Topics in Ring Theory

These notes were written as the continuation of the notes for the course Rings and Algebras. As in the previous course, no originality in results or presentation is claimed although no particular textbook is followed in the presentation.

The course consists of two parts. The first part gives an introduction to **homological algebra** by developing and describing in full detail
- the construction of derived functors;
- in particular the construction of the functors Ext^n_A and to a lesser extent the functors Tor^A_n;
- connection of the Ext functors and extensions of modules, furthermore some applications of the Yoneda-product;
- the most important homological dimensions, together with some results and well-known conjectures.

The second part of the course can be thought of as "applied homological algebra", but instead of standard applications in topology, commutative algebra or group theory, we shall concentrate on the developments in **representation theory of associative algebras**, in particular
- construction and properties of almsot split sequences (also known as Auslander–Reiten sequences);
- the construction, role and properties of the Auslander–Reiten graphs of algebras;
- representation type and Auslander's proof of the first Brauer–Thrall conjecture.

Literature

The following textbooks can be used as a supplement for the course:

A) Homological algebra

1) J. Rotman: *An Introduction to Homological Algebra* (Academic Press, Pure and Applied Mathematics, 1979, Second edition: Springer, Universitext 2008.)
2) C. Weibel: An Introduction to Homological Algebra (CUP, Cambridge Studies in Advanced Mathematics 38, 1994)
3) S. MacLane: Homology (Grundlehren der Mathematischen Wissenschaften 114, Springer 1964, Classics in Mathematics 1994)
4) Yu. Drozd, V. Kirichenko: *Finite Dimensional Algebras* (Springer 1994)

Representation theory of associative algebras

5) I. Assem, D. Simson, A. Skowroski: *Elements of the Representation Theory of Associative Algebras* (CUP, London Mathematical Society Student Texts 65, 2006)
6) M. Auslander, I. Reiten, S. Smalø: *Representation Theory of Artin Algebras* (CUP, Cambridge Studies in Advanced Mathematics 36, 1995)
7) R. Schiffler: *Quiver Representations* (Springer, CMS Books in Mathematics, 2014)
8) K. Erdmann, T. Holm: *Algebras and Representation Theory* (Springer, Undergraduate Mathematics Series, 2018)

Some potential topics for continuing the course:

1) a) group cohomology; b) spectral sequences; c) derived categories…
2) a) hereditary algebras, Coxeter transformations; b) tame algebras…

Throughout the course we shall build upon the content of the course *Rings and Algebras.* The most important concepts of the previous course which are used in the present one: semisimplicity of modules, semisimple rings, radical of a module and of a ring, injective and projective modules, injective hull, projective cover, semiperfect and Artinian rings, categories and functors, the Hom and tensor functors, exact and half exact sequences of modules.

1. Derived functors

We start with recalling the notion of a complex of modules and the category of complexes.

Definition 1.1. A *complex* or *chain complex* in R-Mod is a (doubly) infinite half exact sequence of modules:

$$\cdots \to M_{n+1} \xrightarrow{d_{n+1}} M_n \xrightarrow{d_n} M_{n-1} \to \cdots$$

Thus here $M_n \in R$-Mod is an R-module, $d_n \in \mathrm{Hom}_{R\text{-Mod}}(M_n, M_{n-1})$ is a module homomorphism and $d_n d_{n+1} = 0$ for every $n \in \mathbb{Z}$. Note that the last condition (the half exactess) is equivalent to the condition that $\mathrm{Im}\, d_{n+1} \subseteq \mathrm{Ker}\, d_n$ and sometimes it is denoted symbolically by $d^2 = 0$. The complex above is usually denoted by $(M_\bullet, d_\bullet)$. The maps d_n are called the *boundary maps* or *differentials* of the complex $(M_\bullet, d_\bullet)$.

Definition 1.2. Let R be a ring. Then $C(R)$ denotes the *category of chain complexes* over R where the morphisms are chain maps, i. e. maps $f_\bullet : (M_\bullet, d_\bullet) \to (N_\bullet, \partial_\bullet$ with

$$\begin{array}{lccccccccc}
(M_\bullet, d_\bullet): & \cdots \longrightarrow & M_{n+1} & \xrightarrow{d_{n+1}} & M_n & \xrightarrow{d_n} & M_{n-1} & \longrightarrow \cdots \\
f_\bullet \downarrow & & f_{n+1}\downarrow & \# & f_n \downarrow & \# & f_{n-1}\downarrow & \\
(N_\bullet, \partial_\bullet): & \cdots \longrightarrow & N_{n+1} & \underset{\partial_{n+1}}{\longrightarrow} & N_n & \underset{\partial_n}{\longrightarrow} & N_{n-1} & \longrightarrow \cdots
\end{array}$$

Another, more precise notation would be $C(R\text{-Mod})$. In general, if $\mathcal{K}$ is a full subcategory of R-Mod then $C(\mathcal{K})$ will stand for the subcategory of $C(R)$ whose objects are complexes $(M_\bullet, d_\bullet)$ with $M_i \in \mathrm{Ob}\, CalK$. For example if $\mathcal{P}_R$ denotes the subcategory of projective R-modules then $C(\mathcal{P}_R)$ denotes the category of complexes formed by projective modules.

Definition 1.3. The *n-th homology module* of a complex $(M_\bullet, d_\bullet) \in C(R)$ is defined as

$$H_n(M_\bullet) = \mathrm{Ker}\, d_n / \mathrm{Im}_{d_{n+1}}\,.$$

Note that $H_n(M_\bullet) \leq M_n / \mathrm{Im}\, d_{n+1}$, so the n-th homology module is a slice in M_n.

Example 1.4. a) Let $M \in R$-Mod and consider a *projective resolution* of M:

$$\mathcal{P}(M)^*_\bullet : \qquad \cdots \to P_2 \to P_1 \to P_0 \to M \to 0 \to \cdots$$

with sequence exact, and P_i projective. Then the homology modules are all zero:

$$H_n(\mathcal{P}(M)^*_\bullet) = 0 \text{ for every } n \in \mathbb{Z}.$$

Complexes with all homologies equal to 0 (i. e. exact sequences) are also called *acyclic complexes.*

b) If we delete from the complex $\mathcal{P}(M_\bullet)^*$ the module M then we get the so called *deleted complex* of a resolution of M:

$$\mathcal{P}(M)_\bullet : \qquad \cdots \to P_2 \to P_1 \to P_0 \to 0 \to \cdots$$

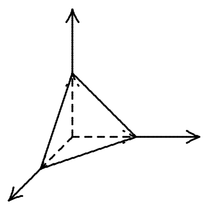
Hence $\mathcal{P}(M)_\bullet$ is a complex with P_i-s projective and

$$H_n(\mathcal{P}(M)_\bullet) = \begin{cases} 0 & \text{for } n \neq 0; \\ m & \text{for } n = 0. \end{cases}$$

Thus the compex $\mathcal{P}(M)_\bullet$ encodes the module M as a homology. The advantage of the deleted resolution complex $\mathcal{P}(M)_\bullet$ over the full resolution complex $\mathcal{P}(M)^*_\bullet$ is that the deleted complex is an object in $C(\mathcal{P}_R)$, the category of projective complexes and this category has many advantages over $C(R)$. Note that very often by projective resolution we shall mean the deleted resolution of M.

c) An example from topology. Let X be a topological space and define then n-simplex Δ_n as the complex hull of the vectors $\{\mathbf{0}, \mathbf{e}_1 \ldots, \mathbf{e}_n\} \subseteq \mathbb{R}^n$ with $\mathbf{e}_i$ being the i-th element of the standard basis of $\mathbb{R}^n$.

Δ_3 :



Then we take

$$S_n(X) = \langle \sigma \mid \sigma : \Delta_n \to X \text{ continuous} \rangle$$

as the free Abelian group generated by all σ-s. Such an element σ in $S(X)$ is called an *n-simplex in X*. Thus for example:

$$S_0(X) = \langle \text{points of } X \rangle$$
$$S_1(X) = \langle \text{curves in } X \rangle$$

If $\sigma \in S_n(X)$ is an n-simplex in X, then we take

$$\partial_n(\sigma) = \sum_{\subset=0}^{n} (-1)^i \sigma_i,$$

where σ_i is the *i-th face* of σ. The map ∂_n can be extended uniquely to a morphism $\partial_n : S_n(X) \to S_{n-1}(X)$ in AB. One can show that $\sigma_{n-1}\sigma_n = 0$, hence we get a complex of Abelian groups

$$S_\bullet(X) : \qquad \cdots \longrightarrow S_{n+1}(X) \xrightarrow{\partial_{n+1}} S_n(X) \xrightarrow{\partial_n} S_{n-1}(X) \xrightarrow{\partial_{n-1}} \cdots$$

Thus $S_\bullet(X)$ is an element of $\mathrm{Ob}(C(\mathrm{AB}))$ or $C(\mathbb{Z})$. It is called the *singular complex of* X and $H_n(S_\bullet(X))$ the *n-th singular homology of* X.

d) Let $F : R\text{-Mod} \to S\text{-Mod}$ be an additive functor and let $(M_\bullet, d_\bullet)$ be a complex. Then by applying F to the elements of the complex and the boundary map, we get a complex $(F_\bullet(M_\bullet), F_\bullet(d_\bullet)) \in \mathrm{Ob}\, C(S)$: the additivity of F implies that the half exactness of the complex $M_\bullet$ is preserved for the obvious sequence $F_\bullet(M_\bullet)$. Actually, the functor $F : R\text{-Mod} \to S\text{-Mod}$ defines the obvious correspondence $F_\bullet : C(R) \to C(S)$ and the functoriality of F means that $F_\bullet$ takes chain maps to chain maps. Thus $F_\bullet$ is a functor and it is easy to show that it is also additive. – The same thing holds if we start with a contravariant functor.

Proposition 1.5. $H_n : C(R) \to R\text{-Mod}$ *is a functor for every* $n \in \mathbb{Z}$.

Proof. We have to define the effect of H_n on morphisms, i.e. on chain maps. So let us consider a chain map $f_\bullet \in \mathrm{Hom}(M_\bullet, N_\bullet)$:

$$\begin{array}{ccccccccccc}
(M_\bullet, d_\bullet): & & \cdots & \longrightarrow & M_{n+1} & \xrightarrow{d_{n+1}} & M_n & \xrightarrow{d_n} & M_{n-1} & \longrightarrow & \cdots \\
{\scriptstyle f_\bullet}\downarrow & & & & {\scriptstyle f_{n+1}}\downarrow & \# & {\scriptstyle f_n}\downarrow & \# & {\scriptstyle f_{n-1}}\downarrow & & \\
(N_\bullet, \partial_\bullet): & & \cdots & \longrightarrow & N_{n+1} & \xrightarrow[\partial_{n+1}]{} & N_n & \xrightarrow[\partial_n]{} & N_{n-1} & \longrightarrow & \cdots
\end{array}$$

and let us now take an element $x \in \mathrm{Ker}\, d_n$. Then we take $[x] = x + \mathrm{Im}\, d_{n+1}$ the image of x in $H_n(M_\bullet)$. We define:

$$H_n(f_\bullet)[x] = [f_n(x)].$$

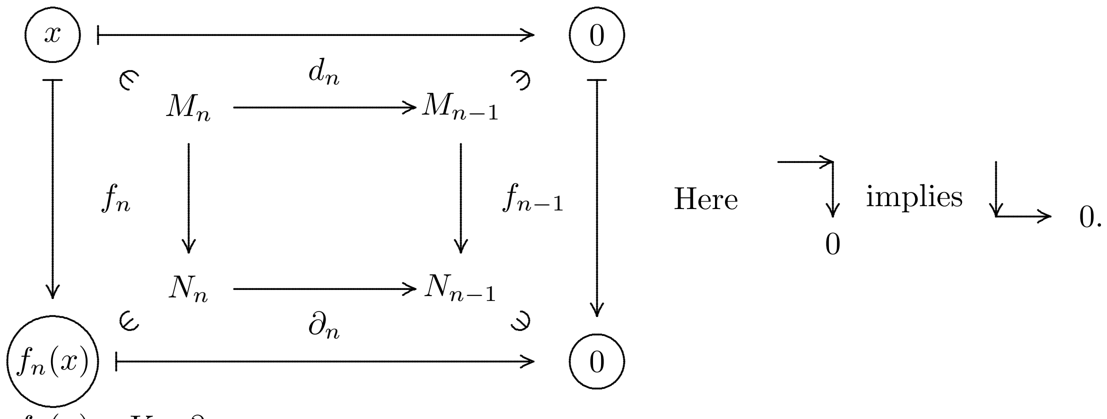
Then we have to show three things:

a) $f_n(x) \in \mathrm{Ker}\, \partial_n$; this will show that our definition of $H_n(f_\bullet)[x]$ is indeed an element of $H_n(N_\bullet)$;

b) $H_n(f_\bullet)$ is well defined, i. e. $H_n(f_\bullet)[x]$ is independent of the particular choice of $x \in [x]$;

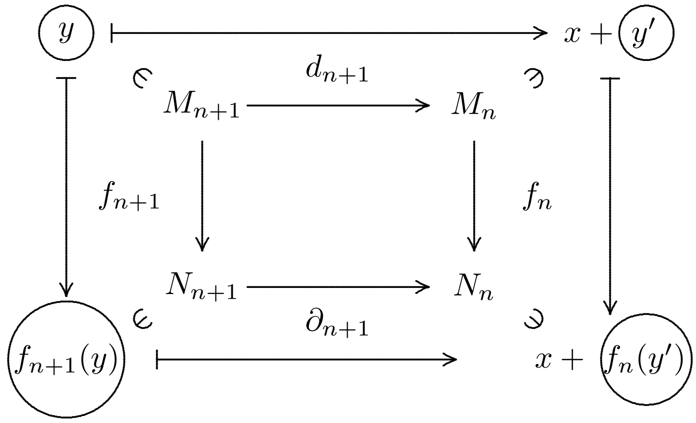
c) H_n is functorial, i. e. H_n preserves $\circ$ and 1.

a) To check the first property, we have the following diagram:



Thus $f_n(x) \in \mathrm{Ker}\, \partial_n$.

b) To show that $H_n(f_\bullet)[x]$ is does not depend on the choice of $x \in [x]$ let us consider the following diagram:



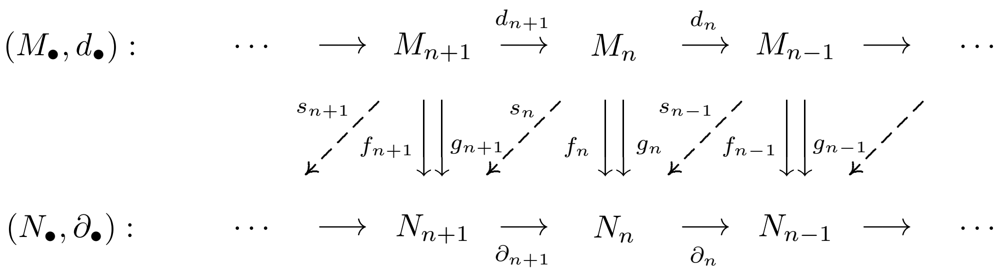
Here if $y' \in \operatorname{Im} d_{n+1}$, thus $[x] = [x + y']$ in $H_n(M_\bullet)$ then there exists $y \in M_{n+1}$ with $d_{n+1}(y) = y'$. But then the commutativity of the diagram implies that $f_n(y') = f_n(d_{n+1}(y)) = \partial_{n+1}(f_{n+1}(y))$, hence $[f_n(x)] = [f_n(x) + f_n(y')] = [f_n(x + y')]$. Thus $[f_n(x)] \in H_n(N_\bullet)$ does not depend on the particular choice of $x \in [x]$.

c) This is clear from the definition of $H_n(f_\bullet)$.

This finishes the proof, showing the H_n is indeed a functor. □

The following definition is motivated by topology, however we shall skip the details and present it for general complexes.

Definition 1.6. Let $(M_\bullet, d_\bullet)$ and $(N_\bullet, \partial_\bullet)$ be two complexes, furthermore let $f_\bullet, g_\bullet \in \operatorname{Hom}_{C(R)}(M_\bullet, N_\bullet)$ be chain maps. Then $f_\bullet$ and $g_\bullet$ are called *homotopic* (denoted by $f_\bullet \sim g_\bullet$) if for every $n \in \mathbb{Z}$ there exists a map $s_n \in \operatorname{Hom}_R(M_n, N_{n+1})$ so that $f_n - g_n = s_{n-1}d_n + \partial_{n+1}s_n$ for every $n \in \mathbb{Z}$:



Thus in a simplified notation and simplified diagram we have:

$$f - g = sd + \partial s \quad \text{or} \quad \downarrow - \downarrow = \nearrow + \swarrow$$

$f_\bullet$ is called 0-*homotopic* if $f_\bullet \sim 0_\bullet$, i. e. $f_\bullet = sd + \partial s$.

The sequence of homomorphisms s_n is called a *chain homotopy* between $f_\bullet$ and $g_\bullet$. Note that a chain homotopy is NOT a chain map between two complexes, i. e. it is not a morphism in the category $C(R)$.

