The translation invariant product measure problem in non-sigma finite case

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Introduction

Product measure space

Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be measure spaces.
A product measure space is the space $X \times Y$ equipped with

- the σ-algebra $\mathcal{A} \otimes \mathcal{B}$ generated by the set $\{A \times B : A \in \mathcal{A}, B \in \mathcal{B}\}$,
- a product measure $\lambda : \mathcal{A} \otimes \mathcal{B} \to \mathbb{R}_0^+$.

Product measure

A measure $\lambda : \mathcal{A} \otimes \mathcal{B} \to \mathbb{R}_0^+$ is a product measure of μ and ν if for all $A \times B$, where $A \in \mathcal{A}$ and $B \in \mathcal{B}$,

$$\lambda(A \times B) = \mu(A)\nu(B).$$

DISTINCT PRODUCT MEASURES ON THE SAME SPACE

Disclaimer: product measure is not necessarily unique. Let $E \in \mathcal{A} \otimes \mathcal{B}$, we define

The primitive product measure:

$$\pi(E) = \inf \left\{ \sum_{n=1}^{\infty} \mu(A_n)\nu(B_n) : \mathcal{A}_n \in \mathcal{A}, B_n \in \mathcal{B}, E \subseteq \bigcup_{n=1}^{\infty} A_n \times B_n \right\}.$$

The completely locally determined (c.l.d) product measure:

$$\rho(E) = \sup \left\{ \pi(E \cap (A \times B)) : \mathcal{A} \in \mathcal{A}, B \in \mathcal{B}; \mu(A), \nu(B) < \infty \right\}.$$

Distinct product measures on the same space

Suppose that

- $X, Y = [0,1]$;
- $\mathcal{A} =$ Lebesgue σ-algebra, $\mathcal{B} = \mathcal{P}([0,1])$;
- $\mu =$ Lebesgue measure, $\nu =$ counting measure.

Consider the set $\Delta = \{(x,x) : x \in [0,1]\}$ in $\mathcal{A} \otimes \mathcal{B}$

$$\Delta = \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{\infty} \left[\frac{k}{n}, \frac{k+1}{n}\right] \times \left[\frac{k}{n}, \frac{k+1}{n}\right]$$

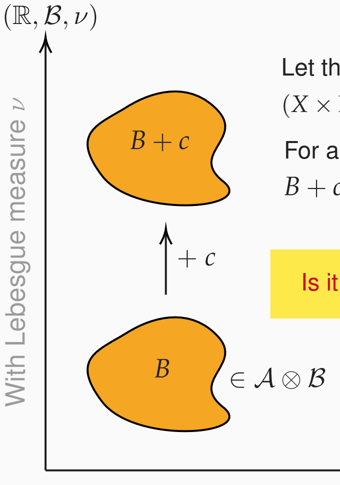
Then, the primitive product measure gives $\pi(\Delta) = +\infty$ and the c.l.d measure gives $\rho(\Delta) = 0$.

INTRODUCTION

Let the product measure space $(X \times \mathbb{R}, \mathcal{A} \otimes \mathcal{B}, \mu \times \nu)$ and a set $B \in \mathcal{A} \otimes \mathcal{B}$ be given.

For any $c \in \mathbb{R}$, define
$B + c \; := \; \{(x, y + c) : \; (x, y) \in B\} \in \mathcal{A} \otimes \mathcal{B}$.

Is it true that $\mu \times \nu \; (B + c) = \mu \times \nu \; (B)$?



$(X, \mathcal{A}, \mu)$

Preliminary Check

Preliminary check

We need that any vertical translate $B + c$ of B is in the product σ-algebra $\mathcal{A} \otimes \mathcal{B}$. Its proof utilises ideas from ...

Construction of a generated σ-algebra

Let X be a set and $\{\emptyset, X\} \subseteq \mathcal{C} \subseteq \mathcal{P}(X)$ be a family of (generating) sets. Let α be an ordinal and λ be a limit ordinal. Define

1. $\mathcal{F}_0 := \mathcal{C}$;
2. $\mathcal{F}_{\alpha+1} := \mathcal{F}_\alpha \cup \left\{\overline{F} : A \in \mathcal{F}_\alpha\right\} \cup \left\{\bigcup_{n\in\mathbb{N}} F_n : F_n \in \mathcal{F}_\alpha\right\}$ and
3. $\mathcal{F}_\lambda := \bigcup_{\alpha<\lambda} F_\alpha.$

Then, $\mathcal{F}_{\omega_1}$ is the generated by $\mathcal{C}$.

