Helly-type theorems and boxes

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Introduction

- If property A holds for any **subfamily** of a family of sets $\mathcal{F}$ that is of a **given finite size** h and property, then some property B holds for the whole **family** $\mathcal{F}$ of **arbitrary finite size** n
- Helly number: h (minimal)

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- **Quantitative Volume Theorem**
  - convex sets
  - lower bound on volume of intersection d^{-2d^2}, (d^{-2d})
  - Helly number: $2d$

Colorful Volume Theorem

- combination of **Colorful Helly** and **Quantitative Volume Theorem**
- convex bodies
- $3d$, $d(d+3)/2$ families
- lower bound on volume of intersection $c^{d^2} d^{-5d^2/2}$, 1
- Helly number: $2d$
- **John's theorem** $\rightarrow$ lexicographic ordering of ellipsoids.

Piercing Boxes

Definition: A set P **pierces** a family of sets $\mathcal{F}$ if for any set $S \in \mathcal{F}$ there is an element $p \in P$ such that $p \in S$. If $|P| = n$, then $\mathcal{F}$ is n-**pierceable**

Theorem (Danzer, Grünbaum). *If $h = h(d, n)$ is the smallest positive integer such that for any finite family $\mathcal{F}$ of axis-parallel boxes in $\mathbb{R}^d$ every h-tuple from $\mathcal{F}$ is n-pierceable implies that $\mathcal{F}$ is n-pierceable then following are the values of h:*

$$h(d,1) = 2$$

$$h(1,n) = n+1$$

$$h(d,2) = \begin{cases} 3d & : \quad 2 \mid d \\ 3d-1 : & \quad 2 \nmid d \end{cases}$$

$$h(2,3) = 16$$

$$h(d,n) = \aleph_0 \quad n \geq 3, (d,n) \neq (2,3)$$

Piercing Boxes

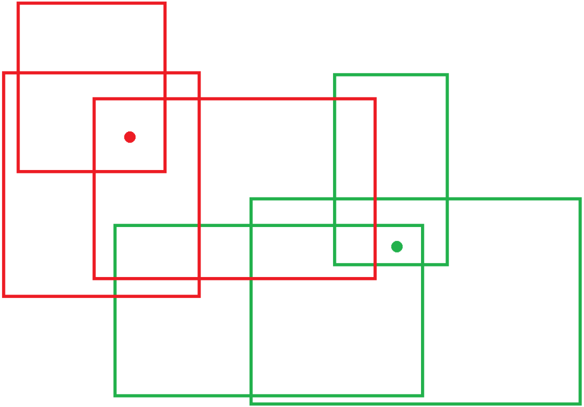

Figure: 2-piercing a family of 2-dimensional boxes

Punching holes into boxes

- combination of piercing and volume theorems $\rightarrow$ punching holes

Definition: For volume set $\mathcal{V} \subset \mathbb{R}_{>0}$ and enumeration $\nu : \mathcal{V} \rightarrow \mathbb{Z}_{>0}$ a family a of d-dimensional boxes $\mathcal{F} = \{\prod_{j=1}^{d} [a_{ij}, b_{ij}] : i \in \mathcal{I}\}$ for some index set $\mathcal{I}$ is $\mathcal{V}, \nu$-**punchable** if there is a family of d-dimensional boxes $\mathcal{H}$ such that

$$\forall v \in \mathcal{V} \quad \nu(v) = |\{H \in \mathcal{H} : \mathrm{Vol}(H) = v\}| \tag{1}$$

$$\forall B \in \mathcal{F} \quad \exists H \in \mathcal{H} \quad H \subset B \tag{2}$$

If (2) holds for some families of boxes $\mathcal{F}, \mathcal{H}$ then $\mathcal{H}$ **punches** $\mathcal{F}$. If the volume set has 1 element $\mathcal{V} = \{v\}$ and $\nu(v) = n$ and there is a family $\mathcal{H}$ for which (1),(2) hold, then $\mathcal{F}$ is n-**punchable**.