Proposition 1.7. *$\sim$, i. e. homotopy of chain maps defines an equivalence relation on $\mathrm{Hom}_{C(R)}(M_\bullet, N_\bullet)$.*

Proof. 1) $f_\bullet \sim f_\bullet$ holds with chain homotopy $s_n = 0$.

2) If $f_\bullet \sim g_\bullet$ with chain homotopy maps s_n, then $g_\bullet \sim f_\bullet$ with chain homotopy maps $t_n = -s_n$.

3) If $f_\bullet \sim g_\bullet$ with chain homotopy s_n and $g_\bullet \sim h_\bullet$ with chain homotopy t_n then $f_\bullet \sim h_\bullet$ with chain homotopy $u_n = s_n + t_n$. □

We should note that $\sim$ is a congruence relation on the group $\mathrm{Hom}_{C(R)}(M_\bullet, N_\bullet)$, i. e. it is compatible with the addition since

$$\left\{f_\bullet \in \mathrm{Hom}_{C(R)}(M_\bullet, N_\bullet) \,|\, f_\bullet \sim 0_\bullet\right\} \leq \mathrm{Hom}_{C(R)}(M_\bullet, N_\bullet)$$

This can be verified using the additivity of the definition of homotopy.

Definition 1.8. We define the *homotopy category* $K(R)$ of chain complexes over the ring R as follows:

$$\mathrm{Ob}\, K(R) = \mathrm{Ob}\, C(R) \quad \textit{and} \quad Hom_{K(R)}(M_\bullet, N_\bullet) = Hom_{C(R)}(M_\bullet, N_\bullet)/\sim\ .$$

Isomorphisms in the homotopy category can be characterized as follows:

Definition 1.9. Two complexes $M_\bullet, N_\bullet \in C(R)$ are *homotopically equivalent* (denoted by $M_\bullet \sim N_\bullet$) if they are isomorphic in $K(R)$, i. e. if there exist morphism $f_\bullet \in \mathrm{Hom}_{C(R)}(M_\bullet, N_\bullet)$ and $g_\bullet \in \mathrm{Hom}_{C(R)}(N_\bullet, M_\bullet)$ such that

$$1)\ g_\bullet f_\bullet \sim 1_{M_\bullet}; \qquad 2)\ f_\bullet g_\bullet \sim 1_{N_\bullet}.$$

Remark 1.10. In what follows we shall invesigate properties of some special additive functors on R-Mod. Note that any such functor has a natural "extensions"to $C(R)$ and $K(R)$. Namely if $F : R\text{-Mod} \to S\text{-Mod}$ is an additive functor then we get functors $F_\bullet : C(R) \to (S)$ and $\tilde{F}_\bullet : K(R) \to K(S)$ in an obvious way. Observe that the additivity of F implies that F not only has an extension to $C(R)$ but this extension actually lives also in $K(R)$.

Proposition 1.11. *Homotopic chain maps induce identical morphisms on homologies, i. e. if $f_\bullet \sim g_\bullet$ for some $f_\bullet, g_\bullet \in \mathrm{Hom}_{C(R)}(M_\bullet, N_\bullet)$ then for every $n \in \mathbb{Z}$ we get $H_n(F_\bullet) = H_n(g_\bullet)$.*

Proof. Suppose $f_\bullet$ and $g_\bullet$ are homotopic maps with maps s_n giving the chain homotopy. Then for $[x] \in H_n(M_\bullet)$ we have:

$$H_n(f_\bullet)[x] = [f_n(x)] = [g_(x) + \underbrace{\partial_{n+1}s_n(x)}_{\in \operatorname{Im}\partial_{n+1}} + \underbrace{s_{n-1}d_n(x)}_{=0 \text{ since } x\in\operatorname{Ker} d_n}] =$$
$$= [g_n(x)] = H_n(g_\bullet)[x].$$

□

Definition 1.12. If $f_\bullet, g_\bullet \in \operatorname{Hom}_{C(R)}(M_\bullet, N_\bullet)$, then we call them *homological* (and this is denoted by $f_\bullet \equiv g_\bullet$) if $H_n(f_\bullet) = H_n(g_\bullet)$ for every $n \in \mathbb{Z}$.

Corollary 1.13. *If two chain maps are homotopic then they are also homological.*

We should note here that the concept of homotopic maps is indeed stronger and in some sense it is a better concept: for example it is preserved by additive functors, furthermore it gives a proper reason for similar behaviour on homologies. As a matter of fact, two maps can be homological for no obvious reasons, too.

The previous statement shows that the homology functors can be defined at the level of homotopy categories. Namely the following holds:

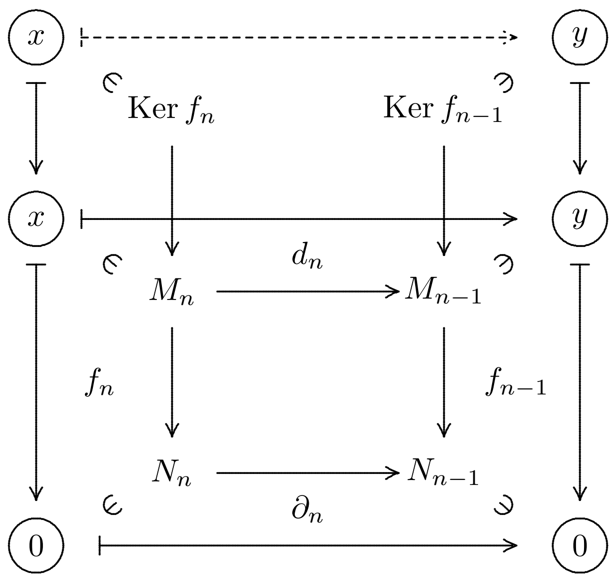
Corollary 1.14. *Homotopically equivalent complexes have isomorphic homologies. (Thus H_n is a functor from $K(R)$ to R-*Mod*.)*

Proof. Suppose $M_\bullet \sim N_\bullet$ and $f_\bullet$ and $g_\bullet$ are the maps showing the equivalence. Thus $g_\bullet f_\bullet \sim 1_{M_\bullet}$ implying that $H_n(g_\bullet)H_n(f_\bullet) = H_n(g_\bullet f_\bullet) = H_n(1_{M_\bullet}) = 1_{H_n(M_\bullet)}$ because H_n is a functor hence preserves composition of maps and the identity. Similarly $H_n(f_\bullet)H_n(g_\bullet) = 1_{H_n(N_\bullet)}$. Thus the maps $H_n(f_\bullet)$ and $H_n(g_\bullet)$ are inverse isomorphisms. □

In order to speak about exact sequence of cmplexes, we need to verify that we can define the kernel and the image of chain maps. Thus we need the following:

Proposition 1.15. *Suppose $f \in \operatorname{Hom}_{C(R)}(M_\bullet, N_\bullet)$. Then we get the following complexes:* $(\operatorname{Ker} f_\bullet, d|_{\operatorname{Ker} f_\bullet}) \in C(R)$ *and* $(\operatorname{Im} f_\bullet, \partial|_{\operatorname{Im} f_\bullet}) \in C(R)$.

Proof. What one has to prove for the kernel sequence is that $d_n(\operatorname{Ker} f_n) \subseteq \operatorname{Ker} f_{n+1}$. But this can be seen from the following diagram:



Here $y \in \operatorname{Ker} f_{n-1}$ holds indeed.

Clearly the sequence formed by the kernel terms is half exact.

One can show the analogue statment for the images, too. □

Remark. Since the 0 sequence obviously acts as a zero object, we can speak about short exact sequences of complexes.

Theorem 1.16. (Long exact sequence of homologies.) *Let*

$$0_\bullet \to M'_\bullet \xrightarrow{f_\bullet} M \xrightarrow{g_\bullet} M'' \to 0_\bullet$$

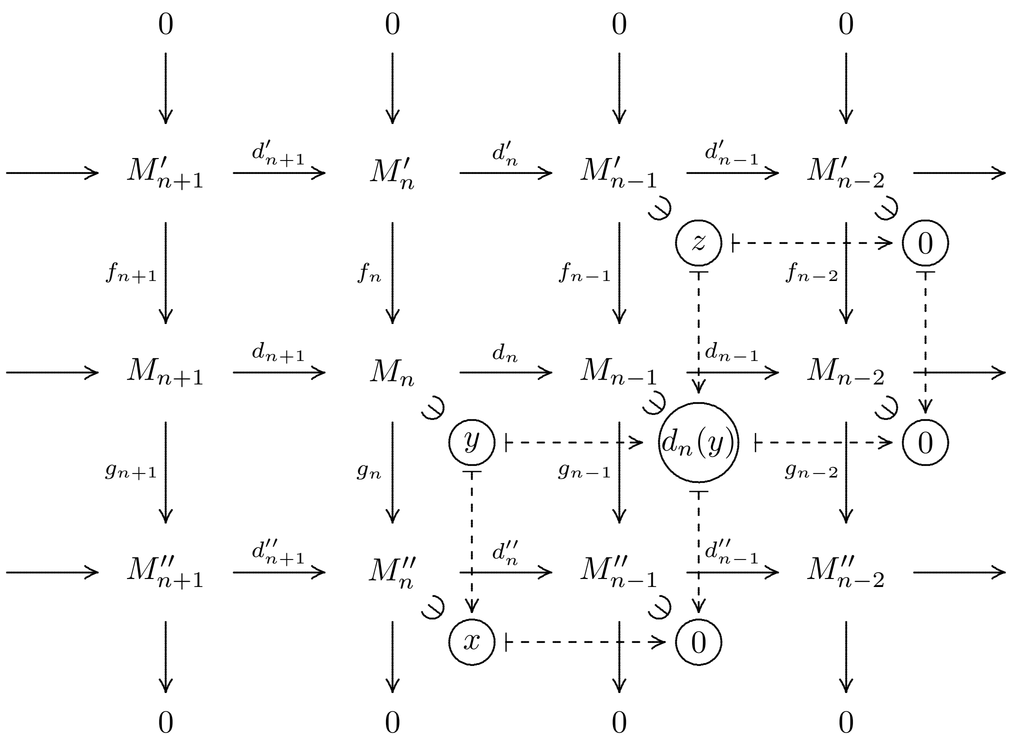
be a short exact sequence in $C(R)$. Then for every $n \in \mathbb{Z}$ there exists a natural map $\partial = \partial_n : H_n(M_\bullet") \to H_{n-1}(M'_\bullet)$ for every $n \in \mathbb{Z}$ so that the following sequence of homologies is exact:

$$\text{(LEX)} \qquad \cdots \xrightarrow{\partial_{n+1}} H_n(M'_\bullet) \xrightarrow{H_n(f_\bullet)} H_n(M_\bullet) \xrightarrow{H_n(g_\bullet)} H_n(M''_\bullet) \xrightarrow{\partial_n} H_{n-1}(M'_\bullet) \xrightarrow{H_{n-1}(f_\bullet)} H_{n-1}(M_\bullet) \xrightarrow{H_{n-1}(g_\bullet)} H_{n-1}(M''_\bullet) \xrightarrow{\partial_{n-1}} \cdots$$

The maps ∂_n are called conecting homomorphisms.

Proof. First we give the construction of the maps $\partial_n : H_n(M_\bullet") \to H_{n-1}(M'_\bullet)$. The method we are going to use throughout the proof is called *diagram chasing*: we follow the path of elements through the diagram by taking images or preimages along the available maps.

One should observe that the homologies $H_n(M_\bullet$”) and $H_{n-1}(M'_\bullet)$ “live” in the modules M''_n and M'_{n-1}. So we start with an element $x \in \operatorname{Ker} d''_n \subseteq M''_n$ and try to find a way to get from x to an element $z \in \ker d'_{n-1} \subseteq M'_{n-1}$. So let us consider the following diagram:



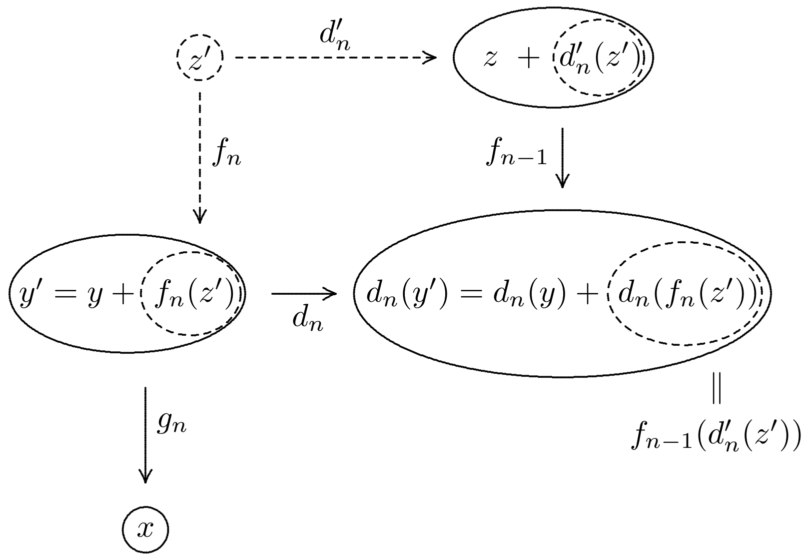
Thus we start with $[x] \in H_n(M''_\bullet)$.

- Here $x \in \operatorname{Ker} d''_n \subseteq M''_n$ and $[x] = x + \operatorname{Im} d''_{n+1}$.
- Since g_n is surjective, we can choose $y \in M_n$ such that $g_n(y) = x$.
- The map d_n takes y into $d_n(y) \in M_{n-1}$.
- If we apply the map g_{n-1} to this element, the commutativity of the diagram implies that $g_{n-1}(d_n(y)) = d''_n(g_n(y)) = d''_n(x) = 0$.
- This gives us that $d_n(y) \in \operatorname{Ker} g_{n-1}$.
- Since the vertical sequences are exact, this implies that $d_n(y) \in \operatorname{Im} f_{n-1}$, thus we can find $z \in M'_{n-1}$ such that $f_{n-1}(z) = d_n(y)$.
- Since f_{n-1} is injective, z is unique.
- Observe also that the commutativity of the diagram implies that $f_{n-2}(d'_{n-1}(z)) = d_{n-1}(f_{n-1}(z)) = d_{n-1}(d_n(y)) = 0$, since $d_{n-1}d_n = 0$.
- Since f_{n-2} is injective, this gives that $d'_{n-1}(z) = 0$, hence $z \in \operatorname{Ker} d'_{n-1}$, hence $[z] = z + \operatorname{Im} d'_n \in H_{n-1}(M\bullet')$ is well-defined.
- Take $\partial_n([x]) = [z]$.

Clearly we get a correspondende $H_n(M''_\bullet) \to H_{n-1}(M'_\bullet)$, but in several steps we made an arbitrary selection. In order to prove the statement of the theorem, we have to show that:

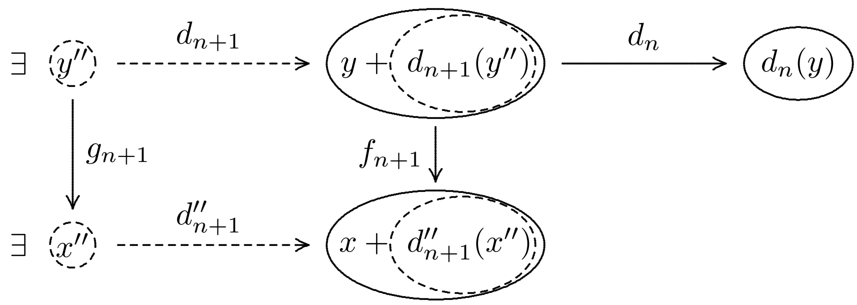
1) ∂_n is well-defined, that is: a) the element $[z]$ does not depend on the particular choice of $y \in g_n^{-1}(x)$ and b) on the choice of $x \in [x]$, furthermore that c) it is a module homomorphism.
2) we have to check the exactness of the sequence (LEX) at three types of elements of the sequence (M'_n, M_n and M''_n) – this altogether means 6 different containments.

1.a) Suppose that instead of y we choose $y' \in M_n$ with the property $g_n(y') = x$. This means that $g_n(y' - y) = g_n(y') - g_n(y) = 0$, implying that $y' - y \in \operatorname{Ker} g_n = \operatorname{Im} f_n$, hence there exists $z' \in M'_n$ such that $y' = y + f_n(z')$. Thus we have the following diagram:



Since f_{n-1} is injective, from the diagram we get that choosing y' in the first step of the definition of $\partial([x])$, we will definitely get $z + d'_n(z')$ in the final step and we have clearly $[z] = [z + d'_n(z')]$. Thus the choice of $[z]$ does not depend on the particular preimage y of x.

1.b) Next we check what happens to z if we replace x by some x' with $[x] = [x']$. This means that there is an element $x'' \in M''_{n+1}$ such that $x' = x + d''_{n+1}(x'')$. Since g_{n+1} s surjective, we get an element $y'' \in M_{n+1}$ with $g_{n+1}(y'') = x''$. The commutativity of the diagarm iplies that we can choose as g_n-preimage of x' the element $y + d_{n+1}(y'')$ as shown in the following diagram:



Here $d_n(y + d_{n+1}(y'')) = s_n(y)$ since $d_n d_{n+1} = 0$. This shows that if we start with x'

instead of x, the image of $\partial_n(x')$ will be $[z]$ for the same element $z \in \operatorname{Ker} d'_{n-1}$. Thus ∂_n is well defined.