The Answer

The Answer to the Problem

The main result

There exists a product measurable space $X \times \mathbb{R}, , \mu \times \nu$ such that for some $c \in \mathbb{R}$ and some measurable set $B \in$, the vertical shift of B by c results in a change in measure. That is, $\mu \times \nu(B) \neq \mu \times \nu(B + c)$.

We will construct a product measure, which utilises the c.l.d. measure.

The Proof

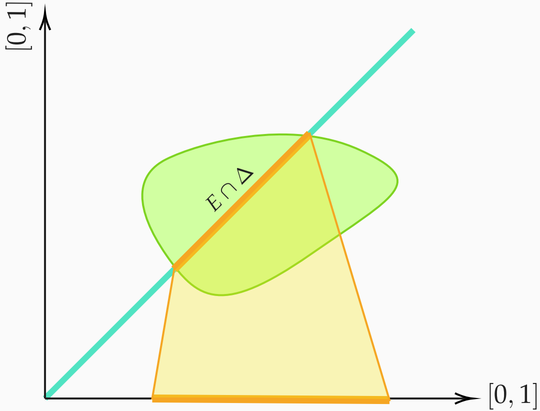
Preparation

Let $\Delta = \{(x,x) : x \in [0,1]\}$ as before. Recall that $\nu : \mathcal{B} \to [0,\infty]$ is the Lebesgue measure on the Borel . Define $f : [0,1] \to [0,1] \times [0,1]$ to be

$$f(x) = (x,x),$$

which is a measurable function on $[0,1]$. Define the set function $\xi : \to [0,1]$ as

$$\xi(E) = \nu(f^{-1}[E \cap \Delta]).$$



$$\xi(E) = \nu\left(f^{-1}(E \cap \Delta)\right)$$

Lebesgue measure of the preimage

Claim: The set function ξ is a measure.

Proof. Trivially, $\xi(\emptyset) = 0$. We now check the σ-additivity property. Let $\{E_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A} \otimes \mathcal{B}$ be a sequence of disjoint sets. Then,

$$\begin{aligned}
\xi\left(\bigcup_{n=0}^{\infty} E_n\right) &= \nu\left(f^{-1}\left[\bigcup_{n=0}^{\infty} E_n \cap \Delta\right]\right) = \nu\left(f^{-1}\left[\bigcup_{n=0}^{\infty} (E_n \cap \Delta)\right]\right) \\
&= \nu\left(\bigcup_{n=0}^{\infty} f^{-1}[E_n \cap \Delta]\right) = \sum_{n=0}^{\infty} \nu\left(f^{-1}[E_n \cap \Delta]\right) \\
&= \sum_{n=0}^{\infty} \xi(E_n).
\end{aligned}$$

■

Proof of the main result

The main result

There exists a product measurable space $X \times \mathbb{R}, , \mu \times \nu$ such that for some $c \in \mathbb{R}$ and some measurable set $B \in$, the vertical shift of B by c results in a change in measure. That is, $\mu \times \nu(B) \neq \mu \times \nu(B+c)$.

Proof. Recall that the c.l.d. product measure is denoted by ρ. Consider the set function $\eta : \mathcal{A} \otimes \mathcal{B} \to [0, \infty]$ given by

$$\eta(E) = \rho(E) + \xi(E).$$

Clearly, η is a measure on $\mathcal{A} \otimes \mathcal{B}$. We claim that η is a product measure.

Case 1. If $\mu(A) < \infty$ and $\nu(B) \leq \infty$, then A has finitely many points since μ is a counting measure. So, $A = \{a_1, ..., a_k\}$ for some $k \in \mathbb{N}$. It holds that

$$A \times B = \{a_1, ..., a_k\} \times B \subseteq \{a_1, ..., a_k\} \times \mathbb{R} = A \times \mathbb{R},$$

