Directed studies 2. report

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Abstract

During my directed studies, I learned the basics of symplectic geometry and geometric quantization. Moreover, I have partially read an article, which is related to polarizations arising in geometric quantization, containing a theorem about the existence of local polarizations. With the help of my supervisor, I managed to show an example of global polarization.

1 Symplectic geometry

Symplectic geometry is the mathematical framework describing classical mechanical systems. This section serves as a summary of the basic principles I needed for this semester. I learned these concepts from a book by John M. Lee called Introduction to Smooth Manifolds [1].

Definition 1.1 (Symplectic manifold). (M, ω) is a symplectic manifold if M is a smooth real manifold equipped with a closed non-degenerate two-form ω, called the **symplectic form**.

Example 1.1 ($\mathbb{R}^{2n}$ is a symplectic manifold). *Consider a point $p \in \mathbb{R}^{2n}$, and we know that $T_p\mathbb{R}^{2n} \cong \mathbb{R}^{2n}$. Consider basis vectors $\{x_i, y_i\}_{i=1}^n$ and the 2-form defined pointwise by*

$$\omega_p(x_i, y_i) = \omega_p(y_i, x_i) = \delta_{ij}, \tag{1}$$

$$\omega_p(x_i, x_j) = \omega_p(y_i, y_j) = 0 \tag{2}$$

*is a symplectic form. The **standard symplectic form** is defined by*

$$\omega_p = \sum_{i=1}^{n} dx_i \wedge dy_i, \tag{3}$$

and it can easily be verified that this is closed and non-degenerate.

Example 1.2. *Any 2-dimensional smooth orientable manifold with a non-vanishing 2-form ω is a symplectic manifold since $d\omega$ is a 3-form.*

In this report, the cotangent bundle is the most important example of a symplectic manifold:

Example 1.3 (Cotangent bundle). *Considering a smooth n-dimensional real manifold Q, the cotangent bundle T^*Q is a $2n$-dimensional symplectic manifold. Considering an open set $U \in Q$ and corresponding coordinate charts $q^k : U \to \mathbb{R}^n$, we have a a corresponding coordinate system $(q^k, p_k) : V \to \mathbb{R}^{2n}$, where V is an open set on T^*Q. Then let*

$$\theta|_U = \sum_{k=1}^{n} p_k dq^k, \tag{4}$$

*which is called the **tautological one-form** or **symplectic potential**, and then*

$$\omega|_U = d\theta|_U = \sum_{k=1}^{n} dp_k \wedge dq^k, \tag{5}$$

and hence $d\omega|_U = 0$.

The non-degeneracy condition of ω yields an isomorphism between the tangent and the cotangent spaces of a symplectic manifold M at $x \in M$. Given a basis vector $v \in T_xM$, we can define

$$T_xM \ni v \mapsto \omega(v, \cdot) \in T_x^*M. \tag{6}$$

Moreover, this can be extended to an isomorphism between TM and T^*M and hence between vector fields and one-forms on M with

$$X \mapsto \omega(X, \cdot) \in \Omega^1(M), \tag{7}$$

where X is a vector field. This means, that for each $f \in C^\infty(M)$ one can define a vector field X_f by

$$df = -\omega(X_f, \cdot). \tag{8}$$

X_f is called **Hamiltonian vector field** of f.

The Lie bracket $[\cdot, \cdot]$ of two Hamiltonian vector fields is also a Hamiltonian vector field given by

$$[X_f, X_g] = X_{\omega(f,g)}, \tag{9}$$

which connects the symplectic form with Lie brackets.

When dealing with classical mechanical systems, one often considers certain submanifolds of symplectic manifolds.

Definition 1.2 (Isotropic subspace)**.** A subspace $(V, \omega|_V)$ of a symplectic vector space (W, ω) is called **isotropic** when $\omega|_V \equiv 0$.

Remark. Such subspace can have a dimension $\frac{1}{2} \dim W$, because ω is non-degenerate.

Definition 1.3 (Lagrangian subspace)**.** An isotropic subspace with maximal dimension (maximally isotropic subspace) is called a **Lagrangian subspace**.

A very similar definition can be given for symplectic manifolds:

Definition 1.4 (Lagrangian submanifold)**.** A submanifold N of a symplectic manifold (M, ω) is called a **Lagrangian submanifold** if $\omega|_N \equiv 0$.