1.c) It is easy to see from the definition of ∂_n that it will be a module homomorphism.

2) Next we turn our attention to showing that the sequence (LEX) is exact.

Half exactness. 2.a) This is clear at $H_n(M_\bullet)$ since $g_\bullet f_\bullet = 0$, hence $H_n(g_\bullet)H_n(f_\bullet) = 0$, since H_n is a functor.

2.b) For half exactness at $H_{n-1}(M'_\bullet)$ we have to show that for $[x] \in H_n(M\bullet'')$ we have $H_{n-1}(f_\bullet)\partial_n([x]) = 0$. But let us recall the defining diagram of ∂_n:

$$
\begin{array}{ccc}
 & & [\,z\,] = \partial_n([x]) \\
 & & \big\downarrow {\scriptstyle f_{n-1}} \\
y & \xrightarrow{d_n} & d_n(y) = f_{n-1}(z) \in \operatorname{Im} d_n \\
\big\downarrow {\scriptstyle g_n} & & \\
[\,x\,] & &
\end{array}
$$

Thus we have:

$$[x] \xrightarrow{\partial_n} [z] \xrightarrow{f_{n-1}} [f_{n-1}(z)] = [d_n(y)] = 0, \quad \text{since } d_n(y) \in \operatorname{Im} d_n.$$

2.c) Half exactness at $H_n(M''_\bullet)$. Take $y \in \operatorname{Ker} d_n$ and consider the diagram of the previous case:

$$
\begin{array}{ccc}
 & & [\,z\,] = \partial_n([x]) \\
 & & \big\downarrow {\scriptstyle f_{n-1}} \\
y & \xrightarrow{d_n} & d_n(y) = 0 \\
\big\downarrow {\scriptstyle g_n} & & \\
[\,x\,] & &
\end{array}
$$

Here we have:

$$[y] \xrightarrow{g_n} [x] \xrightarrow{\partial_n} [z]$$

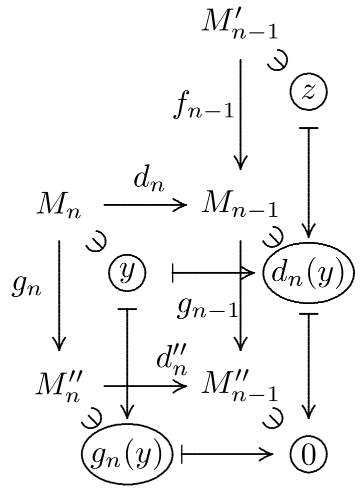
and since $d_n(y) = f_{n-1}(z) = 0$ and f_{n-1} is injective, we get that $z = 0$.

Thus we proved thet the sequence (LEX) is half exact.

Exactness. 2.d) Let us prove exactness at $H_{n-1}(M\bullet')$. Suppose we have:

$$\begin{array}{ccccc} H_n(M''_\bullet) & \xrightarrow{\partial_n} & H_{n-1}(M'_\bullet) & \xrightarrow{H_{n-1}(f_\bullet)} & H_{n-1}(M_\bullet) \\ & & \cup & & \cup \\ & & [z] & \longmapsto & 0 \end{array}$$

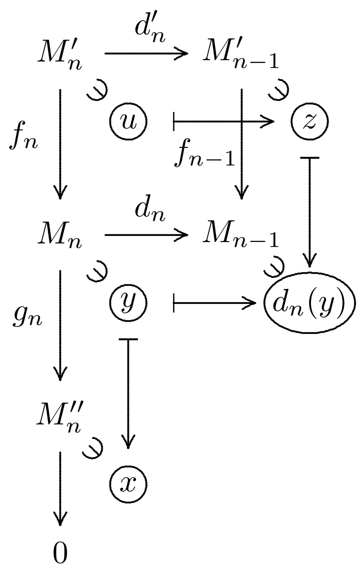
This means that $f_{n-1}(z) = d_n(y)$ for some $y \in M_n$. But by recalling the definition of ∂_n we get that $[z] = \partial_n([g_n(y)]$, thus $[z] \in \operatorname{Im}\partial_n$. Note that one still has to check that $g_n(y) \in \operatorname{Ker} d''_n$ (in order to make sure that $[g_n(y)] \in H_n(M''_\bullet)$). But this follows from the commutativity of the diagram and the exactness of the vertical sequences: $d''_n(g_n(y)) = g_{n-1}(d_n(y)) = g_{n-1}(f_{n-1}(z)) = 0$. Here is the site of our "diagram chasing":



2.e) Exactness at $H_n(M''_\bullet)$. Let us suppose that we have:

$$\begin{array}{ccccc} H_n(M_\bullet) & \xrightarrow{H_n(g_\bullet)} & H_n(M''_\bullet) & \xrightarrow{\partial_n} & H_{n-1}(M'_\bullet) \\ & & \cup & & \cup \\ & & [x] & \longmapsto & 0 \end{array}$$

This means that if $z \in M'_{n-1}$ is given by the definition of ∂_n then there exists $u \in M'_n$ for which $z = d'_n(u)$:



Let us now take $y' = y - f_n(u)$. Then $g_n(y') = g_n(y) + g_n(f_n(u)) = g_n(y) = x$ since $g_n f_n = 0$. On the other hand:

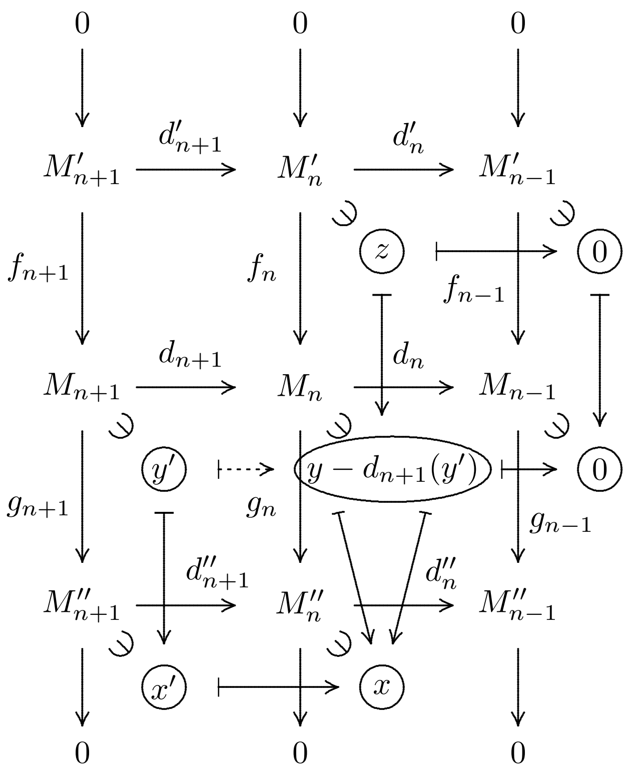
$$\begin{aligned} d_n(y') &= d_n(y) - d_n(f_n(u)) = d_n(y) - f_{n-1}(d'_n(u)) = \\ &= d_n(y) - f_{n-1}(z) = 0 \end{aligned}$$

This means that $y' \in \operatorname{Ker} d_n$, thus $[y'] \in H_n(M_\bullet)$ and hence $x = g_n(y')$ implies that $[x] \in \operatorname{Im} H_n(g_\bullet)$. This shows the exactness at $H_n(M''_\bullet)$.

2.f) Finally, let us prove the exactness at $H_n(M_\bullet)$. Suppose that for $y \in \operatorname{Ker} d_n \subseteq M_n$ we have:

$$\begin{array}{ccccc} H_n(M'_\bullet) & \xrightarrow{H_n(f_\bullet)} & H_{n-1}(M_\bullet) & \xrightarrow{H_n(g_\bullet)} & H_{n-1}(M''_\bullet) \\ & & \cup & & \cup \\ & & [y] & \longmapsto & 0 \end{array}$$

This means that $H_n(g_\bullet)([y]) = 0$, that is, if $x = g_n(y) \in M''_n$ then there exists an element $x' \in M''_{n+1}$ such that $x = d''_n(x') = x$:



- Since g_{n+1} is surjective, there exists $y' \in M_{n+1}$ such that $g_{n+1}(y') = x'$.
- Consider now the element $y - d_{n+1}(y') \in M_n$. The commutativity of the diagram gives: $g_n(y = d_{n+1}(y')) = g_n(y) - g_n(d_{n+1}(y')) = x - d''_{n+1}(g_{n+1}(y')) = x - d_{n+1}(x') = x - x = 0$.
- Since the vertical sequences are exact, this gives an element $z \in M'_n$ such that $f_n(z) = y - d_{n+1}(y')$.

- We have $d_n(d_{n+1}(y')) = 0$ and y defines a homology, thus both elements belong to $\operatorname{Ker} d_n$, and we have $[y] = [y - d_{n+1}(y')]$.
- The commutativity of the diagram gives that $f_{n-1}(d'_n(z)) = d_n(f_n(z)) = d_n(y - d_{n+1}(y')) = d_n(y) - d_n(d_{n+1}(y')) = 0 - 0 = 0$.
- The injectivty of f_{n-1} implies that $d'_n(z) = 0$.
- Thus $[z] \in H_n(M'_\bullet)$, furthermore $f_n([z]) = [y - d_{n+1}(y')] = [y]$.
- Hence $[y] \in \operatorname{Im} H_n(f_\bullet)$.

This shows the exactness at $H_n(M_\bullet)$ as well. □

Since we defined the connecting homomorphisms ∂_n "uniformly", i. e. without taking into account any special features of the exact sequence, we get the follwoing statement:

Proposition 1.17. *In the previous situation the connecting homomorphisms are natural in the following sense. Suppose we have two short exact sequences of complexes with chain maps $\alpha_\bullet$, $\beta_\bullet$ and $\gamma_\bullet$ making the following diagram commutative:*

$$\begin{array}{ccccccccc}
0_\bullet & \longrightarrow & M'_\bullet & \xrightarrow{f_\bullet} & M_\bullet & \xrightarrow{g_\bullet} & M''_\bullet & \longrightarrow & 0_\bullet \\
 & & {\scriptstyle\alpha_\bullet}\downarrow & & {\scriptstyle\beta_\bullet}\downarrow & & {\scriptstyle\gamma_\bullet}\downarrow & & \\
0_\bullet & \longrightarrow & N'_\bullet & \xrightarrow{u_\bullet} & N_\bullet & \xrightarrow{v_\bullet} & N''_\bullet & \longrightarrow & 0_\bullet
\end{array}$$

Then the following diagram, containing the two corresponding long exact sequences of homologies, is also commutative:

$$\begin{array}{ccccccccc}
\longrightarrow & H_n(M_\bullet) & \xrightarrow{H_n(g_\bullet)} & H_n(M''_\bullet) & \xrightarrow{\partial_n} & H_{n-1}(M'_\bullet) & \xrightarrow{H_{n-1}(f_\bullet)} & H_{n-1}(M_\bullet) & \longrightarrow \\
 & {\scriptstyle H_n(\beta_\bullet)}\downarrow & & {\scriptstyle H_n(\gamma_\bullet)}\downarrow & \# & \downarrow{\scriptstyle H_{n-1}(\alpha_\bullet)} & & \downarrow{\scriptstyle H_{n-1}(\beta_\bullet)} & \\
\longrightarrow & H_n(N_\bullet) & \xrightarrow[H_n(v_\bullet)]{} & H_n(N''_\bullet) & \xrightarrow[\delta_n]{} & H_{n-1}(N'_\bullet) & \xrightarrow[H_{n-1}(u_\bullet)]{} & H_{n-1}(N_\bullet) & \longrightarrow
\end{array}$$

Proof. Straightforward calculation. Note that only the commutativity of the middle square is in question, the rest follows from the fact that H_n is a functor. □

A simple and useful application of the previous theorem is the following well-known lemma.

Corollary 1.18. (Snake lemma.) *Suppose that we have the following commutative diagram of modules, with horizontal rows being exact:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\
 & & {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\gamma}\downarrow & & \\
0 & \longrightarrow & A & \xrightarrow[f']{} & B & \xrightarrow[g']{} & C & \longrightarrow & 0
\end{array}$$

Then there exists a connecting homomorphsm ∂ making the following sequence exact:

$$0 \to \operatorname{Ker}\alpha \xrightarrow{f} \operatorname{Ker}\beta \xrightarrow{g} \operatorname{Ker}\gamma \xrightarrow{\partial} \operatorname{Coker}\alpha \xrightarrow{\bar{f}'} \operatorname{Coker}\beta \xrightarrow{\bar{g}'} \operatorname{Coker}\gamma \to 0.$$

Note that for a map $\chi : X \to X'$ the cokernel of χ is $\operatorname{Coker}\chi = X'/\operatorname{Im}\chi$.

Proof. Let $X \in \{A, B, C\}$ with $\chi : X \to X'$ as given above. Define the complex $X_\bullet$ as follows:

$$X_\bullet : \qquad \cdots \to 0 \to X \xrightarrow{\chi} X' \to 0 \to \cdots$$

This gives us a short exact sequence of complexes:

$$(*) \qquad 0_\bullet \to A_\bullet \xrightarrow{f_\bullet} B_\bullet \xrightarrow{g_\bullet} C_\bullet \to 0_\bullet$$

It is easy to see that $H_1(X_\bullet) = \operatorname{Ker}\chi$ and $H_0(X_\bullet) = \operatorname{Coker}\chi$. This if we take the log exact sequence of homologies, induced by the short exact sequence of complexes in $(*)$ then we get the statement. □

An easy application of the Snake lemma gives the following well-known statement:

Proposition 1.19. (3 × 3 lemma.) *Suppose we have the following diagram with exact columns and excat middle row. Then the top row is exact if and only if the bottom row is exact:*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A_1 & \to & A_2 & \to & A_3 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & B_1 & \to & B_2 & \to & B_3 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C_1 & \to & C_2 & \to & C_3 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Proof. The statement follows from the Snake lemma. □

Let us now turn our attention to constructing the so-called derived functors of an additive functor. In some sense these functors will measure in some sense, how far is the functor from being exact.

Definition 1.20. 1) Let us take a projective resolution of the module M:

$$\begin{array}{llcccccccccc}
\mathcal{P}(M)_\bullet : & \cdots \longrightarrow & P_2 & & \longrightarrow & P_1 & & \longrightarrow & P_0 & & \longrightarrow 0 \longrightarrow \cdots \\
 & & & \searrow^{p_2} & \nearrow & & \searrow^{p_1} & \nearrow & & \searrow^{p_0} & \\
 & & & K_2 & & & K_1 & & & M = K_0 & \\
 & & \nearrow & & \searrow & \nearrow & & \searrow & & & \searrow \\
 & & 0 & & & 0 & & & 0 & & 0
\end{array}$$

Here the diagram is commutative and the sequences $0 \to K_{i+1} \to P_i \to K_i \to 0$ are exact. The module K_i is called the *i-th syzygy module* of M. If R is (left) perfect then M has a *minimal projective resolution*: in this case for every i the map $P_i \xrightarrow{p_i} K_i \to 0$ is a projective cover, i. e. we assume that $\operatorname{Ker} p_i = K_{i+1} \ll P_i$. Let us recall that R is left perfect if for example R is (left or right) Artinian; in particular minimal projective resolutions exist over finite dimensional algebras. The uniqueness of the projective cover implies the uniqueness of the minimal projective resolution (if it exists). Note thet projective resolutions are not unique in general since if

$$\cdots \xrightarrow{p_{i+2}} P_{i+1} \xrightarrow{p_{i+1}} P_i \xrightarrow{p_i} \cdots$$

is part of a projective resolution then

$$\cdots \xrightarrow{\binom{p_{i+2}}{0}} P_{i+1} \oplus Q \xrightarrow{\begin{pmatrix} p_{i+1} & 0 \\ 0 & 1_Q \end{pmatrix}} P_i \oplus Q \xrightarrow{(p_i\ 0)} \cdots$$

also gives a projective resolution for arbitrary projective module Q.

2) We may also consider the injective coresolution of M:

$$\begin{array}{lcccccccccccc}
\mathcal{I}(M)^\bullet : & \cdots & \longrightarrow & 0 & \longrightarrow & & I_0 & \longrightarrow & I_1 & \longrightarrow & I_2 & \longrightarrow & \cdots \\
& & & & & {\scriptstyle i_0}\nearrow & & \searrow \ {\scriptstyle i_1}\nearrow & & \searrow \ {\scriptstyle i_2}\nearrow & & & \\
& & & & M = C_0 & & & C_1 & & C_2 & & & \\
& & & \nearrow & & & \nearrow & & \searrow \ \nearrow & & \searrow & & \\
& & & 0 & & & 0 & & 0 & & 0 & &
\end{array}$$

Here the corresponding short exact sequences are $0 \to C_j \to I_j \to C_{j+1} \to 0$ and the modules C_i are called *co-syzygy modules* of M. Note that unlike projective resolutions where rojective covers may not exist, for every module M we can choose a minimal injective coresoluion by taking $0 \to C_j \to I_j$ to be an injective envelope, i. e. $C_j \trianglelefteq I_j$ for every j.

