The translation invariant product measure problem in non-sigma finite case

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1 Introduction

Let $(X, \mathcal{A}, \mu)$ be a measure space with σ-algebra $\mathcal{A}$ and measure μ, and $(\mathbb{R}, \mathcal{B}, \nu)$ be a real measure space with the Borel σ-algebra $\mathcal{B}$ and the Lebesgue measure ν. Denote their product measure space by $(X \times \mathbb{R}, \mathcal{A} \otimes \mathcal{B}, \mu \times \nu)$, where the product measure is arbitrary. Define a product measure using the definition given by D.H. Fremlin in [1]. The set function $\mu \times \nu : \mathcal{A} \otimes \mathcal{B} \to [0, \infty]$ is a product measure iff it is a measure and for every measurable rectangle $A \times B$, where $A \in \mathcal{A}$ and $B \in \mathcal{B}$, we have

$$\mu \times \nu\,(A \times B) = \mu(A)\nu(B).$$

We shall fix these measure spaces throughout the article.

The Lebesgue measure is know to be translation-invariant. One question we may ask is whether a product measure $\mu \times \nu$ inherits this property in the sense that any shift of a measurable set $B \in \mathcal{A} \otimes \mathcal{B}$ along the real axis does not alter the measure. Formally, we conjecture

1.1 Conjecture. *Let the product measure space $(X \times \mathbb{R}, \mathcal{A} \otimes \mathcal{B}, \mu \times \nu)$ be arbitrary and a set $B \in \mathcal{A} \otimes \mathcal{B}$ be given. For any $c \in \mathbb{R}$, define the vertical shift of B by c as the set*

$$B + c \coloneqq \{(x, y + c) : (x, y) \in B\} \in \mathcal{A} \otimes \mathcal{B}.$$

Then, $\mu \times \nu\,(B + c) = \mu \times \nu\,(B)$.

If the measure space $(X, \mathcal{A}, \mu)$ is σ-finite, then the conjecture holds trivially as the product measure is unique. This unique product measure is obtained through the Carathéodory's extension theorem. As for the non-σ-finite case, we will show that the conjecture is not true.

2 Completely locally determined product measure

Let $(X, \mathcal{A}, \mu) = ([0, 1], \mathcal{B}, \mu)$, where $\mathcal{B}$ is the Borel σ-algebra and μ is the counting measure. Then, we may define the measurable space of $(X, \mathcal{A}, \mu)$ and $(\mathbb{R}, \mathcal{B}, \nu)$. Let

$$\pi(E) = \inf \left\{ \sum_{n=0}^{\infty} \mu \times \nu\,(A_n \times B_n) : \{A_n\}_{n \in \mathbb{N}} \subseteq X, \{B_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}, E \subseteq \bigcup_{n=0}^{\infty} A_n \times B_n \right\}$$

be the product measure space obtained through the Carathéodory's extension theorem.

Another candidate as a product measure is the completely locally determined product measure (c.l.d), which the reader may refer to [1] for further details. The c.l.d product measure is given by

$$\rho(E) = \{\pi(E \cap (A \times B)) : A \in \mathcal{A}, B \in \mathcal{B}, \mu(A) < \infty, \nu(B) < \infty\}.$$

On the diagonal $\Delta = \{(x, x) : x \in [0, 1]\}$, which can be written as

$$\Delta = \bigcap_{n=1}^{\infty} \bigcup_{n=1}^{\infty} \left[\frac{k}{n}, \frac{k+1}{n}\right] \times \left[\frac{k}{n}, \frac{k+1}{n}\right] \in \mathcal{A} \otimes \mathcal{B},$$

we have $\pi(\Delta) = \infty$ and $\rho(\Delta) = 0$.

3 Counterexample measure

We will construct a product measure, which utilises the c.l.d. measure. Let $\Delta = \{(x, x) : x \in [0, 1]\}$ as before. Recall that $\nu : \mathcal{B} \to [0, \infty]$ is the Lebesgue measure on the Borel σ-algebra . Define $f : [0, 1] \to [0, 1] \times [0, 1]$ to be

$$f(x) = (x, x),$$

which is a measurable function on $[0, 1]$. As every preimage $f^{-1}[E]$ of a measurable set $E \in \mathcal{A} \otimes \mathcal{B}$ is measurable in $\mathcal{B}$, we can safely define the set function $\xi : \mathcal{A} \otimes \mathcal{B} \to [0, 1]$ as

$$\xi(E) = \nu(f^{-1}[E \cap \Delta]).$$

We claim that ξ is a measure. Trivially, $\xi(\emptyset) = 0$. We now check the σ-additivity property. Let $\{E_n\}_{n \in \mathbb{N}} \subseteq \mathcal{A} \otimes \mathcal{B}$ be a sequence of disjoint sets. Then,