Example 1.4. *Consider a manifold Q. Then Q is a Lagrangian submanifold of T^*Q.*

Locally, any Lagrangian submanifold is given by n functions $\{F_k\}_{i=1}^n$ such that

$$\omega(X_{F_k}, X_{F_l}) = 0, \text{ where } k, l = 1, \dots, n. \tag{10}$$

When the functions $\{F_k\}_{i=1}^n$ satisfy this property, it is said that they are **in involution**.

2 Geometric quantization

Geometric quantization is a framework in modern physics which aims to associate a quantum system with a given classical system [2]. This procedure relies on several axioms, which prescribe how functions on a symplectic manifold (classical observables) can be mapped to operators (quantum observables) on a Hilbert space. For our purposes, considering the cotangent bundle as the symplectic manifold is enough. In this setup, quantization refers to an assignment

$$\mathcal{Q} : C^\infty(M) \to (\mathcal{H} \to \mathcal{H}), \tag{11}$$

where $M = T^*Q$ is the cotangent bundle of a smooth manifold Q and $\mathcal{H}$ is a Hilbert space. In classical mechanics, Q is usually referred to as configuration space and T^*Q as phase space. The quantization axioms are [2, Section 3]:

- **Q1**: $\mathbb{R}$-linearity,
$$\mathcal{Q}(rf + g) = r\mathcal{Q}(f) + \mathcal{Q}(g), \qquad \forall r \in \mathbb{R}, f, g \in C^\infty(T^*M). \tag{12}$$
- **Q2**: Unitality (identity-preserving),
$$\mathcal{Q}(1) = \mathbb{1}, \text{ where } \mathbb{1} \text{ is the identity operator on } \mathcal{H}. \tag{13}$$

- **Q3**: Hermitian,

$$\mathcal{Q}(f)^* = \mathcal{Q}(f). \tag{14}$$

- **Q4**: The quantum condition,

$$[\mathcal{Q}(f), \mathcal{Q}(g)] = -i\hbar\mathcal{Q}(\omega(f,g)), \tag{15}$$

  where $\hbar$ is the Planck constant.

- **Q5**: If $\{f_i\}_{i=1}^n$ is a complete set of classical observables, then $\{\mathcal{Q}(f_i)\}_{i=1}^n$ is a complete set of quantum observables. For this axiom, we need the following two definitions:

  Definition 2.1 (Complete set of classical observables)**.** Let Q be a smooth manifold and $M := T^*Q$. A set of classical observables $\{f_i\}_{i=1}^n$, $f_i \in C^\infty(M)$ are **complete** if any $f \in C^\infty(M)$ with

$$\omega(f, f_i) = 0, \qquad \forall i = 1, \dots, n, \tag{16}$$

  is a constant function.

  Definition 2.2 (Complete set of quantum observables)**.** A set of quantum observables is **complete** if any operator commuting with all of them is a multiple of the identity operator

The problem with these axioms is that $Q4$ and $Q5$ cannot be simultaneously true. However, it is "satisfactory" if these axioms are true for physically relevant observables, like p^2. Hence, we can restrict the algebra of classical observables $(C^\infty(M), \omega)$ to a subalgebra, containing only the physically relevant ones.

The idea is to try to associate functions on the phase space to its Hamiltonian vector field, i.e. one could define

$$\mathcal{Q}(f) = -i\hbar X_f - \theta(X_f) + f, \tag{17}$$

which satisfies **Q1-Q4**, and here the Hilbert space is $L^2(M, \omega)$. The problem with this approach is that it fails to satisfy **Q5** and fails to reproduce observables quadratic in p.

In the cotangent bundle, the tautological one-form θ is globally defined. When defining $\mathcal{Q}$ above, we could also have chosen $\theta + dg$ for some function g on M instead of θ in (17). One can compensate for this by multiplying functions on M by the phase factor $\exp(ig/\hbar)$, but this could not be performed on the real-valued M. This suggests to regard $\mathcal{Q}(f)$ as operators acting on a trivial complex line bundle L over M equipped with a connection D defined by

$$D := d - \frac{i}{\hbar}\theta, \tag{18}$$

and transition functions $\exp(-i/\hbar)$.