Proposition 1.21. *Let $\mathcal{P}(M)_\bullet$ and $\mathcal{P}(M')_\bullet$ be (deleted) projective resolutions of the modules $M, M. \in R$-*Mod*. Then there is a bijection between* $\operatorname{Hom}_R(M, M')$ *and* $\operatorname{Hom}_{K(R)}(\mathcal{P}(M)_\bullet, \mathcal{P}(M')_\bullet)$ *which is also an isomorphism of Abelian groups. In particular:*

1) *every morphism $f_\bullet : \mathcal{P}(M)_\bullet \to \mathcal{P}(M')_\bullet$ induces a morphism $F : M \to M'$;*
2) *every morphism $\varphi : M \to M'$ induces a morphism $\varphi_\bullet : \mathcal{P}(M)_\bullet \to \mathcal{P}(M')_|$comp, which is unique up to homotopy;*
3) *the two correspondences are inverses to each other, moreover the correspondences are additive.*

Proof. 1) For $f_\bullet : \mathcal{P}(M) \to \mathcal{P}(M')_\bullet$ take the map $f = H_0(f_\bullet) : H_0(\mathcal{P}(M)_\bullet) = M \to H_0(\mathcal{P}(M')_\bullet) = M'$. Note that homotopic chain maps induce the same maps on homologies hence this gives a correspondence from $\operatorname{Hom}_{K(R)}(\mathcal{P}(M)_\bullet, \mathcal{P}(M')_\bullet)$.

2) Suppose now that we have a morphism $\varphi : M \to M'$. Consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_1 & \longrightarrow & P_0 & \xrightarrow{\tilde{p}_0} & M & \longrightarrow & 0 \\ & & \Big\downarrow\vdots & & \Big\downarrow\vdots\,\exists\varphi_0 & & \Big\downarrow\varphi & & \\ 0 & \longrightarrow & K_1' & \longrightarrow & P_0' & \xrightarrow[\tilde{p}_0']{} & M' & \longrightarrow & 0 \end{array}$$

Here $\exists\varphi_0$ since P_0 is projective. Furthermore $\varphi(K_1) \subseteq K_1'$ since $\tilde{p}_0'\varphi_0(K_1) = \varphi\tilde{p}_0(K_1) = \varphi(0) - 0$ and the bottom sequence is exact. This gives us a map $\tilde{\varphi}\big|K_1 : K_1 \to K_1'$, making the diagram commutative.

We can now repeat the procedure with replacing $M \xrightarrow{\varphi} M'$ by $K_1 \xrightarrow{\varphi_0|K_1} K_1'$. This results in a sequence of morphisms φ_i, giving us the required chain map $\varphi_\bullet$. This chain map is called the *lifting of* φ. It is clear that if we apply to $\varphi_\bullet$ the corrsepondence in the first part, we will get $\varphi \in \mathrm{Hom}_R(M, M')$, as required.

We still have to prove that any two liftings are homotopic. Once we get this, we shall not only get the coorespondence from R-Mod to the homotopy category $K(R)$ but this will also prove that the two correspondences are inverses to each other also in the other direction. Now, the additivity of the correspondence obviously holds, so to show that any two liftings are homotopic, it is enough to prove that any lifting of 0 is nullhomotopic. In order to show this let us consider the full projective resolutions and the following diagram:

$$\begin{array}{lcccccccccccc} \mathcal{P}(M)^*_\bullet: & \cdots & \longrightarrow & P_2 & \xrightarrow{p_2} & P_1 & \xrightarrow{p_1} & P_0 & \xrightarrow{\tilde{p}_0} & M & \longrightarrow & 0 \\ & & & \varphi_2\Big\downarrow & \overset{s_1}{\swarrow} & \varphi_1\Big\downarrow & \overset{s_0}{\swarrow} & \varphi_0\Big\downarrow & & 0\Big\downarrow & & \\ \mathcal{P}(M')^*_\bullet: & \cdots & \longrightarrow & P_2 & \xrightarrow[p_2']{} & P_1 & \xrightarrow[p_1']{} & P_0 & \xrightarrow[\tilde{p}_0']{} & M & \longrightarrow & 0 \end{array}$$

Here $\mathrm{Im}\,\varphi_0 \subseteq \mathrm{Ker}\,\tilde{p}_1 = \mathrm{Im}\,p_1$, since $\tilde{p}_0\varphi_0 = 0\tilde{p}_0 = 0$. Thus by the projectivity of P_0 implies that there exists $s_0 : P_0 \to P_1'$ for which $\varphi_0 = p_1' s_0 \ \big(= p_1' s_0 + 0p_0 \big)$. (Thus we choose $s_{-1} = 0$.)

Next we take the function $\varphi_1 - s_0 p_1 : P_1 \to p_1'$. Here from the commutativity of the diagram we get:

$$p_1'(\varphi_1 - s_0 p_1) = p_1'\varphi_1 - \underbrace{p_1' s_0}_{\varphi_0} p_1 = p_1'\varphi_1 - \varphi_0 p_1 = 0$$

Thus $\mathrm{Im}(\varphi_1 - s_0 p_1) \subseteq \mathrm{Ker}\,p_1' = \mathrm{Im}\,p_2'$. Hence the projectivity of P_1 implies that there exists $s_1 : P_1 \to P_2'$ such that $p_2' s_1 = \varpi - s_0 p_1$, that is: $\varphi_1 = p_2' s_1 + s_0 p_1$.

We continue by induction, applying the same method in the next step for $\varphi_2 - s_1 p_2$. Thus we proved that any two liftings of φ are homotopic. As we observed this also means that if we start with a chain map $\varphi_\bullet$, attach to it the map $H_0(\varphi_\bullet)$ defined on the 0-th homology and then take the lifting of this map, denoted by $H_0(\varphi_\bullet)_\bullet$, then $\varphi_\bullet$ and $H_0(\varphi_\bullet)_\bullet$ are homotopic. This proves that indeed we get a bijection between $\mathrm{Hom}_R(M, M;)$ and $\mathrm{Hom}_{K(R)}(\mathcal{P}(M)_\bullet, \mathcal{P}(M')_\bullet)$. □

Corollary 1.22. *If $\mathcal{P}_1(M)_\bullet$ and $\mathcal{P}_2(M)_\bullet$ are two projective resolutions of M, then $\mathcal{P}_1(M)_\bullet$ and $\mathcal{P}_2(M)_\bullet$ are homotopcally equivalent.*

Proof. Let $f_\bullet : \mathcal{P}_1(M)_\bullet \to \mathcal{P}_2(M)_\bullet$ be a lifting of 1_M, and similarly $g_\bullet : \mathcal{P}_2(M)_\bullet \to \mathcal{P}_1(M)_\bullet$ be also a lifting of 1_M. Then $g_\bullet f_\bullet$ and $f_\bullet g_\bullet$ are both liftings of 1_M, hence they are both homotopic with the obvious liftings, the identity map of these chain complexes, i. e. $g_\bullet f_\bullet \sim 1_{\mathcal{P}_1(M)_\bullet}$ and $f_\bullet g_\bullet \sim 1_{\mathcal{P}_2(M)_\bullet}$. Thus the two resolutions are homotopically equivalent. □

Remark 1.23. We just showed that the correspondence $M \to \mathcal{P}(M)_\bullet$, i. e. assigning to a module its projective resolution in the homotopy category is well-defined and actually it provides a functor $\mathcal{P} : R\text{-Mod} \to K(R)$. This functor is *full* and *faithful*, i. e. for every $M, M' \in R$-Mod we get an isomorphism $\mathcal{P} : \mathrm{Hom}_R(M, M') \to \mathrm{Hom}_{K(R)}(\mathcal{P}(M)_\bullet, \mathcal{P}(M')_\bullet$. Thus we can think of R-Mod as a full subcategory of the homotopy category.

Proposition 1.24. *Let us take an additive functor $F : R\text{-Mod} \to S\text{-Mod}$, furthermore, for the modules $M, M' \in R$-Mod we take – as before – projective resolutions $\mathcal{P}(M)_\bullet$ and $\mathcal{P}(M')_\bullet$. For a morphism $f \in \mathrm{Hom}_R(M, M')$ let $f_\bullet$ denote the lifting of f to a morphism between the resolution complexes. Then:*

- *a) for fixed M the module $H_n(F(\mathcal{P}(M)_\bullet)$ is independent of the particular choice of resolution;*
- *b) the morphism $H_n(F(f_\bullet))$ depends only on f but it is independent of the particular choice of resolutions.*

Proof. The aditive functor $F : R\text{-Mod} \to S\text{-Mod}$, when applied to the elements of a complex will obviously take this complex to a complex since by additivity half exactness is preserved. Furthermore homotopy is defined additively hence homotopic complexes will be mapped to homotopic ones. (We can denote by $\bar{F}$ the extension of F to complexes and $\tilde{F}$ can denote the action of F on the homotopy category. However for simplicity verey often we shall use F for each these functors.) Since homotopic sequences have ismorphic homologies and any two projective resolutions are homotopic, the homologies $H_n(F(\mathcal{P}(M)_\bullet$ will not depend on the particular projective resolution, but only on M. Similarly, different lifitings of f will be homotopic, hence they induce the same mapping on homologies. □

Definition 1.25. Let $F : R\text{-Mod} \to S\text{-Mod}$ be an additive functor, $M, M' \in R$-Mod, $\mathcal{P}(M)_\bullet$ a projective resolution of M, $f \in \mathrm{Hom}_R(M, M')$ and $f_\bullet$ a lifting of f to $\mathcal{P}(M)_\bullet$.

We define the *n-th left derived functor* of F as follows:

$$(L_nF)(M) = H_n(F(\mathcal{P}(M)_\bullet))$$
$$(L_nF)(f) = H_n(F(f_\bullet))$$

Thus we have the following correspondence:

$$\begin{array}{ccccccccc} M & & \mathcal{P}(M)_\bullet\colon \cdots & \longrightarrow & P_n & \longrightarrow & P_{n-1} & \longrightarrow & \cdots \\ f\downarrow & \Longrightarrow & & & f_n\downarrow & & f_{n-1}\downarrow & & \Longrightarrow \\ M' & & \mathcal{P}(M')_\bullet\colon \cdots & \longrightarrow & P'_n & \longrightarrow & P'_{n-1} & \longrightarrow & \cdots \end{array}$$

$$\begin{array}{cccccccc} & F(\mathcal{P}(M)_\bullet)\colon \cdots & \longrightarrow & F(P_n) & \longrightarrow & F(P_{n-1}) & \longrightarrow & \cdots \\ \Longrightarrow & & & F(f_n)\downarrow & & F(f_{n-1})\downarrow & & \Longrightarrow \\ & F(\mathcal{P}(M')_\bullet)\colon \cdots & \longrightarrow & F(P'_n) & \longrightarrow & F(P'_{n-1}) & \longrightarrow & \cdots \end{array}$$

$$\begin{array}{cc} & H_n(F(\mathcal{P}(M)_\bullet)) \\ \Longrightarrow & \downarrow H_n(F(f_\bullet)) \\ & H_n(F(\mathcal{P}(M')_\bullet)) \end{array}$$

The fact that this gives a functor can be seen from the following:

$$\begin{array}{ccccccccc} & M & \xmapsto{\mathcal{P}} & \mathcal{P}(M)_\bullet & \xmapsto{\tilde{F}} & F(\mathcal{P}(M)_\bullet) & \xmapsto{H_n} & H_n(F(\mathcal{P}(M)_\bullet)) \\ L_nF\colon & \pitchfork & & \pitchfork & & \pitchfork & & \pitchfork \\ & R\text{-Mod} & \longrightarrow & K(R) & \longrightarrow & K(S) & \longrightarrow & S\text{-Mod} \end{array}$$

Thus $L_nF = H_n \circ \tilde{F} \circ \mathcal{P}$, hence it is a functor.

If we start from an injective coresolution, we get the *right derived functors* R_nF. Thus $R^n H^n \circ \tilde{F} \circ \mathcal{I}$.

In case G is contravariant then we use the injective coresolution fro obtaining L_nG, while R^nG can be obtained by starting with the projective resolution.

Example 1.26. For $N \in R$-Mod take $G = \mathrm{Hom}_R(-.N) : R\text{-Mod} \to \mathrm{AB}$. Then $R^nG(M) \in \mathrm{AB}$ will be denoted by $\mathrm{Ext}^n_R(M,N)$ and it is called the *n-th extension module* of M and N. Thus to construct $\mathrm{Ext}^n_R(M,N)$ we have the following steps:

$$M \quad\Longrightarrow\quad \mathcal{P}(M)_\bullet \quad\Longrightarrow\quad \mathrm{Hom}_R(\mathcal{P}(M)_\bullet, N) \quad\Longrightarrow\quad H^n(\mathrm{Hom}_R(\mathcal{P}(M)_\bullet, N))$$

Proposition 1.27. *If the additive functor is right exact then $L_0F \simeq F$. If G is left exact then $R^0G \simeq G$.*

Proof (sketch). If F is right exact then end of the image of the full projective resolution of M under the action of F remains exact:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 & & \text{is exact} \\ & & & & \Big\Downarrow & F \text{ is right exact} & & & & & \\ \cdots & \longrightarrow & F(P_1) & \longrightarrow & F(P_0) & \longrightarrow & F(M) & \longrightarrow & 0 & & \text{is exact} \end{array}$$

Thus if we take the 0-th homology of the image of the deleted projective resolution $\cdots \to F(P_1) \to F(P_0) \to 0$ then it is isomorphic to $F(M)$.

We leave it to the reader to check the naturality of the isomorphism between the functors L_0F and F. □

Corollary 1.28. $\mathrm{Ext}^0_R(M,N) \simeq \mathrm{Hom}_R(M,N)$.

Proposition 1.29. (Horsehsoe lemma) *Let $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ be a short exact sequence in R-Mod. Then the sequence*

$$0 \to \mathcal{P}(M')_\bullet \xrightarrow{f_\bullet} \mathcal{P}(M)_\bullet \xrightarrow{g_\bullet} \mathcal{P}(M'')_\bullet \to 0$$

is also exact in the homotopy category $K(R)$. (Here $f_\bullet$ and $g_\bullet$ are the liftings of f and f, respectively, to the deleted complexes of projective resolutions.) In other words this means that the functor $\mathcal{P} : R\text{-Mod} \to K(R)$ is exact.

Proof. It is enough to show that there exist resolutions making the above sequence exact. We will show actually that to every resolution $\mathcal{P}(M')_\bullet$ and $\mathcal{P}(M'')_\bullet$ we can choose $\mathcal{P}(M)_\bullet = \mathcal{P}(M')_\bullet \oplus \mathcal{P}(M'')_\bullet$. Namely let us consider the following diagram (resembling a ,,horseshoe"):

$$\begin{array}{ccccccccc} & & P'_1 & & & & P''_1 & & \\ & & \downarrow & & & & \downarrow & & \\ & & P'_0 & & & & P''_0 & & \\ & & \downarrow & & & & \downarrow & & \\ 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & \downarrow & & & & \downarrow & & \\ & & 0 & & & & 0 & & \end{array}$$

Thus we want to fill the middle column by elements of the form $P'_i \oplus P''_i$. To this end let us consider the following diagram, obtained by taking the last two terms of the (complete) projective resolutions $\mathcal{P}(M')$ and $\mathcal{P}(M'')$ together with the kernel terms (first syzygies):

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_1' & \longrightarrow & K_1 & \longrightarrow & K_1'' & \longrightarrow & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_0' & \longrightarrow & P_0' \oplus P_o'' & \longrightarrow & P_0'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \tilde{p}_0'} & & \downarrow{\scriptstyle \tilde{p}_0} & \swarrow{\scriptstyle \varphi} & \downarrow{\scriptstyle \tilde{p}_0''} & & \\
0 & \longrightarrow & M' & \xrightarrow{f} & M & \xrightarrow{g} & M'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & & & 0 & &
\end{array}
$$

Here projecivity of P_0'' implies that we have φ such that $g\varphi = \tilde{p}_0''$. Hence we can define the map $\tilde{p}_o$ as the sum of the maps $f\tilde{p}_0'$ and φ. If we write to the top row that kernels of the morphisms $\tilde{p}_0', \tilde{p}_0$ and $\tilde{p}_0''$, the exactness of the top sequence follows from the Snake lemma, and the same lemma implies also that $\tilde{p}_o$ is surjective. Thus the above diagram exists and is commutative. Inductively we can repeat the process, this time applying the argument to the exact sequence of syzygies. □

Proposition 1.30. *Let us denote by $\mathcal{P}(R)$ the full subcategory of R-*Mod *consisting of projective R-modules, and let $C(\mathcal{P}(R))$ and $K(\mathcal{P}(R))$ denote the category of complexes (resp. the homotopy category of complexes) consisting of projective modules. Suppose that the the following sequence is exact in $C(\mathcal{P}(R))$ or $K(\mathcal{P}(R))$:*

$$0 \to P_\bullet' \to P_\bullet \to P\bullet'' \to 0$$

*If $F : R$-*Mod $\to S$-Mod *is an additive functor, then*

$$0 \to F(P_\bullet') \to F(P_\bullet) \to F(P\bullet'') \to 0$$

is also exact.

