i \in \mathcal{M}_{\mathcal{A}}$ such that $M_i \subset A$ is possible only if one of following statements holds:

- $i = 1$;
- $2 \le i \le m-1$ and $b_{i+1} > b_i + 1$;
- $i = m$ and $b_1 > b_m + 1$.

Let t be the largest number of such A among $\mathcal{A}\backslash\mathcal{M}_{\mathcal{A}}$ and $|\mathcal{A}\backslash\mathcal{M}_{\mathcal{A}}| = l$. Since there exists at most one element $A' \in \mathcal{A}\backslash\mathcal{M}_{\mathcal{A}}$ such that $M' \subset A'$ for any $M' \in \mathcal{M}_{\mathcal{A}}$, we have $m = |\mathcal{M}_{\mathcal{A}}| \le 2(l-t)+t = 2l-t$ and $l \le \frac{m+t}{2}$. Also, up to modulo n, $b_1 = n + |M_1| > b_m \ge m - 1 + |M_1| + t - 1$, which implies $t \le n - m + 1$.

Thus, $|\mathcal{A}| = m + l \le m + \frac{n+1}{2} \le \frac{3}{2}n - \frac{1}{2} \le \left\lfloor \frac{3n}{2} \right\rfloor$, where $|\mathcal{A}| = \left\lfloor \frac{3n}{2} \right\rfloor$ holds if and only if $m = n-1$ and n is odd.

Now we extend the result in the case that $a_1a_2a_3\dots a_{m-1}a_m$ are not consecutive. Note that we can consider $a_1a_2a_3\dots a_{m-1}a_m$ as a union of some consecutive segments.

Assume the permutation is constructed by d segments: $a_1^j a_2^j \dots a_{m_j}^j$ $j = 1, 2, \dots, d$. Without loss of generality, assume that $a_1^1 = 1$ and $a_{m_d}^d < n$. For each $j = 1, 2, \dots, d$ and $1 \le i \le m_j$, let M_i^j be the interval with starting point a_i^j. Let k_i^j be the length of M_i^j and b_i^j be the endpoint of M_i^j. Then for any $A \in \mathcal{A}\backslash\mathcal{M}_{\mathcal{A}}$, exactly one element $M_i^j \in \mathcal{M}_{\mathcal{A}}$ such that $M_i^j \subset A$ is possible only if one of following statements holds:

- $i = 1$;
- $2 \le i \le m_j - 1$ and $b_{i+1}^j > b_i^j + 1$;
- $i = m_j$ and $b_{m_j}^j + 1 < b_1^{j+1}$.

Let t_j be the largest number of such A with respect to $\{M_1^j, M_2^j, \dots, M_{m_j}^j\}$. One has $m = m_1 + m_2 + \dots + m_d \le n - d$, $t = t_1 + t_2 + \dots + t_d$ and inequality

$$t_j - 1 + b_1^j + m_j - 1 \le b_{m_j}^j \le b_1^{j+1} - 1$$

Note that $b_1^j = a_1^j - 1 + k_1^j$ and $b_1^{j+1} = a_1^{j+1} - 1 + k_1^{j+1}$. Thus,

$$t_j + a_1^j + k_1^j + m_j - 1 \le a_1^{j+1} + k_1^{j+1}. \tag{2.1}$$

Sum (2.1) over all j, up to modulo n, we have

$$\sum_{j=1}^{d}\left(t_j+a_1^j+k_1^j+m_j-1\right)\leq\sum_{j=1}^{d}\left(a_1^{j+1}+k_1^{j+1}\right).$$

Therefore, $t+m-d\leq n$ and similarly, $|\mathcal{A}\backslash\mathcal{M}_{\mathcal{A}}|\leq\frac{m+t}{2}\leq\frac{n+d}{2}$. Then $|\mathcal{A}|=|\mathcal{M}_{\mathcal{A}}|+|\mathcal{A}\backslash\mathcal{M}_{\mathcal{A}}|\leq(n-d)+\frac{n+d}{2}=\frac{3}{2}n-\frac{d}{2}<\lfloor\frac{3}{2}n\rfloor$, as desired. □

Lemma 2.5. *If* $|\mathcal{A}|=\left\lfloor\frac{3}{2}n\right\rfloor$, *then* $|A|>\frac{n}{2}$ *for any* $A\in\mathcal{A}$.

Proof. From Lemmas 2.3 and 2.4, we know $|\mathcal{A}|=\left\lfloor\frac{3}{2}n\right\rfloor$ if and only if $|\mathcal{M}_{\mathcal{A}}|=n-1$ and n is odd or $|\mathcal{M}_{\mathcal{A}}|=n$, in particularly, when $|\mathcal{M}_{\mathcal{A}}|=n$ the result is trivial.

When $|\mathcal{M}_{\mathcal{A}}|=n-1$, n is odd, Let $\mathcal{M}_{\mathcal{A}}=\{M_1,M_2,\ldots,M_{n-1}\}$ and M_i has starting point i and endpoint b_i. Since $\mathcal{M}_{\mathcal{A}}$ is inclusion-free, $b_i<b_{i+1}$ for any $1\leq i\leq n-2$, where the inequality is considered modulo n. Then $|M_i|=b_i-i+1\leq b_{i+1}-(i+1)+1=|M_{i+1}|$. So $|M_1|\leq|M_2|\leq\cdots\leq|M_{n-1}|$ and $b_{n-1}\leq n+|M_1|-1$. Thus, $|M_{n-1}|\leq|M_1|+1$.

For odd n, suppose that $|M_1|=k<\frac{n}{2}$. Since $\mathcal{M}_{\mathcal{A}}$ is an intersecting family then for M_{k+1}, we have $b_{k+1}\geq n+1$ which means $|M_{k+1}|\geq n+1-k$. Combining with $|M_{n-1}|\leq|M_1|+1=k+1$, we have $n+1-k\leq k+1$ which implies $n\leq 2k<n$, which is impossible. □

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