Now the Hilbert space is the square-integrable functions on the sections of the complex line bundle L, and we can think of quantum-mechanical wave functions as sections of L. However, the problem with the resulting Hilbert space is that it is "too large", since e.g. in the Schrödinger picture of quantum mechanics, the wave functions only depend on coordinates of Q, which is a Lagrangian submanifold. Motivated by this, we need to restrict these functions to a certain Lagrangian submanifold. Hence, considering ψ a section of L, we need to search for sections X of a subbundle $P \subset TM^{\mathbb{C}}$ such that they are covariantly constant along X, i.e.,

$$D(X)\psi = 0, \tag{19}$$

which implies that

$$[D(X), D(Y)]\psi = 0, \qquad \forall X, Y \text{ section of } P. \tag{20}$$

The curvature of L Ω can be calculated by using (18):

$$\Omega(X, Y) = i([D(X), D(Y)] - D([X, Y])) = \frac{1}{\hbar}\omega(X, Y). \tag{21}$$

Hence, the condition (20) can be written as

$$\left(D([X, Y]) - \frac{i}{\hbar}\omega(X, Y)\right)\psi = 0, \qquad \forall X, Y \text{ section of } P. \tag{22}$$

which is automatically satisfied if $[X, Y]$ is also a section of P (P is an integrable subbundle) and $\omega(X, Y) = 0$ (Lagrangian subbundle).

Therefore, in geometric quantization one is interested in certain subbundles of $TM^{\mathbb{C}}$, these are called polarizations [2, Section 5.1]:

Definition 2.3 (Polarization)**.** Let (M, ω) be a symplectic manifold. A **polarization** P of (M, ω) is an integrable Lagrangian subbundle of the complexified tangent bundle $TM^{\mathbb{C}}$.

However, the problem is that the quantization function $\mathcal{Q}$ does not preserve this property. A function $f \in C^\infty(M)$ is called *quantizable* when for all ψ covariantly constant along a polarization P, then $\mathcal{Q}(f)\psi$ is also covariantly constant. However, even the simple observable p^2 is not quantizable. There is a way around this called the *Blattner-Kostant-Sternberg construction* [2, Section 6.1], but this construction is not well understood.

3 Involutive structures

Polarizations in geometric quantization are called involutive structures in differential geometry. Lempert investigated these involutive structures and found a condition for the local existence of involutive structures [3], and the results presented in this section are from this research. The main objective of our investigation is to find examples of global involutive structures. For this, we need the following definition:

Definition 3.1 (Involutive structure)**.** An involutive structure on a smooth manifold N is a smooth subbundle $P \subset \mathbb{C}TN = \mathbb{C} \otimes_{\mathbb{R}} TN$ that is involutive in the sense that the Lie bracket of its (smooth, local) sections is again a section.

Example 3.1. *A CR (Cauchy-Riemann) manifold M is a differentiable manifold together with a subbundle $L \subset \mathbb{C} \otimes_{\mathbb{R}} TM$ such that*

$$[L, L] \subseteq L, \qquad \textit{(involutivity, or integrability)} \tag{23}$$

$$L \cap \overline{L} = \{0\}, \tag{24}$$

and this subbundle is also called a ***CR structure****.*

Example 3.2 (Complex structure)**.** *Involutive structure with $P \oplus \overline{P} = \mathbb{C}TN$ is a complex structure, P corresponding to $(1,0)$-vectors.* [1]

Certain manifolds admitting these structures are called involutive manifolds:

Definition 3.2 (Involutive manifold)**.** An involutive manifold is a smooth manifold N endowed with an involutive structure P and is denoted by (N, P).

Remark. The involutive structure is very similar to the CR structure, only the (24) is omitted. Note, that this condition is equivalent to the definition given by 2.3.

Definition 3.3 (Involutive map)**.** Suppose (N, P) and (M, Q) are involutive manifolds. Then an involutive map is a C^1 map f where

$$f_* Q \subset P. \tag{25}$$

The following Lemma [3, Lemma 3.1] is concerned about involutive structures and smooth maps:

Lemma 3.1. *Let X, Y be smooth manifolds, $P \subset \mathbb{C}TX$, $Q \subset \mathbb{C}TY$ and a smooth map $\phi : X \to Y$ such that $P_x = \phi_*^{-1} Q_{\phi(x)}$. Then if Q is involutive, P is also involutive.*

Now assume we are given complex manifolds C, Z and a smooth map $\epsilon : N \times C \to Z$ such that $\epsilon^x = \epsilon(x, \cdot)$ is holomorphic for all $x \in N$. In the article of Lempert [3], a certain symmetry of geodesics is also accounted for, but this part is not needed for this work and is omitted for brevity. Given such map ϵ, the following theorem [3, Theorem 3.4] can be shown:

Theorem 1. *Let $c \in C$ and $\psi = \psi_c = \epsilon(\cdot, c) : N \to Z$. If $P = P(c) = \psi_*^{-1} T^{1,0} Z \subset \mathbb{C}TN$ is a subbundle, then it is an involutive structure.*

This theorem gives an involutive structure locally, using certain analyticity assumptions.