Proof. Since the modules in the initial complexes are all projective, and any short exact sequence of projective modules splits, the short exact sequences at each level are split exact. But any additive functor will map a split exact sequence into a split exact sequence since direct sums can be described via additive expressions of morphisms. Thus we get that the second sequence is also (split) exact .

Corollary 1.31. (Long exact sequence of dervied functors) *Let us take a short exact sequence in* R-Mod*:*

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0.$$

If $F : R\text{-Mod} \to S\text{-Mod}$ *is an additive functor then there exist connecting homomorphisms* ∂_n *for each* n *so that the following sequence is also exact in* S-Mod*:*

$$\cdots \xrightarrow{\partial_{n+1}} L_nF(M') \xrightarrow{L_nF(f)} L_nF(M) \xrightarrow{L_nF(g)} L_nF(M'') \xrightarrow{\partial_n}$$
$$L_{n-1}F(M') \xrightarrow{L_{n-1}F(f)} L_{n-1}F(M) \xrightarrow{L_{n-1}F(g)} L_{n-1}F(M'') \xrightarrow{\partial_{n-1}} \cdots$$

Proof. The Horseshoe lemma (Lemma 1.29) implies that the sequence of complexes

$$0 \to \mathcal{P}(M')_\bullet \xrightarrow{f_\bullet} \mathcal{P}(M)_\bullet \xrightarrow{g_\bullet} \mathcal{P}(M'')_\bullet \to 0$$

is also exact. By Proposition 1.30 we get that the following sequence is exact:

$$0 \to F(\mathcal{P}(M')_\bullet) \xrightarrow{F(f_\bullet)} F(\mathcal{P}(M)_\bullet) \xrightarrow{g_\bullet} F(\mathcal{P}(M'')_\bullet) \to 0.$$

The statement will now follow from Theorem 1.16 about the long exact sequence of homologies. □

Corollary 1.32. *1) Suppose* F *is an additive functor. Then:*

F *is right exact* $\Leftrightarrow$ $F \simeq L_0F$*.*

F *is left exact* $\Leftrightarrow$ $F \simeq R^0F$*.*

2) Suppose that the additive functor is right exact. Then

F *is exact* $\Leftrightarrow$ $0 \simeq L_1F$*.*

3) Suppose that the additive functor is left exact. Then

F *is exact* $\Leftrightarrow$ $0 \simeq R^1F$*.*

Proposition 1.33. (Axiomatic characterization of derived functors) *Suppose* F *is an additive, right exact functor,* $\{\Phi_n \mid n \geq 0\}$ *a class of additive functors for which the following properties hold:*

1) $\Phi_0 \simeq F$*;*

2) $\Phi_n(P) = 0$ *for every* $n > 0$ *and every projective module* P*;*

3) For every short exact sequence

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0.$$

*there exist connecting homomorphisms Δ_n for each n so that the following sequence is also exact in S-*Mod*:*

$$\cdots \xrightarrow{\Delta_{n+1}} \Phi_n(M') \xrightarrow{\Phi_n(f)} \Phi_n(M) \xrightarrow{\Phi_n(g)} \Phi_n(M'') \xrightarrow{\Delta_n} \Phi_{n-1}(M') \xrightarrow{\Phi_{n-1}(f)} \Phi_{n-1}(M) \xrightarrow{\Phi_{n-1}(g)} \Phi_{n-1}(M'') \xrightarrow{\Delta_{n-1}} \cdots$$

Then $\Phi_n \simeq L_nF$, i. e. the three properties listed above characterize the sqeuence of derived functors.

Proof. (Sketch only, for a more detailed proof see MacLane's Homology.)

The first and third properties hold for the sequence of functors L_nF by Proposition 1.27 and Corollary 1.31. The second property is also easy: if P is projective then we may take the deleted projective resolution of P to consist of only one nonzero term, i. e. $\mathcal{P}(P)_\bullet : \ \cdots \to 0 \to P \to 0 \to \cdots$. Then $F(\mathcal{P}(P)_\bullet) : \ \cdots \to 0 \to F(P) \to 0 \to \cdots$. Hence the homologies $H_n(F(\mathcal{P}(P)_\bullet))$ are automatically 0 for $n > 0$.

To show the opposite direction suppose that Φ_n satisfies properties 1)-3). Consider now for an arbitrary module M the end of a projective resolution:

$$0 \to K_1 \xrightarrow{\alpha} P_0 \xrightarrow{\beta} M \to 0$$

Thus the above sequence is exact and P_0 projective. By properties 2) and 3) applied to the families L_nF and Φ_n we get the following:

$$\begin{array}{ccccccccccc}
\Phi_1(P_0) & \longrightarrow & \Phi_1(M) & \xrightarrow{\Delta_1} & \Phi_0(K_1) & \xrightarrow{\Phi_0(\alpha)} & \Phi_0(P_0) & \xrightarrow{\Phi_0(\beta)} & \Phi_0(M) & \longrightarrow & 0 \\
\downarrow{\scriptstyle =0} & & \vdots & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} & & \\
L_1F(P_0) & \longrightarrow & L_1F(M) & \xrightarrow{\partial_1} & F(K_1) & \xrightarrow{F(\alpha)} & F(P_0) & \xrightarrow{F(\beta)} & F(M) & \longrightarrow & 0
\end{array}$$

Then $\Phi_1(M) \simeq \operatorname{Ker} \Phi_0(\alpha) \simeq \operatorname{Ker} F(\alpha) \simeq L_1F(M)$ and we can proceed by induction for $n > 1$:

$$\begin{array}{ccccccc}
\Phi_{n+1}(P_0) & \longrightarrow & \Phi_{n+1}(M) & \xrightarrow{\Delta_{n+1}} & \Phi_n(K_1) & \xrightarrow{\Phi_n(\alpha)} & \Phi_n(P_0) \\
\downarrow{\scriptstyle =0} & & \vdots & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle =0} \\
L_{n+1}F(P_0) & \longrightarrow & L_{n+1}F(M) & \xrightarrow{\partial_n} & L_nF(K_1) & \xrightarrow{L_nF(\alpha)} & L_nF(P_0)
\end{array}$$

Since $\Phi_n(K_1) \simeq L_nF(K_1)$ by induction, we conclude that $\Phi_{n+1}(M) \simeq L_{n+1}F(M)$, as required. □

2. The Ext and Tor functors. Extensions of modules and the Ext functor. The Yoneda product

Our next goal is to investigate the right derived functors of the Hom functor. Here the first thing to clarify is the relationship of the derived functors of the covariant and the contravariant Hom functors. Let us start with the observation that both functors map R-Mod to $\mathbb{Z}$-Mod $= AB$, hence the images of a module under these derived functors will be in general Abelian groups, however in many cases one will have additional structure on them; for example if R is an algebra over a field K then the Hom-sets will have a vector space structure as well, hence the derived functors will also give vector spaces.

Notation 2.1. To simplify the notation, we shall use the notation $h_M = \mathrm{Hom}_R(M, -)$ and $h_N^\circ = \mathrm{Hom}_R(-, N)$ for the covariant and contravariant Hom functors, respectively. Let us observe that both functors are left exact, hence we shall construct their right derived functors: $R^n H_M$ and $R^n h_N^\circ$. In particular for $R^n h_M$ we shall use injetive coresolutions, while for $R^n h_N^\circ$ projective resolutions in the following way:

Constructing $R^n h_M(N)$. Take an injective coresolution of N and apply the functor $h_M = \mathrm{Hom}(M, -)$:

$$\begin{array}{lccccccccc}
\mathcal{I}(N)_\bullet : & 0 & \longrightarrow & \Big(N & \longrightarrow \Big) & I_0 & \longrightarrow & I_1 & \longrightarrow & \cdots \\
 & & & & & \Downarrow h_M(-) & & & & \\
h_M(\mathcal{I}(N)_\bullet) : & 0 & \longrightarrow & \Big(h_M(N) & \longrightarrow \Big) & h_M(I_0) & \longrightarrow & h_M(I_1) & \longrightarrow & \cdots
\end{array}$$

Cohomologies of the latter sequence give the values of $R^n h_M(N)$.

Constructing $R^n h_N^\circ(M)$. Take a projective resolution of M and apply the functor $h_N^\circ = \mathrm{Hom}(-, N)$:

$$\begin{array}{lccccccccc}
\mathcal{P}(M)_\bullet : & \cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \Big(\longrightarrow & M \Big) & \longrightarrow & 0 \\
 & & & & & \Downarrow h_M(-) & & & & \\
h_M(\mathcal{P}(M)_\bullet) : & 0 & \longrightarrow & \Big(h_N^\circ(M) & \longrightarrow \Big) & h_N^\circ(P_0) & \longrightarrow & h_N^\circ(P_1) & \longrightarrow & \cdots
\end{array}$$

Cohomologies of the second sequence give the values of $R^n h_N^\circ(M)$.

Theorem 2.2. *For any pair of modules $M, N \in R$-Mod we have:*

$$R^n h_M(N) \simeq R^n h_N^\circ(M).$$

The common value of these two functors will be denoted by $\mathrm{Ext}_R^n(M, N)$.

Proof. We shall use the axiomatic decsription of the sequence of derived functors (Proposition 1.33): we will show that the the correspondence $N \mapsto R^n h_N^\circ(M)$ is also functorial, and by defining $\Phi_n(N) = R^n h_N^\circ(M)$ for a fixed module M we get a family of (covariant) functors satisfying the axioms for the right derived functors of h_M. This will give us the equality $R^n h_N^\circ(M) = R^n h_M(N)$.

Let us first recall the construction of $\Phi_n(N)$. Thus

- we take a projective resolution of the (fixed) module M:

$$\mathcal{P}(M)_\bullet : \qquad \cdots \longrightarrow P_1 \xrightarrow{p_1} P_0 \left(\xrightarrow{\tilde{p}_0} M \right) \to 0$$

- Apply the functor $\mathrm{Hom}_R(-, N) = h_N^\circ$ to this sequence:

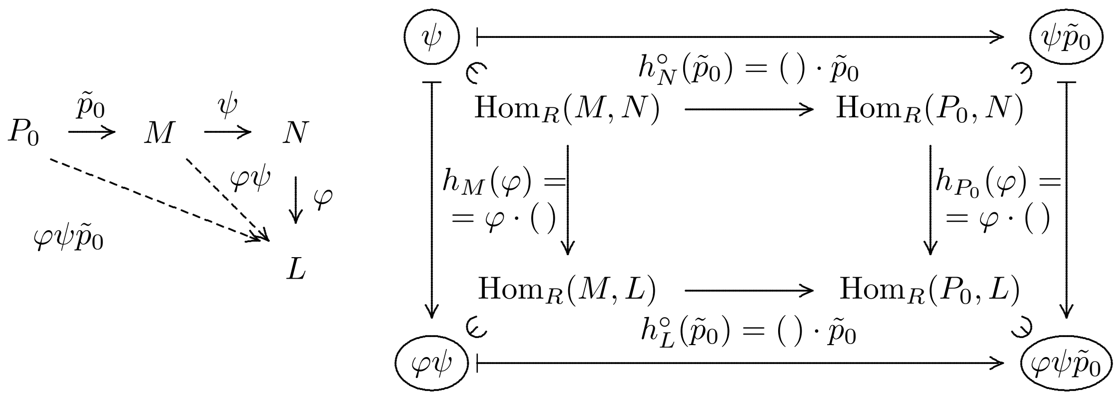
$$h_N^\circ(\mathcal{P}(M)_\bullet) : \quad 0 \to \left(\mathrm{Hom}_R(M, N) \xrightarrow{h_N^\circ(\tilde{p}_o)} \right) \mathrm{Hom}_R(P_0, N) \xrightarrow{h_n^\circ(p_1)} \mathrm{Hom}_R(P_1, N) \to \cdots$$

- take the n-th cohomology of this sequence to get $\Phi_n(N)$.

We want to make this correspondence a functor by defining it also on morphisms. For $\varphi : N \to L$ we need a morphisms $\Phi_n(\varphi) : \Phi_n(N) \to \Phi_n(L)$. But we have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}_R(M,N) & \xrightarrow{h_N^\circ(\tilde{p}_0)} & \mathrm{Hom}_R(P_0,N) & \xrightarrow{h_N^\circ(p_1)} & \mathrm{Hom}_R(P_1,N) & \longrightarrow & \cdots \\
 & & h_M(\varphi) \Big\downarrow & \# & h_{P_0}(\varphi) \Big\downarrow & \# & h_{P_1}(\varphi) \Big\downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}_R(M,L) & \xrightarrow[h_L^\circ(\tilde{p}_0)]{} & \mathrm{Hom}_R(P_0,L) & \xrightarrow[h_L^\circ(p_1)]{} & \mathrm{Hom}_R(P_1,L) & \longrightarrow & \cdots
\end{array}$$

The commutativity of the above diagram follows from the definitions of the corresponding maps. For example the commutativity of the first square can be seen from the following diagram:



Thus we get a chain map from $h_N^\circ(\mathcal{P}(M)_\bullet)$ to $h_L^\circ(\mathcal{P}(M)_\bullet)$ which induces a map on the cohomologies:

$$\Phi_n(N) \xrightarrow{\Phi_n(\varphi)} \Phi_n(L)$$

and this makes Φ_n a functor.

Let us now show that the sequence of functors $\{\Phi_n \mid n \in \mathbb{N}\}$ satisfies the requirements for the right derived functors of h_M.

1) The left exactness of the functor h_N° gives that

$$\Phi_0(N) \simeq \mathrm{Hom}_R(M, N) \simeq h_M(N).$$

This can be seen by aplying the h_N° functor to the full projective resolution $\mathcal{P}(M)^*_\bullet$: here the beginning of the sequence is exact, hence the 0-th cohomology of the image $h_N^\circ(\mathcal{P}(M)_\bullet)$ of the deleted complex is $\mathrm{Hom}_R(M, N)$.

2) $\Phi_n(N) = 0$, if $n > 0$ and N is injective. In this case the functor h_N° is exact hence the image of the complete projective resolution complex $h_N^\circ(\mathcal{P}(M)^*_\bullet)$ is exact, and this means that the higher cohomologies of the image of the deleted complex also vanish.

3) We will show now the existence of the long exact sequence for the functors Φ_n: this is the third property of the axiomatic characterization of derived functors. – Thus let us fix a projective resolution $\mathcal{P}(M)_\bullet$ for the module M and apply the family of functors $\mathrm{Hom}(\mathcal{P}(M)_\bullet, -)$ to an arbitrary short exact sequence $0 \to N' \to N \to N''$. We get a short exact sequence

$$0 \to \mathrm{Hom}_R(\mathcal{P}(M)_\bullet, N') \to \mathrm{Hom}_R(\mathcal{P}(M)_\bullet, N) \to \mathrm{Hom}_R(\mathcal{P}(M)_\bullet, N'') \to 0$$

in K(R) and this sequence is exact since each of the terms of the complex $\mathcal{P}(M)_\bullet$ is projective. This means that in the following commutative diagram the vertical columns are exact:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}(P_0, N') & \longrightarrow & \mathrm{Hom}(P_0, N') & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}(P_0, N) & \longrightarrow & \mathrm{Hom}(P_0, N) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}(P_0, N'') & \longrightarrow & \mathrm{Hom}(P_0, N'') & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

Thus we get a long exact sequence on the cohomologies of these horizontal complexes:

$$\cdots \to \Phi_n(N') \to \Phi_n(N) \to \Phi_n(N'') \xrightarrow{\partial_n} \Phi_{n+1}(N') \to \cdots$$

.

By Proposition 1.33, i. e. the characterization of derived functors we get that $R^n h_N^\circ(M) = \Phi_n(N) \simeq R^n h_M(N)$. Thus $\mathrm{Ext}^n_R(M, N)$ can be ontained in two different ways. □

Definition 2.3. Let $M \in$ Mod-R and $N \in R$-Mod. For $t_M : M \underset{R}{\otimes} - : R\text{-Mod} \to \text{AB}$ and $\bar{t}_N : - \underset{R}{\otimes} N : \text{Mod-}R \to \text{AB}$ we define:

$$L_n t_M(N) \simeq L_n \bar{t}_N(M) = \operatorname{Tor}_n^R(M, N).$$

(The proof of the fact that the first isomorphism holds is similar to the proof of the previous theorem.)

Let us summarize an important consequence of the construction of the Ext and Tor functors: in what follows these exact sequences will frequently appear.