$$\begin{aligned}
\xi\left(\bigcup_{n=0}^{\infty} {}^{\cdot}\, E_n\right) &= \nu\left(f^{-1}\left[\bigcup_{n=0}^{\infty} {}^{\cdot}\, E_n \cap \Delta\right]\right) \\
&= \nu\left(f^{-1}\left[\bigcup_{n=0}^{\infty} {}^{\cdot}\, (E_n \cap \Delta)\right]\right) \\
&= \nu\left(\bigcup_{n=0}^{\infty} {}^{\cdot}\, f^{-1}[E_n \cap \Delta]\right) \\
&= \sum_{n=0}^{\infty} \nu\left(f^{-1}[E_n \cap \Delta]\right) \\
&= \sum_{n=0}^{\infty} \xi(E_n).
\end{aligned}$$

That is, ξ is indeed a measure on $\mathcal{A} \otimes \mathcal{B}$. We now proceed to the main result.

3.1 Theorem. *There exists a product measurable space $(X \times \mathbb{R}, \mathcal{A} \otimes \mathcal{B}, \mu \times \nu)$ such that for some $c \in \mathbb{R}$ and some measurable set $B \in \mathcal{A} \otimes \mathcal{B}$, the vertical shift of B by c results in a change in measure. That is, $\mu \times \nu(B) \neq \mu \times \nu(B + c)$.*

Proof. We assume the notions previously defined in this section. Consider the set function $\eta : \mathcal{A} \otimes \mathcal{B} \to [0, \infty]$ given by

$$\eta(E) = \rho(E) + \xi(E).$$

Since η is a sum of measures on $\mathcal{A} \otimes \mathcal{B}$, we have that η is also a measure on $\mathcal{A} \otimes \mathcal{B}$. We remain to prove that η is a product measure. For this, we consider the following cases for a measurable rectangle $A \times B$, where $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

- If $\mu(A) < \infty$ and $\nu(B) \leq \infty$, then A has finitely many points since μ is a counting measure. So, $A = \{a_1, ..., a_k\}$ for some $k \in \mathbb{N}$. It holds that

$$A \times B = \{a_1, ..., a_k\} \times B \subseteq \{a_1, ..., a_k\} \times \mathbb{R} = A \times \mathbb{R},$$

  and hence,

$$\Delta \cap (A \times B) \subseteq \Delta \cap (A \times \mathbb{R}) = \{(x, x) : x = a_1, ..., a_k\}.$$

  Using monotonicity of measure,

$$\xi(A \times B) \leq \xi(A \times \mathbb{R}) = \nu(f^{-1}[\Delta \cap (A \times \mathbb{R})]) = \nu(\{a_1, ..., a_k\}) = 0.$$

  Therefore, $\eta(A \times B) = \rho(A \times B) + \underbrace{\xi(A \times B)}_{0} = \rho(A \times B) = \mu(A)\nu(B)$.

- If $\mu(A) = \infty$ and $\nu(B) > 0$, then $\rho(A \times B) = \mu(A)\nu(B) = \infty$. Therefore,

$$\eta(A \times B) = \underbrace{\rho(A \times B)}_{\infty} + \underbrace{\xi(A \times B)}_{\geq 0} = \underbrace{\rho(A \times B)}_{\infty} = \mu(A)\nu(B).$$

- If $\mu(A) = \infty$ and $\nu(B) = 0$, then $\rho(A \times B) = \mu(A)\nu(B) = 0$. It holds that

$$f^{-1}[\Delta \cap (A \times B)] \subseteq f^{-1}[\Delta \cap (\mathbb{R} \times B)] = B \cap [0, 1]$$

  By monotonicity of measure,

$$\xi(A \times B) = \nu(f^{-1}[\Delta \cap (A \times B)]) \leq \nu(B \cap [0, 1]) \leq \nu(B) = 0.$$

  Thus, $\eta(A \times B) = \rho(A \times B) + \xi(A \times B) = 0 = \mu(A)\nu(B)$.

Therefore, η is indeed a product measure. Furthermore, $\eta(\Delta) = \rho(\Delta) + \xi(\Delta) = 0 + 1 = 1$. However, $\eta(\Delta + 1) = \rho(\Delta + 1) + \xi(\Delta + 1) = 0 + 0 = 0$. ■

References

[1] David H. Fremlin. *Measure Theory*. Vol. 2. Colchester, UK: Torres Fremlin, 2001. isbn: 978-0-9538129-7-4.