We are concerned with the following example during our investigations [3, Example 4.2]:

[1] $(1,0)$-vectors are vectors annihilating all anti-holomorphic functions. Conversely, $(0,1)$-vectors annihilate holomorphic functions.

Example 3.3. *We consider first-order ODEs (vector fields) and let Q be a n-dimensional manifold. Let $\pi : TQ \to Q$ be the bundle projection. Let N consist of trajectories of a fixed real analytic vector field ξ on TQ. We also know that*

$$\pi_* \xi(x) = x. \tag{26}$$

If U is an open subset of Q such that $TU \cong U \times \mathbb{R}^n$, then a vector field ξ on TU can be written as

$$\xi(q,p) = (q, F(q,p)), \tag{27}$$

where $F : TU \to \mathbb{R}^n$. This means that the trajectories must be solutions to the second-order ODE

$$\ddot{q} = F(q, \dot{q}), \tag{28}$$

which also means that the trajectories are in one-to-one correspondence with TQ.

One can complexify this setup to acquire

$$\pi^{\mathbb{C}} : TQ^{\mathbb{C}} \to Q^{\mathbb{C}} \tag{29}$$

and a holomorphic vector field $\xi^{\mathbb{C}}$ such that Q, TQ are maximally real, analytic submanifolds of $Q^{\mathbb{C}}, TQ^{\mathbb{C}}$, $\pi^{\mathbb{C}}|_{TQ} = \pi$ and $2\operatorname{Re}\xi^{\mathbb{C}}|_{TQ} = \xi$.

Now let C be a connected, simply connected, bounded neighbourhood of $0 \in \mathbb{C}$ and let $\epsilon > 0$ be such that the trajectories $x : (-\epsilon, \epsilon) \to TQ$ extend to $\bar{C}$, and call the sets of these trajectories N. Let us denote this extension $x \in N$ by $\tilde{x}$. Then if $x \in N$ is a trajectory of the vector field ξ, then $\tilde{x}$ is a trajectory of the holomorphic vector field $\xi^{\mathbb{C}}$. By identifying all $x \in N$ by $x(0)$, N can be identified by an open subset of TM.

Fix $c \in C$ and define

$$\psi = \psi_c : N \to Q^{\mathbb{C}} \tag{30}$$
$$x \mapsto \tilde{x}(c). \tag{31}$$

For Theorem 1 to apply, it has to be shown that $\psi_*^{-1} T^{10} Q^{\mathbb{C}}$ is really a subbundle:

Proposition 3.1. *$\psi_*^{-1} T^{10} Q^{\mathbb{C}} \subset \mathbb{C}TN$ is a subbundle.*

Example provides a construction where involutive structures are local. We want to show examples, where it is not only local but global. The reason for doing this is that polarizations defined from local complex structures may be intractable in general. Dropping condition of demanding the involutive bundle to define a complex structure pointwise and only requiring a global polarization to exist, we may have more

The idea is to search global involutive structures in well-studied classical mechanical systems. However, instead of using the tangent bundle TQ, we will use the cotangent bundle $M := T^*Q$, which naturally has a symplectic structure (see 1.3), however, the tangent bundle may not always be symplectic.

3.1 Global involutive structure of the complexified harmonic oscillator

Consider a real analytic manifold Q and its cotangent space $M := T^*Q \cong \mathbb{R}^2$. Let us define the Hamiltonian function $H \in C^\infty(M)$ of the harmonic oscillator as

$$H(q,p) = \frac{1}{2}(q^2 + p^2). \tag{32}$$

The Hamiltonian vector field X_H (ξ in Lempert's article) is

$$X_H(q,p) = p\partial_q - q\partial_p \tag{33}$$

Trajectories will be integral curves of this vector field. Consider $(q_0, p_0) \in M$. We can solve the Hamilton equations to acquire the unique trajectory from this point, which is of the form