Corollary 2.4. *Let is take the following exact sequence of R-modules in R-Mod:*

$$0 \to {}_RX \to {}_RY \to {}_RZ \to 0.$$

Then we get the following exact sequences:

1) for arbitrary ${}_RM \in R$-Mod the following sequence is exact:

$$\begin{array}{ccccccccc} 0 & \to & \operatorname{Hom}_R(M,X) & \to & \operatorname{Hom}_R(M,Y) & \to & \operatorname{Hom}_R(M,Z) & \to \\ & \to & \operatorname{Ext}^1_R(M,X) & \to & \operatorname{Ext}^1_R(M,Y) & \to & \operatorname{Ext}^1_R(M,Z) & \to \\ & \to & \operatorname{Ext}^2_R(M,X) & \to & \operatorname{Ext}^2_R(M,Y) & \to & \cdots & \end{array}$$

2) for arbitrary ${}_RN \in R$-Mod the following sequence is exact:

$$\begin{array}{ccccccccc} 0 & \to & \operatorname{Hom}_R(Z,N) & \to & \operatorname{Hom}_R(Y,N) & \to & \operatorname{Hom}_R(X,N) & \to \\ & \to & \operatorname{Ext}^1_R(Z,N) & \to & \operatorname{Ext}^1_R(Y,N) & \to & \operatorname{Ext}^1_R(X,N) & \to \\ & \to & \operatorname{Ext}^2_R(Z,N) & \to & \operatorname{Ext}^2_R(Y,N) & \to & \cdots & \end{array}$$

3) for arbitrary $M_R \in$ Mod-R the following sequence is exact:

$$\begin{array}{ccccccccc} & \cdots & \to & \operatorname{Tor}^R_2(M,Y) & \to & \operatorname{Tor}^R_2(M,Z) & \to & \\ \to & \operatorname{Tor}^R_1(M,X) & \to & \operatorname{Tor}^R_1(M,Y) & \to & \operatorname{Tor}^R_1(M,Z) & \to & \\ \to & M \underset{R}{\otimes} X & \to & M \underset{R}{\otimes} Y & \to & M \underset{R}{\otimes} Z & \to & 0 \end{array}$$

4) If we start with a short exact sequence of right R-modules, then for an arbitrary module ${}_RM \in R$-Mod we may get a similar exact sequence for the functors $- \underset{R}{\otimes} M$ and $\operatorname{Tor}^R_n(-, M)$.

Our next aim is to give a different interpretation for the functors Ext^n_R and Ext^1_R in particular. This will also reveal the source of the name of the functor (Ext for "extension").

Definition 2.5. Let $M, N \in R$-Mod be R-modules. We shall say that a short exact sequence $\mathcal{E} : 0 \to N \xrightarrow{\alpha} K \xrightarrow{\beta} M \to 0$ is *an extension of the module M by the module N.* A *morphism between two extensions* is a triple of R-homomorphisms (ν, κ, μ) which is a morphism (i. e. a chain map) between the two complexes:

$$
\begin{array}{ccccccccccc}
\mathcal{E}: & & 0 & \longrightarrow & N & \xrightarrow{\alpha} & K & \xrightarrow{\beta} & M & \longrightarrow & 0 \\
(\nu,\kappa,\mu)\big\downarrow & & & & \nu\big\downarrow & & \kappa\big\downarrow & & \mu\big\downarrow & & \\
\mathcal{E}': & & 0 & \longrightarrow & N' & \xrightarrow[\alpha']{} & K' & \xrightarrow[\beta']{} & M' & \longrightarrow & 0
\end{array}
$$

We will say that *the extensions $\mathcal{E}$ and $\mathcal{E}'$ are equivalent* ($\mathcal{E} \equiv \mathcal{E}'$) if there exists a morphism $(1_N, \kappa, 1_M)$ between them. Note that the 5-lemma implies that in this case κ must be an isomorphism and the middle terms of the two extensions are isomorphic.

Remark 2.6. It may happen that two extensions $\mathcal{E} : 0 \to N \to J \to M \to 0$ and $\mathcal{E}' : 0 \to N \to K' \to M \to 0$ are not equivalent. This could be the case even with $K \simeq K'$. For example:

$$
\begin{array}{ccccccccccc}
 & & & & (a) & \longmapsto & (3b)\;(b) & \longmapsto & (c) & & \\
 & & & & \cap & & \cap\quad\cap & & \cap & & \\
\mathcal{E}: & & 0 & \longrightarrow & \mathbb{Z}_3^+ = \langle a\rangle & \xrightarrow{\alpha} & \mathbb{Z}_9^+ = \langle b\rangle & \xrightarrow{\beta} & \mathbb{Z}_3^+ = \langle c\rangle & \longrightarrow & 0 \\
 & & & & \| & & \not\exists \; \big\downarrow & & \| & & \\
\mathcal{E}': & & 0 & \longrightarrow & \mathbb{Z}_3^+ = \langle a\rangle & \xrightarrow[\alpha]{} & \mathbb{Z}_9^+ = \langle b\rangle & \xrightarrow[\beta]{} & \mathbb{Z}_3^+ = \langle c\rangle & \longrightarrow & 0 \\
 & & & & \cup & & \cup\quad\cup & & \cup & & \\
 & & & & (a) & \longmapsto & (3b)\;(b) & \longmapsto & (c) & &
\end{array}
$$

Here we cannot complete the diagram with a vertical arrow between the middle terms of the exact sequences to keep the diagram commutative. One should observe that equivalence of extensions is a stronger concept than what could be considered an "isomorphism" (i. e. where there exists a morphism triple with all terms being isomorphisms). The obstacle here was that once we fix an isomorphism between the end terms then the completion becomes impossible; however we could easily provide an isomorphism between the two extensions. But note that if one has a morphism between the middle terms and one of the end terms then we could easily find a morphism for the other end term, too (see the exercises).

Definition 2.7. Let us denote by $\mathrm{Ex}(M, N)$ the set of equivalence clases of extensions of M by N. Please, observe that this is a set (and not a class): the cardinality of the middle term is of such extensions is bounded.

We would like to make this correspondence a bifunctor. Thus we have to define its action on morphisms.

1) For $\mu : M' \to M$ and $\mathcal{E} \in \mathrm{Ex}(M, N)$ we define $\mathcal{E}\mu \in \mathrm{Ex}(M', N)$ as follows. If $\mathcal{E} : 0 \to N \xrightarrow{\alpha} K \xrightarrow{\beta} M \to 0$, then take the pullback of the maps $\beta : K \to M$ and $\mu : M' \to M$:

$$\begin{array}{ccc} & & M' \\ & & \downarrow{\scriptstyle\mu} \\ K & \xrightarrow[\beta]{} & M \end{array} \qquad \Rightarrow \qquad \begin{array}{ccc} \tilde{K} & \xrightarrow{\beta'} & M' \\ \downarrow{\scriptstyle\kappa} & & \downarrow{\scriptstyle\mu} \\ K & \xrightarrow[\beta]{} & M \end{array}$$

(For the definition of the pullback we refer to the exercises.) From the construction of the pullback we get that if β is a surjection then β' is also a surjection, moreover the maps β and β' have isomorphic kernels, creating the following commutative diagram with exact rows.

$$\begin{array}{lccccccccc} \mathcal{E}\mu : & 0 & \longrightarrow & N & \xrightarrow{\alpha'} & \tilde{K} & \xrightarrow{\beta'} & M' & \longrightarrow & 0 \\ & & & \| & & \downarrow{\scriptstyle\kappa} & & \downarrow{\scriptstyle\mu} & & \\ \mathcal{E} : & 0 & \longrightarrow & N & \xrightarrow[\alpha]{} & K & \xrightarrow[\beta]{} & M & \longrightarrow & 0 \end{array}$$

Here the top row will be the extension $\mathcal{E}\mu \in \mathrm{Ex}(M', N)$. This defines the action of Ex on morphisms. (We will still have to prove that this process is functorial.)

2) Dually, for $\nu : N \to N'$ take the exact sequence for $\nu\mathcal{E} \in \mathrm{Ex}(M, N')$ which can be obtained by a suitable pushout diagram:

$$\begin{array}{lccccccccc} \mathcal{E} : & 0 & \longrightarrow & N & \xrightarrow{\alpha} & K & \xrightarrow{\beta} & M & \longrightarrow & 0 \\ & & & {\scriptstyle\nu}\downarrow & & {\scriptstyle\kappa}\downarrow & & \| & & \\ \nu\mathcal{E} : & 0 & \longrightarrow & N' & \xrightarrow[\alpha']{} & \bar{K} & \xrightarrow[\beta']{} & M & \longrightarrow & 0 \end{array}$$

Here the first commutative square is obtained as a pushout diagram:

$$\begin{array}{ccc} N & \xrightarrow{\alpha} & K \\ {\scriptstyle\nu}\downarrow & & \\ N' & & \end{array} \qquad \Rightarrow \qquad \begin{array}{ccc} N & \xrightarrow{\alpha} & K \\ {\scriptstyle\nu}\downarrow & & {\scriptstyle\kappa}\downarrow \\ N' & \xrightarrow[\alpha']{} & \bar{K} \end{array}$$

Dually to the previous case we get that $\mathrm{Coker}\,\alpha \simeq \mathrm{Coker}\,\alpha'$ hence we can indentify the end term of the short exact sequences given above.

In principle one should check (but it is easy) that the correspondence is compatible with the equivalence of extensions, i. e.

(i) for $\mu : M' \to M$ we get a (contravariant) correspondence $\mathrm{Ex}(M, N) \to \mathrm{Ex}(M', N)$, $\mathcal{E} \mapsto \mathcal{E}\mu$;

(ii) for $\nu : N \to N'$ we get a (covariant) correspondence $\mathrm{Ex}(M, N) \to \mathrm{Ex}(M, N)$, $\mathcal{E} \mapsto \nu\mathcal{E}$.

We still have to show that the correspondence is functorial, i. e. it preserves the composition of morphisms. In this respect the follwoing lemma will be helpful.

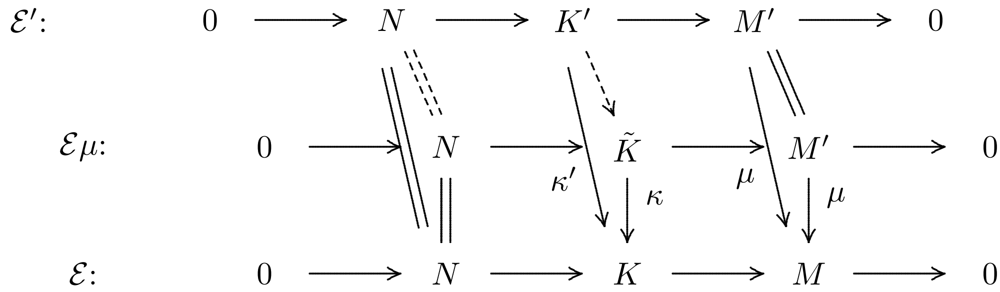
Proposition 2.8. *Let $\mathcal{E}$, $\mathcal{E}\mu$ and $\nu\mathcal{E}$ be extensions as defined above.*

1) If there exists a morphism $\mathcal{E}' \xrightarrow{(1_N,\kappa',\mu)} \mathcal{E}$ then $\mathcal{E}' \equiv \mathcal{E}\mu$.

2) If there exists a morphism $\mathcal{E} \xrightarrow{(\nu,\kappa'',1_M)} \mathcal{E}''$ then $\mathcal{E}'' \equiv \nu\mathcal{E}$.

This implies that the "shape" of morphism between these extensions already determines the extensions obtained from the pullback and pushout diagrams. In particular we get that $\mathrm{Ex}(-,-)$ *is a functor in both variables since it preserves the composition of morphisms.*

Proof. We shall give a proof only for the first statement, the second can be shown by a dual argument. Thus let us consider the following diagram:

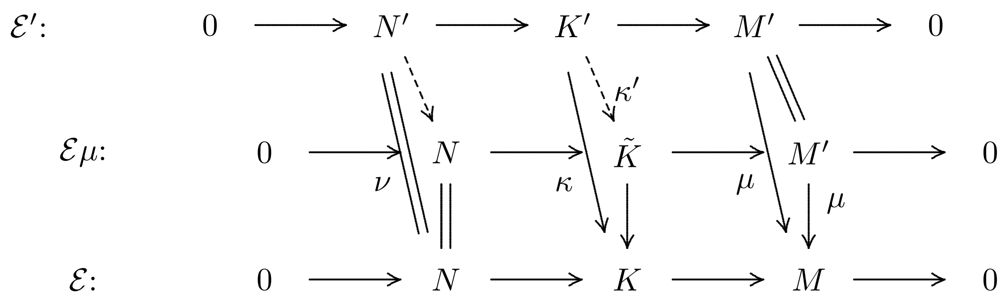


Here the pushout property of $\tilde{K}$ implies that there exists the map $K' \to \tilde{K}$, making the diagram commutative. But if we have the morphism between K' and $\tilde{K}$, furthermore the identity between M and M, connecting the top two extensions, then we get also a morphism between the kernel terms N and this has to be the identity. hence we get that $\mathcal{E}' \equiv \mathcal{E}\mu$.

The previous statement implies that $(\mathcal{E}\mu)\rho \equiv \mathcal{E}(\mu\rho)$. This shows that $\mathrm{Ex}(-, N)$ is indeed a functor. ☐

Proposition 2.9. *Every morphism $\mathcal{E}' \xrightarrow{(\nu,\kappa,\mu)} \mathcal{E}$ can be factored through the extension $\mathcal{E}\mu \equiv \nu\mathcal{E}'$.*

Proof. We have the following diagram:



Here the pullback property implies the existence of the map $\kappa' : K' \to \tilde{K}$ and then the restriction of κ to N' gives $N' \to N$, making the diagram commutative. Hence we must have a morphism $\mathcal{E}' \xrightarrow{(\nu,\kappa',1_M)} \mathcal{E}\mu$, giving the required factorization. Since the shape of this morphism is the same as the morphism from $\mathcal{E}'$ to $\nu\mathcal{E}$, the previous statement implies that $\mathcal{E}\mu \equiv \nu\mathcal{E}'$. □

Proposition 2.10. *For any extension $\mathcal{E} \in \mathrm{Ex}(M,N)$ and morphisms $\mu : M' \to M$ and $\nu : N \to N'$ we have*

$$(\nu\mathcal{E})\mu \equiv \nu(\mathcal{E}\mu).$$

In particular this implies that $\mathrm{Ex}(-,-)$ *is a bifunctor.*

Proof. We can take the following diagram:

$$\begin{array}{lccccccccc}
\mathcal{E}\mu: & 0 & \longrightarrow & N & \longrightarrow & \tilde{K} & \longrightarrow & M' & \longrightarrow & 0 \\
 & & & \| & & \downarrow & & \downarrow \mu & & \\
\mathcal{E}: & 0 & \longrightarrow & N & \longrightarrow & K & \longrightarrow & M & \longrightarrow & 0 \\
 & & & \nu\downarrow & & \downarrow & & \| & & \\
\nu\mathcal{E}: & 0 & \longrightarrow & N & \longrightarrow & \bar{K} & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

The composition of the morphisms $\mathcal{E}\mu \to \mathcal{E}$ and $\mathcal{E} \to \nu\mathcal{E}$ gives us a morphisms $\mathcal{E}\mu \xrightarrow{(\nu,\kappa,\mu)} \nu\mathcal{E}$. Using the previous proposition we get a factorization through $\nu(\mathcal{E}\mu) \equiv (\nu\mathcal{E})\mu$. □

Remark 2.11. So far we have obtained that $\mathrm{Ex}(-,-) : R\text{-Mod} \times R\text{-Mod} \to \mathrm{SET}$ is a bifunctor. (The equivalence classes of extensions formed a set.) We would like to define an Abelian group structure on $\mathrm{Ex}(M,N)$.

Definition 2.12. For arbitrary modules ${}_RM$ and ${}_RN$. let us define the following maps:

$$\begin{array}{llllll}
\Delta_M : M & \to & M \oplus M, & m & \mapsto & (m,m) \quad (\textit{diagonal map}) \\
\nabla_N : N \oplus N & \to & N, & (n_1,n_2) & \mapsto & n_1+n_2 \quad (\textit{codiagonal map})
\end{array}$$

Definition 2.13. If $\mathcal{E}_i \in \mathrm{Ex}(M_i, N_i$ for $1 = 1,2$ then we define in a natural way the *direct sum of these extensions*: $\oplus\mathcal{E}_i \in \mathrm{Ex}(\oplus M_i, \oplus N_i)$.

Definition 2.14. (Baer sum of extensions) Let $\mathcal{E}_1, \mathcal{E}_2 \in \mathrm{Ex}(M,N)$. We define the *Baer sum* of these extensions as:

$$\mathcal{E}_1 + \mathcal{E}_2 = \nabla_N(\mathcal{E}_1 \oplus \mathcal{E}_2)\Delta_M \in \mathrm{Ex}(M,N)$$

In particular:

$$\begin{array}{lccccccccc}
(\mathcal{E}_1 \oplus \mathcal{E}_2)\Delta_M : & 0 & \longrightarrow & N \oplus N & \longrightarrow & \tilde{K} & \longrightarrow & M & \longrightarrow & 0 \\
 & & & \| & & \downarrow & & \Delta_M \downarrow & & \\
(\mathcal{E}_1 \oplus \mathcal{E}_2) : & 0 & \longrightarrow & N \oplus N & \longrightarrow & K_1 \oplus K_2 & \longrightarrow & M \oplus M & \longrightarrow & 0 \\
 & & & \nabla_N \downarrow & & \downarrow & & & & \\
\nabla_N(\mathcal{E}_1 \oplus \mathcal{E}_2)\Delta_M : & 0 & \longrightarrow & N & \longrightarrow & \bar{K} & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

Note that the associativity property, given in Proposition 2.10 implies that the sum is well-defined.

Theorem 2.15. *1)* $\operatorname{Ex}(M,N)$ *is an Abelian group wiith respect to the Baer sum. The* 0 *element is the class of split extensions, while the additive inverse of* $\mathcal{E}$ *can be written as* $(-1_N)\mathcal{E}$.