Results

Statement 1: *For a family of intervals* $\mathcal{F} = \{I_i = [a_i, b_i] \subset \mathbb{R} : i \in \mathcal{I}\}$ *if any subfamily of* $n+1$*-elements is n-punchable, then* $\mathcal{F}$ *is n-punchable.*
Statement 2: *For any dimension* d *there is a family* $\mathcal{F}$ *of* d*-dimensional boxes such that any* $3d$*-tuple is 2-punchable, but* F *is only* $\{\varepsilon\}, 2$*-punchable for any* $\varepsilon > 0$.
Corollary: *In any Helly-type theorem about 2-punching boxes, the Helly number has to be at least* $3d+1$.

Proof of Statement 1

Minkowski difference

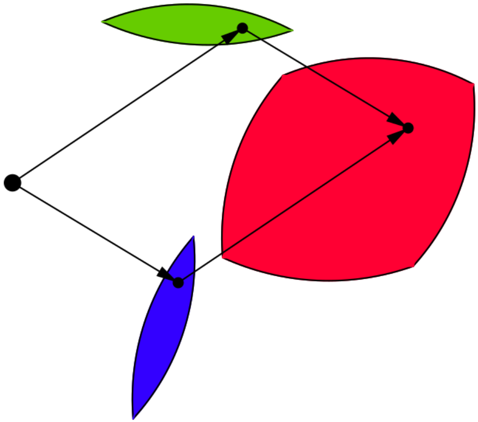

Figure: Minkowski addition, difference

n-piercing intervals

Proof of Statement 2

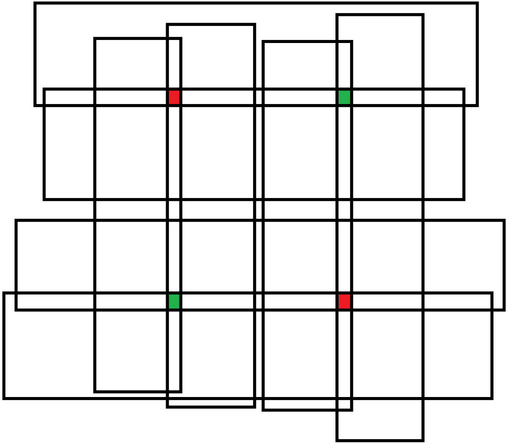

Figure: Construction for $d = 2$. Punching pairs are of the same color.

Proof of Statement 2

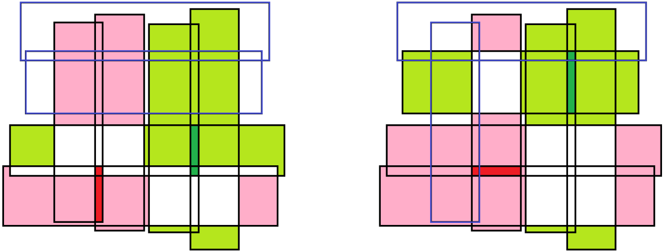

Figure: Any 6-tuple can be punched by 2 big boxes.

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- **Definition:** An intersection-connected family of d-dimensional boxes is **intersection-punchable** if there is a family of d-dimensional boxes $\mathcal{H}$ which **punches** $\mathcal{F}$ such that

$$\forall B \in \mathcal{F} \quad \exists H \in \mathcal{H} \quad \exists B' \in \mathcal{F} \setminus \{B\} \quad H \subset B \cap B' \qquad (3)$$

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- **Definition:** A family of d-dimensional boxes is n-**sum**-s-**punchable** if there is a family of d-dimensional boxes $\mathcal{H}$ that **punches** $\mathcal{F}$ such that

$$\sum_{H \in \mathcal{H}} \text{Vol}(H) = s \tag{4}$$

$$|\mathcal{H}| = n \tag{5}$$

References

Damásdi, G., Viktória Földvári, V. & Naszódi,M. (2020). Colorful Helly-type theorems for the volume of intersections of convex bodies. Journal of Combinatorial Theory.

Chakraborty. S., Ghosh, A. & Nandi, S. (2022). Coloful Helly Theorem for Piercing Boxes with Two Points.