$$x(t) = (p_0 \sin t + q_0, p_0 \cos t). \tag{34}$$

To define $\psi(x) = \pi^{\mathbb{C}}(\tilde{x}(i))$ as in Example 3, we define the holomorphic extension of M as $M^{\mathbb{C}} \cong \mathbb{C}^2$ and $\pi^{\mathbb{C}} : M^{\mathbb{C}} \to Q^{\mathbb{C}}$, since $\tilde{x}$ is the trajectory corresponding to $H^{\mathbb{C}}$. The complexification of H is given by

$$H^{\mathbb{C}}(z_1, z_2) = \frac{1}{2}(z_1^2 + z_2^2), \tag{35}$$

where we can similarly calculate the corresponding Hamiltonian vector field

$$X_H^{\mathbb{C}}(z_1, z_2) = z_2 \partial_{z_1} - z_1 \partial_{z_2}, \tag{36}$$

and one can easily verify that $2\,\mathrm{Re}\,X_H^{\mathbb{C}} = X_H$ indeed. The complexified trajectories are the integral curves corresponding to $X_H^{\mathbb{C}}$, which are given by

$$\partial_w z_1(w) = z_2(w), \tag{37}$$
$$\partial_w z_2(w) = -z_1(w). \tag{38}$$

The trajectories in the complexified space will be

$$\tilde{x}(c) = (z_2 \sin c + z_1, z_2 \cos c) \tag{39}$$

and hence

$$\psi_c(x) = \pi^{\mathbb{C}}(\tilde{x}(c)) = \pi^{\mathbb{C}}(z_2 \sin c + z_1, z_2 \cos c) = z_2 \sin c + z_1 \in M^{\mathbb{C}}. \tag{40}$$

and then

$$\psi_i(x) = z_1 + i z_2 \sinh(1). \tag{41}$$

Now we need to determine the pushforward $\psi_* : TN \to M^{\mathbb{C}}$, consider $v \in T_{(a,b)}M$ and write $v = \alpha\partial_q + \beta\partial_p$. Consider a path $\chi : \mathbb{R} \to N$ such that

$$\chi(s) = (a + s\alpha, b + s\beta) \in N, \tag{42}$$

we know that $\chi(0) = (a, b)$ and $\chi'(0) = v = (\alpha, \beta)$. The pushforward is

$$(\psi_* v)_{(a,b)} = \frac{d}{ds}\psi(\chi(s))|_{s=0} = \frac{d}{ds}\left(a + s\alpha + i(b + s\beta)\sinh(1)\right)|_{s=0} = \alpha + i\beta\sinh(1) = \alpha\partial_q + \beta\sinh(1)\partial_{\eta_q}. \tag{43}$$

For this to be holomorphic, we need that

$$\psi_*((v_1 + iw_1)\partial_q + (v_2 + iw_2)\partial_p) = (v_1 + iw_1)\partial_q + (v_2 + iw_2)\sinh(1)\partial_{\eta_q} \overset{!}{=} \frac{1}{2}\xi\partial_q - \frac{i}{2}\xi\partial_{\eta_q} \tag{44}$$
$$\implies (v_1 + iw_1) = i(v_2 + iw_2)\sinh(1) \tag{45}$$
$$\implies v_1 = -w_2\sinh(1), \quad w_1 = v_2\sinh(1), \tag{46}$$

which yields that

$$P = \{\sinh(1) z\partial_q - iz\partial_p : \ z \in \mathbb{C}\} \tag{47}$$

is a global involutive structure (polarization).

It turned out, that a global polarization can be acquired easier if one considers that ψ is a diffeomorphism, since then the subbundle $T^{10}M^{\mathbb{C}}$ can be defined as the involutive structure using this diffeomorphism. Given $(q, p) \in M \cong \mathbb{R}^2$, $\psi : \mathbb{R}^2 \to \mathbb{C}$

$$\psi_i((q, p)) = q + ip\sinh(1), \tag{48}$$

which is clearly bijective and differentiable with the differentiable inverse.

References

[1] John M. Lee. Introduction to smooth manifolds. 2000.

[2] Matthias Blau. Symplectic geometry and geometric quantization 1. 2007. Available at https://ncatlab.org/nlab/files/BlauGeometricQuantization.pdf.

[3] László Lempert. Adapted complex and involutive structures. https://arxiv.org/abs/1901.03388, 2019.