2) $\operatorname{Ex}(-,-)$ *is a biadditive bifunctor from* R-Mod *into* AB*. In particular:*

$$\nu(\mathcal{E}_1+\mathcal{E}_2)=\nu\mathcal{E}_1+\nu\mathcal{E}_2 \quad \textit{and} \quad (\mathcal{E}_1+\mathcal{E}_2)\mu=\mathcal{E}_1\mu+\mathcal{E}_2\mu; \qquad (*)$$

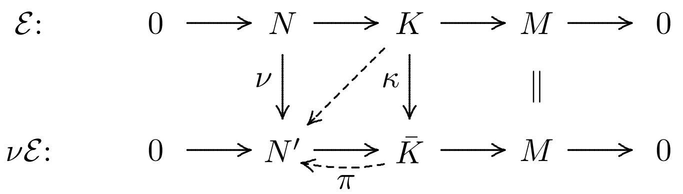
$$(\nu_1+\nu_2)\mathcal{E}=\nu_1\mathcal{E}+\nu\mathcal{E}_2 \quad \textit{and} \quad \mathcal{E}(\mu_1+\mu_2)=\mathcal{E}\mu_1+\mathcal{E}\mu_2 \qquad (**)$$

Proof. The proof will be added at a later stage. □

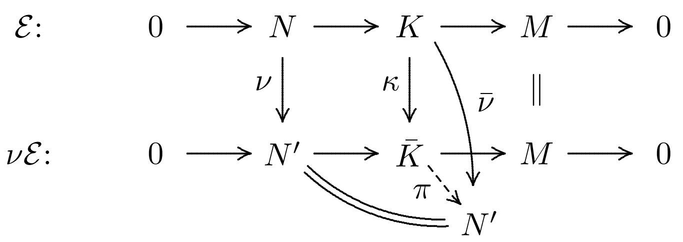
Next we show how the previus constructions may explain the non-exactness of the Hom functors.

Proposition 2.16. *Let us consider the extension* $\mathcal{E}: 0\to N\to K\to M\to 0$ *and a morphism* $\nu: N\to N'$. *Then* ν *can be extended to a morphism* $\bar\nu: K\to N'$ *if and only if the sequence* $\nu\mathcal{E}$ *is split, i. e.* $\nu\mathcal{E}=0$ *in* $\operatorname{Ex}(M,N)$. *(The dual statement can also be formulated and proved.)*

Proof. (⇐) Suppose $\nu\mathcal{E}$ is split exact. Then the splitting map $\pi:\tilde K\to N'$ gives an extension to ν by taking $\bar\nu: K\xrightarrow{\kappa}\bar K\xrightarrow{\pi}N'$:



(⇒) Let us suppose that we can find an extension $\bar\nu: K\to\nu'$. Then the pushout property will give us the required splitting map $\pi:\bar K\to N'$:



Definition 2.17. Let $\mathcal{E}\in\operatorname{Ex}(M,N)$, $\nu\in\operatorname{Hom}(N,N')$ and $m\in\operatorname{Hom}(M',M)$. Define the following correspondences:

$$\begin{array}{rccccc} \mathcal{E}_*: & \operatorname{Hom}(M',M) & \longrightarrow & \operatorname{Ex}(M',N), & \mu & \mapsto \ \mathcal{E}\mu \\ \mathcal{E}^*: & \operatorname{Hom}(N,N') & \longrightarrow & \operatorname{Ex}(M,N'), & \nu & \mapsto \ \nu\mathcal{E} \end{array}$$

Proposition 2.18. *Let us take an extension* $\mathcal{E} : \to N \xrightarrow{\alpha} K \xrightarrow{\beta} \xrightarrow{M} \to 0$. *Then for an arbitrary module* N' *we get the following exact sequence:*

$$0 \longrightarrow \mathrm{Hom}(M,N') \xrightarrow{\beta^*} \mathrm{Hom}(K,N') \xrightarrow{\alpha^*} \mathrm{Hom}(N,N') \xrightarrow{\mathcal{E}^*}$$
$$\xrightarrow{\mathcal{E}^*} \mathrm{Ex}(M,N') \xrightarrow{\cdot\beta} \mathrm{Ex}(K,N') \xrightarrow{\cdot\alpha} \mathrm{Ex}(N,N').$$

Dually, we can get a similar exact sequence for arbitrary M' *and the* $\mathcal{E}_*$ *map.*

Proof. The exactness is well-known for the beginning of the sequence while the exactness at $\mathrm{Hom}(N,N')$ is an immediate consequence of Proposition 2.16.

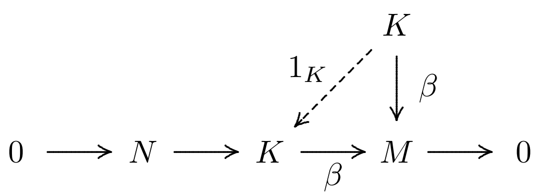
The halfexactness at $\mathrm{Ex}(K,N')$ can be seen from the following diagram:

$$\begin{array}{ccccc}
\mathrm{Ex}(M,N') & \xrightarrow{\cdot\beta} & \mathrm{Ex}(K,N') & \xrightarrow{\cdot\alpha} & \mathrm{Ex}(N,N') \\
\Cup & & \Cup & & \Cup \\
\mathcal{E} & \longmapsto & \mathcal{E}\beta & \longmapsto & \mathcal{E}\beta\alpha = \mathcal{E}0 = 0
\end{array}$$

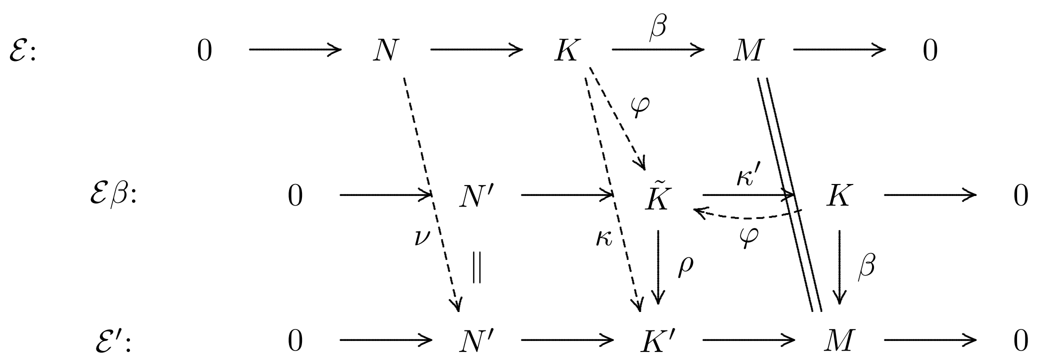
Next, the halfexactness at $\mathrm{Ex}(K,N')$ follows from a similar diagram:

$$\begin{array}{ccccc}
\mathrm{Hom}(N,N') & \xrightarrow{\mathcal{E}^*} & \mathrm{Ex}(M,N') & \xrightarrow{\cdot\beta} & \mathrm{Ex}(K,N') \\
\Cup & & \Cup & & \Cup \\
\nu & \longmapsto & \nu\mathcal{E} & \longmapsto & (\nu\mathcal{E})\beta
\end{array}$$

Note that here $(\nu\mathcal{E})\beta = \nu(\mathcal{E}\beta)$ and Proposition 2.16 implies that $\mathcal{E}\beta$ splits, since 1_K is obviously a lifting of β in the following diagram:



Let us now show that the sequence is exact at $\mathrm{Ex}(M,N')$. Suppose that for $\mathcal{E}' \in \mathrm{Ex}(M,N')$ we have $\mathcal{E}'\beta = 0$. We have to find a $\nu \in \mathrm{Hom}(N,N')$ for which $\mathcal{E}^*(\nu) = \nu\mathcal{E} = \mathcal{E}'$. Thus let us look at the following diagram:



Here $\mathcal{E}\beta = 0$ implies that there exists a splitting map $\varphi : K \to \tilde{K}$ (i. e. for this map we have $\beta\kappa' = 1_K$) and this gives a morphism $\mathcal{E} \xrightarrow{(\nu,\kappa,1_M)} \mathcal{E}$ where $\kappa = \rho\varphi$ and $\nu = \kappa\big|_N$. But we know that the existence of such a morphism characterizes $\nu\mathcal{E}$ (see Proposition 2.8), hence $\mathcal{E}' = \nu\mathcal{E}$, as required.

Finally we come to proving the exactness a $\mathrm{Ex}(K, N')$. Thus let us take $\mathcal{E}'' \in \mathrm{Ex}(K, N')$ for which $\mathcal{E}''\alpha = 0$. Then we have the following:

$$\begin{array}{lccccccccc}
\mathcal{E}''\alpha: & 0 & \longrightarrow & N' & \longrightarrow & \tilde{K} & \longrightarrow & N & \longrightarrow & 0 \\
 & & & \| & & \downarrow & {\scriptstyle \exists\varphi:\, N \dashrightarrow \tilde{K}}\quad {\scriptstyle \exists\rho:\, N \dashrightarrow K''} & \downarrow \alpha & & \\
\mathcal{E}'': & 0 & \longrightarrow & N' & \longrightarrow & K'' & \xrightarrow{\kappa''} & K & \longrightarrow & 0 \\
 & & & \| & & \downarrow & & \downarrow \beta & & \\
\mathcal{E}': & 0 & \longrightarrow & N' & \longrightarrow & K''/\operatorname{Im}\rho & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

Here the existence of a splitting map φ implies the existence of a lifting of α, denoted by ρ. The commutativity implies that $\beta\kappa''\rho = \beta\alpha = 0$, hence one can factor $\beta\kappa$ through $K''/\operatorname{Im}\rho$. One should check that the kernel of this morphism is N', i. e. the extension $\mathcal{E}'$ is well-defined. The morphism between $\mathcal{E}''$ and $\mathcal{E}'$ implies that $\mathcal{E}'' = \mathcal{E}'\beta$, as required. □

Remark 2.19. The morphisms $\mathcal{E}_*$ and $\mathcal{E}^*$ are natural.

Theorem 2.20. $\mathrm{Ex}(M, N) \simeq \mathrm{Ext}^1_R(M, N)$ *for any* $M, N \in R$-Mod.

Proof. Take a short exact sequence $\mathcal{E} : 0 \to K \to P \to M \to 0$ where P is projective. Thus $\mathcal{E}$ is the firts step of a projective resolution of M. Then we have the following long exact sequences:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathrm{Hom}(M,N) & \longrightarrow & \mathrm{Hom}(P,N) & \longrightarrow & \mathrm{Hom}(K,N) & \xrightarrow{\partial} & \mathrm{Ext}^1(M,N) & \longrightarrow & \mathrm{Ext}^1(P,N) \\
 & & \| & & \| & & \| & & & & \| \\
 & & & & & & & & & & 0 \\
 & & & & & & & & & & \| \\
0 & \longrightarrow & \mathrm{Hom}(M,N) & \longrightarrow & \mathrm{Hom}(P,N) & \longrightarrow & \mathrm{Hom}(K,N) & \xrightarrow[\mathcal{E}^*]{} & \mathrm{Ex}(M,N) & \longrightarrow & \mathrm{Ex}(P,N)
\end{array}$$

Here we know that $\mathrm{Ext}^1_R(P, N) = \mathrm{Ex}(P, N) = 0$ for every P projective module and any module N. This implies that $\mathrm{Ext}^1_R(M, N) \simeq \mathrm{Ex}(M, N)$. □

Definition 2.21. Let $M, N \in R$-Mod be R-modules. We shall say that an exact sequence

$$\mathcal{E} : 0 \to N \to X_{n-1} \to \cdots \to X_1 \to X_0 \to M \to 0$$

is *an extension of length n of the module M by the module N*. A *morphism between two extensions* is a chain map $(\nu, \xi_{n-1}, \ldots, \xi_0, \mu)$ between the two complexes. We shall call two extensions $\mathcal{E}$ and $\mathcal{E}'$ *equivalent* (and denote this by $\mathcal{E} \equiv \mathcal{E}'$) if there exists a sequence of morphisms

$$\mathcal{E} \xrightarrow{\Gamma_0} \mathcal{E}_1 \xleftarrow{\Gamma_1} \mathcal{E}_2 \xrightarrow{\Gamma_2} \cdots \xleftarrow{\Gamma_s} \mathcal{E}'$$

with $\Gamma_i = (1_N, \xi^i_{n-1}, \dots, \xi^i_0, 1_M)$. Thus we take the symmetric and transitive closure of the relation defined by the existence of a morphism of the given type. We should note that in case of extensions of length 1 the 5-lemma implies that the middle homomorphisms is an isomorphism, hence the equivalence of extensions of length 1 is much simpler.

Definition 2.22. For $\mathcal{E} : 0 \to N \to X_{k-1} \to \cdots \to X_0 \to M \to 0$ and $\mathcal{E}' : 0 \to Y_{\ell-1} \to \cdots \to Y_0 \to K \to 0$ we define the *Yoneda product* of these two extensions as

$$\mathcal{E} \circ \mathcal{E}' : 0 \to N \to X_{k-1} \to \cdots \to X_0 \to Y_{\ell-1} \to \cdots Y_0 \to K.$$

It is easy to see that forming the Yoneda product is associative and it is compatible with the equivalence of extensions.

A routine calculation leads us to the following statement:

Proposition 2.23. *Every extension of length n is the Yoneda product of n short exact sequences. Furthermore, if $\mathcal{E} = \mathcal{E}_{n-1} \circ \mathcal{E}_{n-2} \circ \cdots \circ \mathcal{E}_0$ and $\mathcal{E}' = \mathcal{E}'_{n-1} \circ \mathcal{E}'_{n-2} \circ \cdots \circ \mathcal{E}'_0$ is such Yoneda decomposition of two extensions of length n then $\mathcal{E}$ and $\mathcal{E}'$ are equivalent if and only if the Yoneda decomposition of $\mathcal{E}'$ can be obtained from the decomposition of $\mathcal{E}$ by repeated application of the following steps:*

(i) replace $\tilde{\mathcal{E}}_i$ by $\overline{\mathcal{E}}_i$ where $\tilde{\mathcal{E}}_i$ and $\overline{\mathcal{E}}_i$ are equivalent short exact sequences;
(ii) replace $\tilde{\mathcal{E}}_i\alpha \circ \tilde{\mathcal{E}}_{i-1}$ by $\tilde{\mathcal{E}}_i \circ \alpha\tilde{\mathcal{E}}_{i-1}$;
(iii) replace $\tilde{\mathcal{E}}_i \circ \alpha\tilde{\mathcal{E}}_{i-1}$ by $\tilde{\mathcal{E}}_i\alpha \circ \tilde{\mathcal{E}}_{i-1}$.

Definition 2.24. We shall denote by $\mathrm{Ex}^n(M,N)$ the set(?) of equivalence classes of extensions of length n of m by N.

Definition 2.25. If $\mathcal{E} = \mathcal{E}_{n-1} \circ \cdots \circ \mathcal{E}_0 \in \mathrm{Ex}^n(M,N)$, furthermore $\mu : M' \to M$ and $\nu : N \to N'$, then we define $\mathcal{E}\mu = \mathcal{E}_{n-1} \circ \cdots \circ \mathcal{E}_0\mu \in \mathrm{Ex}^n(M',N)$ and $\nu\mathcal{E} = \nu\mathcal{E}_{n-1} \circ \cdots \circ \mathcal{E}_0 \in \mathrm{Ex}^n(M,N')$. One should check that this definition is correct, i. e. it is compatible with the equivalence of extensions.

Proposition 2.26. Ex^n *is a bifunctor from* R-Mod $\times$ R-Mod *to* SET.

Proof. We have to prove only that $\mathrm{Ex}(M,N)$ is indeed a set (and not a class). Namely, unlike with short exat sequences, i. e. extensions of length 1 where the cardinality of the modules being part of the short exact sequence is bounded from above (once we fix the end terms), in the case of extensions of length at least 2 we do not have such bound: we can always add a module as a direct summand to consecutive terms in the middle terms of the extension:

$$0 \to N \to X_{n-1} \to X_{n-2} \to \cdots \to X_0 \to M \to 0 \in \mathrm{Ex}^n(M,N)$$
$$\Downarrow$$
$$0 \to N \to X_{n-1} \oplus Y \to X_{n-2} \oplus Y \to \cdots \to X_0 \to M \to 0 \in \mathrm{Ex}^n(M,N)$$

Thus modules of arbitrarily large cardinality may occur among the middle terms of an extension.