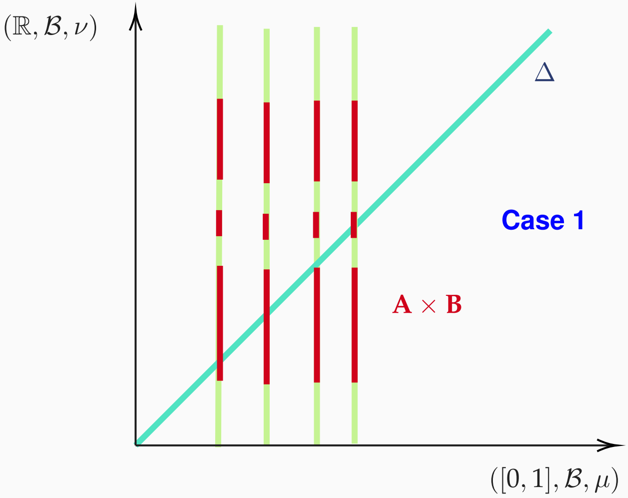
and hence,

$$\Delta \cap (A \times B) \subseteq \Delta \cap (A \times \mathbb{R}) = \{(x, x) : x = a_1, ..., a_k\}.$$

Using monotonicity of measure,

$$\xi(A \times B) \leq \xi(A \times \mathbb{R}) = \nu(f^{-1}[\Delta \cap (A \times \mathbb{R})]) = \nu(\{a_1, ..., a_k\}) = 0.$$

Therefore, $\eta(A \times B) = \rho(A \times B) + \underbrace{\xi(A \times B)}_{0} = \rho(A \times B) = \mu(A)\nu(B)$.



$(\mathbb{R}, \mathcal{B}, \nu)$
Δ
Case 1
$\mathbf{A} \times \mathbf{B}$
$([0,1], \mathcal{B}, \mu)$

Case 2. If $\mu(A) = \infty$ and $\nu(B) > 0$, then $\rho(A \times B) = \mu(A)\nu(B) = \infty$. Therefore,

$$\eta(A \times B) = \underbrace{\rho(A \times B)}_{\infty} + \underbrace{\xi(A \times B)}_{\geq 0} = \underbrace{\rho(A \times B)}_{\infty} = \mu(A)\nu(B).$$

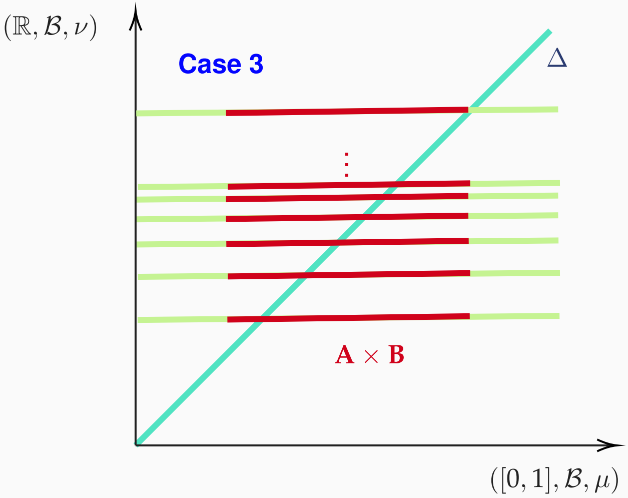
Case 3. If $\mu(A) = \infty$ and $\nu(B) = 0$, then $\rho(A \times B) = \mu(A)\nu(B) = 0$. It holds that

$$f^{-1}[\Delta \cap (A \times B)] \subseteq f^{-1}[\Delta \cap (\mathbb{R} \times B)] = B \cap [0, 1]$$

By monotonicity of measure,

$$\xi(A \times B) = \nu(f^{-1}[\Delta \cap (A \times B)]) \leq \nu(B \cap [0, 1]) \leq \nu(B) = 0.$$

Thus, $\eta(A \times B) = \rho(A \times B) + \xi(A \times B) = 0 = \mu(A)\nu(B)$.


$(\mathbb{R}, \mathcal{B}, \nu)$
Case 3
Δ
$\vdots$
$\mathbf{A} \times \mathbf{B}$
$([0,1], \mathcal{B}, \mu)$

Therefore, η is indeed a product measure.
Furthermore, $\eta(\Delta) = \rho(\Delta) + \xi(\Delta) = 0 + 1 = 1$. However, $\eta(\Delta + 1) = \rho(\Delta + 1) + \xi(\Delta + 1) = 0 + 0 = 0$. ■

The Next Step

The next step

The proof of the construction of non-translation-invariant product measure also implies that there can be infinitely many product measures for a given product measure space. The result provided an example to the following problem.

The number of product measures

Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two measure spaces. Let $(X \times Y, \mathcal{A} \otimes \mathcal{B})$ be their product measurable space.
Then, prove or disprove that the number of product measures on $\mathcal{A} \otimes \mathcal{B}$ is either one or infinite.

Thank you for your attention!

Bibliography

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