We will show that every extension of length n is equivalent to one where the modules in the interior of the extension have fixed cardinalities (depending on M and N, of course). Namely let us take an arbitrary extension $\mathcal{E} \in \mathrm{Ex}^n(M,N)$ plus a fixed projective resolution of M and then take the lifting of the idenity map of M to this resolution:

$$\begin{array}{lccccccccccccc}
\tilde{\mathcal{E}}\colon & 0 & \to & K_n & \to & P_{n-1} & \to & \cdots & \to & P_0 & \to & M & \to & 0 \\
 & & & \nu \downarrow & & \nu_{n-1} \downarrow & & & & \nu_0 \downarrow & & \| & & \\
\mathcal{E}\colon & 0 & \to & N & \to & X_{n-1} & \to & \cdots & \to & X_0 & \to & M & \to & 0
\end{array}$$

Let us recall that the lifting $(\nu, \nu_{n-1}, \ldots \nu_0)$ – as in some earlier situations – exists since the modules P_i are projective. Let us now write both sequences as the Yoneda product of short exact sequences and we will insert into this diagram the sequence $\nu\tilde{\mathcal{E}} = \nu(\tilde{\mathcal{E}}_{n-1} \circ \cdots \circ \tilde{\mathcal{E}}_0)$:

$$\begin{array}{lcccccccccccccccccc}
\tilde{\mathcal{E}}\colon & 0 & \to & K_n & \to & P_{n-1} & \to & K_{n-1} & \to & 0 & \to & K_{n-1} & \to & P_{n-2} & \to \cdots \to & P_0 & \to & M & \to 0 \\
 & & & \nu\downarrow & \searrow^{\nu_{n-1}} & \downarrow \overline{\nu}_{n-1} & & \| & & & & \| & & \| & & \| & & \| & \\
\nu\tilde{\mathcal{E}}\colon & 0 & \to & N & \to & \tilde{P}_{n-1} & \to & K_{n-1} & \to & 0 & \to & K_{n-1} & \to & P_{n-2} & \to \cdots \to & P_0 & \to & M & \to 0 \\
 & & & \| & & \downarrow \mu_{n-1} & & \downarrow \mu & & & & \downarrow \mu & & \downarrow \nu_{n-2} & & \downarrow \nu_0 & & \| & \\
\mathcal{E}\colon & 0 & \to & N & \to & X_{n-1} & \to & L_{n-1} & \to & 0 & \to & L_{n-1} & \to & X_{n-2} & \to \cdots \to & X_0 & \to & M & \to 0
\end{array}$$

The existence of μ_{n-1} and μ follows from the fact that the morphism from $\tilde{\mathcal{E}}_{n-1}$ to $\mathcal{E}_{n-1}$ where the first module homomorphism is ν can be factored through $\nu\tilde{\mathcal{E}}_{n-1}$.

Thus we get a morphism $(1_N, \mu_{n-1}\nu_{n-2}, \ldots, \nu_0, 1_M) : \ \nu\tilde{\mathcal{E}} \to \mathcal{E}$ implying that these two extensions are equivalent. On the other hand we can see that the cardinality of the modules appearing in $\nu\tilde{\mathcal{E}}$ are bounded from above (once we fix the projective resolution of M), hence the equivalence classes of extensions of length n form a set. □

The following consequence of the previous argument is worth noting:

Corollary 2.27. *Every extension $\mathcal{E} \in \mathrm{Ex}^n(M,N)$ is equivalent to an extension of the type*

$$\mathcal{E}' : \ 0 \to N \to Y_{n-1} \to P_{n-2} \to \cdots \to P_0 \to M \to 0$$

where P_i is projective for $0 \le i \le n-2$.

A routine calculation shows the following statement.

Proposition 2.28. *If one defines the Baer sum on* $\mathrm{Ex}^n(M,N)$ *similarly to the length one case then* $\mathrm{Ex}^n(M,N)$ *becomes an Abelian group and* $\mathrm{Ex}^n(-,-)$ *is a bifunctor to* AB.

Theorem 2.29. *For every* $M, N \in R\text{-Mod}$ *we have* $\mathrm{Ex}^n(M,N) \simeq \mathrm{Ext}^n_R(M,N)$ *and this isomorphism is natural in both variables.*

Proof. We shall only give the isomorphism between the two sets.

Let us fix a projective resolution $\mathcal{P}(M)^*_\bullet$ of the module M. For $\mathcal{E} \in \mathrm{Ex}^n(M,N)$ we take the lifting of id_M to $\mathcal{P}(M)_\bullet$ as we have seen in the proof of Proposition 2.26:

$$\begin{array}{lccccccccccccc}
\mathcal{P}(M)^*_\bullet\colon & P_{n+1} & \xrightarrow{\partial_{n+1}} & P_n & \xrightarrow{\partial_n} & P_{n-1} & \xrightarrow{\partial_{n-1}} & \cdots & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & {\scriptstyle 0}\downarrow & & {\scriptstyle g_n}\downarrow & & {\scriptstyle g_{n-1}}\downarrow & & & & {\scriptstyle g_0}\downarrow & & \| & & \\
\mathcal{E}\colon & 0 & \longrightarrow & N & \longrightarrow & X_{n-1} & \longrightarrow & \cdots & \longrightarrow & X_0 & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

We assign to $\mathcal{E}$ the homology class of g_n in $\mathrm{Hom}(P_n, N)$ in the complex $\mathrm{Hom}(\mathcal{P}(M)_\bullet, N)$: by definition the n-th cohomology of this complex is $\mathrm{Ext}^n_R(M,N)$. Of course, one has to check first that the mapping is well-defined.

First we have to show that $g_n \in \mathrm{Ker}\,\mathrm{Hom}(\partial_{n+1}, 1_N)$ where

$$\mathrm{Hom}(\partial_{n+1}, 1_N) : \mathrm{Hom}(P_n, N) \longrightarrow \mathrm{Hom}(P_{n+1}, N)$$

is the right multiplication by ∂_{n+1}. But the image of g_n under this map is $g_n\partial_{n+1}$ and the commutativity of the above diagram implies that this is equal to 0.

On the other hand we know that any other lifting g'_n if id_M is homotopic to g_n: this means that $g_n - g'_n = s_{n-1}\partial_n \in \mathrm{Im}\,\mathrm{Hom}(\partial_n, 1_N)$. Thus the mapping $\mathcal{E} \mapsto [g_n]$ gives a well-defined correspondence from $\mathrm{Ex}^n(M,N)$ to $H^n(\mathrm{Hom}_R(\mathcal{P}(M)_\bullet, N) = \mathrm{Ext}^n_R(M,N)$, as required.

We urge the reader to find explicitly the inverse of this correspondence. □

We shall conclude this section by defining the concept of homological dual of an algebra.

Definition 2.30. Let A be a finite dimensional algebra over a field and take the semisimple top of A, defined as ${}_A\hat{S} = A/\operatorname{rad} A$. We may form the vector space $A^* = \bigoplus_{i=0}^{\infty} \mathrm{Ext}^i_A(\hat{S}, \hat{S})$ and define a multiplication of A^* via the Yoneda product. In this way we get a graded algebra structure on A^*: this is called the *extension algebra* or *homological dual* of A. Interesting cases arise when $A^* \simeq A$ (in this case we call A self dual), or when $A^{**} \simeq A$.

3. Homological dimensions

Definition 3.1. Let $M \in R$-Mod be an arbitrary R-module. We will say that the *projective dimension of* M is n and we will denote this by $pd\, M = n$ if there exists a projective resolution of M of length n, i. e. an exact sequence

$$0 \to P_n \xrightarrow{p_n} P_{n-1} \xrightarrow{p_{n-1}} \cdots \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \to 0$$

with P_i projective for $0 \leq i \leq n$, furthermore no projective resolution of length smaller than n exists. In particular this means that the kernels of the maps p_i are not projective for $i < n$. If no finite resolution exists then we will say that M is of *i*nfinite projective dimension ($pd\, M = \infty$).

Definition 3.2. The *i*njective dimension of the module $M \in R$-Mod is n (denoted by $id\, M = n$) if there exists an injective (co)resolution of M of length n, i. e. an exact sequence

$$0 \to M \xrightarrow{i_0} \to I_0 \xrightarrow{i_1} \cdots \xrightarrow{i_{n-1}} I_{n-1} \xrightarrow{i_n} I_n \to 0$$

with I_i injective for $0 \leq i \leq n$, furthermore no injective coresolution of length smaller than n exists. In particular this means that the cokernels of the maps i_i are not injective for $i < n$. If no finite coresolution exists then we will say that M is of *i*nfinite injective dimension ($id\, M = \infty$).

Remark 3.3. It is immediate that $pd\, M = 0$ iff M is projective and similarly, $id\, M - 0$ iff M is injective.

Example 3.4. a) for $\mathbb{Z}_2 \in$ AB we have $0 \to \mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z} \to \mathbb{Z}_2 \to 0$, showing that $pd\, \mathbb{Z}_2 \leq 1$. On the other hand $\mathbb{Z}_2$ is obviously not projective, hence $pd\, \mathbb{Z}_2 = 1$.

b) Take $R = \mathbb{Z}_4$ and consideer $\mathbb{Z}_2$ as an R-module. Then we have the following exact sequence: $0 \to \mathbb{Z}_2 \xrightarrow{2\cdot} \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2 \to 0$. The kernel is clearly not projective (it is annihilated by $2 \in R$), hence we get an infinite projective resolution:

$$\cdots \xrightarrow{2\cdot} \mathbb{Z}_4 \xrightarrow{2\cdot} \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2 \to 0$$

It is rather tempting to say that $pd\, {}_R\mathbb{Z}_2 = \infty$, however at this momment we do not know whether another projective resolution could terminate in a finite number of steps. Thus for the time being – only from the existence of one infinite projective resolution with non-projective kernel terms – we cannot conclude that there is no finite resolution. We shall need a stronger characterization for this.

Proposition 3.5. *The following are equivalent for a module* ${}_RM$*:*

(1) $pd\, M = n$.

(2) there exists $N \in R$-Mod *so that* $\mathrm{Ext}^n_R(M, N) \neq 0$ *but for every* $N \in R$-Mod *we have* $\mathrm{Ext}^{n+1}_R(M, N) = 0$.

(3) for every $0 \leq i \leq n$ *there exists* $N_i \in R$-Mod *so that* $\mathrm{Ext}^i_R(M, N_i) \neq 0$ *but for every* $N \in R$-Mod *and every* $m > n$ *we have* $\mathrm{Ext}^m_R(M, N) = 0$.

Proof. (1) ⇒ (2) Suppose we have a projective resolution

$$\mathcal{P}_\bullet : \ 0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0.$$

Then from the definition of derived functors we get that $\mathrm{Ext}^m_R(M, -)-$ for every $m > n$ since in this case the m-th term of the complex $\mathrm{Hom}_R(\mathcal{P}_\bullet, N)$ is 0, hence the m-th cohomology is also 0. On the other handm since there is no shorter projective resolution, we get that the syzygies $K_1, \ldots, K_{n-1}$ are non-projective:

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & P_n & \longrightarrow & P_{n-1} & \longrightarrow & P_{n-2} & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & M \longrightarrow 0 \\ & & & & \searrow & \nearrow & & \searrow & & \nearrow & & & \\ & & & & & K_{n-1} & & K_{n-2} & & K_1 & & & \end{array}$$

But by considering the long exact sequence of homologies obtained from the short exact sequences $0 \to K_i \to P_{i-1} \to K_{i-1} \to 0$ (with $K_0 = M$) we get that

$$\mathrm{Ext}^i_R(K_0, N) \simeq \mathrm{Ext}^{i-1}_R(K_1, N) \simeq \cdots \simeq \mathrm{Ext}^1_R(K_{i-1}, N)$$

(The previous argument is often called the *dimension shifting* argument.) Now, for $i \leq n$ we have that K_{i-1} is non-projective, hence there exists N_i such that $\mathrm{Ext}^1_R(K_{i-1}, N_i) \neq 0$. But then $\mathrm{Ext}^i_R(M, N_i) \neq 0$ holds.

(3) ⇒ (2) is obvious.

(2) ⇒ (1) Since $\mathrm{Ext}^{n+1}_R(M, -) = 0$, the dimension shofting argument gives us that $\mathrm{Ext}^1_R(K_n, -) = 0$ for any projective resolution, implying that the n-th syzygy is projective. This gives that $pdimM \leq n$. On the other hand, if there was a shorter projetive resolution, i. e. $pd\, M < n$, then by referring once more to the definition of dervied fnctors, this would ipmply that $\mathrm{Ext}^n_R(M, -) = 0$, contradicting (2). Thus $pd\, M = n$. □

A dual argiment gives that the following characterizations of injective dimension.

Proposition 3.6. *The following are equivalent for a module* $_RM$*:*

(1) $id\, N = n$.
(2) there exists $M \in R\text{-Mod}$ *so that* $\mathrm{Ext}^n_R(M, N) \neq 0$ *but for every* $M \in R\text{-Mod}$ *we have* $\mathrm{Ext}^{n+1}_R(M, N) = 0$.
(3) for every $0 \leq i \leq n$ *there exists* $M_i \in R\text{-Mod}$ *so that* $\mathrm{Ext}^i_R(M_i, N) \neq 0$ *but for every* $M \in R\text{-Mod}$ *and every* $m > n$ *we have* $\mathrm{Ext}^m_R(M, N) = 0$. □

Corollary 3.7. *By taking different projective resolutions of a fixed module* M*, the kernels are projective for the same indices for each pojective resolution.*

Proof. The dimension shifting argument gives that $\mathrm{Ext}^i_R(M, N) \simeq \mathrm{Ext}^1(K_{i-1}, N)$ for the syzygy K_{i-1} in an arbitrary resolution and thus the projectivity of K_{i-1} is equivalent to the condition that $\mathrm{Ext}^i_R(M, -) = 0$. □

The following result gives a few basic connections between projective dimensions of the members of a short exact sequence.

Proposition 3.8. *Let $0 \to N \to K \to M \to)$ be a short exact sequence.*

(1) If at least two of the projective dimnesions $pd\, N, pd\, K$ and $pd\, M$ are finite then so is the third dimension.

(2) If $pd\, K > pd\, N$, then $pd\, K = pd\, M$.

(3) If $pd\, K < pd\, N$, then $pd\, M = pd\, N + 1$.

(4) If $pd\, K = pd\, N$, then $pd\, M \leq pd\, N + 1$.

(5) $pd\, K \leq \max\{pd\, N, pd\, M\}$.

Proof. The proof is based on the long exact sequence of Hom and Ext groups. Let us show the process on one of the statements, for example let us prove (3). Suppose $pd\, K < pd\, N$. In particular this implies that $pd\, K = n < \infty$. Then we have the following exact sequence:

$$\cdots \mathrm{Ext}_R^k(K, X) \to \mathrm{Ext}_R^k(N, X) \to \mathrm{Ext}_R^{k+1}(M, X) \to \mathrm{Ext}_R^{k+1}(K, X) \to \cdots$$

Then for every $n < k \leq pd\, N$ we get that $\mathrm{Ext}_R^k(K, -) = 0$, hence $\mathrm{Ext}_R^k(N, X) \simeq \mathrm{Ext}_R^{k+1}(M, X)$. Since for every $n < k \leq pd\, N$ there exists a module X_k such that $\mathrm{Ext}_R^k(N, X_k) \neq 0$, we get that $pd\, M \geq pd\, N + 1$, furthermore – since we already know that $pd\, M \geq pd\, K + 2$ – we get that $pd\, N \geq pd\, M - 1$. This implies that $pd\, M = pd\, N + 1$. □

The homological complexity of modules, measured by projective and injective dimension gives also a measure for the complexity of the base ring.

Definition 3.9. The (left) global dimension of a ring R is defined as:

$$l\,gl\,dim\, R = \sup\{pd\, M \mid M \in R\text{-Mod}\} = \sup\{id\, N \mid N \in R\text{-Mod}\}.$$

One may define the right global dimension of a ring ($r\,gl\,dim\, R$) in an a similar fashion.

Proposition 3.10. *The left global dimension of an algebra is well defined, i. e. the supremum of projective dimensions and the supremum of injective dimensions if R-modules will coincide.*

Proof. In view of the characterization of projective and injective dimensions, both numbres are equal to the number: $\sup\{n \mid \exists M, N \in R\text{-Mod}\ \ \mathrm{Ext}_R^n(M, N) \neq 0\}$. □

Example 3.11. (1) R is semisimple $\Leftrightarrow l\,gl\,dim\, R = 0$.

(2) R is left hereditary $\Leftrightarrow l\,gl\,dim\, R \leq 1$. (Let us recall that R is called left hereditary iff the homomorphic image of a (left) injective R-module is always injective, or equivalently, if submdules of projective left R-modules are projective.)

Proposition 3.12. (Auslander) $l\,gl\,dim\, R = \sup\{pd\, R/I \mid {}_RI \leq {}_RR\}$. *This means that to compute the global dimension of the ring, it is enough to check the projective dimensions of cyclic modules.*

Proof. If the supremum defined above is infinite then we are done. So we may assume that the supremum of projective dimensions of cyclic modules is finite and it is equal to n. We have to show that $\mathrm{Ext}_R^{n+1}(-,-)=0$. Let us take arbitrary modules $M,N\in R$-Mod and compute $\mathrm{Ext}_R^{n+1}(M,N)$. To do this, let us start with an injective coresolution of N:

$$\begin{array}{ccccccccccccccc} 0 & \longrightarrow & N & \longrightarrow & I_0 & & \longrightarrow & & I_1 & \longrightarrow & \cdots \longrightarrow & I_n & & \longrightarrow & I_{n+1} \longrightarrow \cdots \\ & & & & & \searrow & & \nearrow & & & \searrow \quad \nearrow & & \searrow \quad \nearrow & & \\ & & & & & & C_1 & & & & C_n & & C_{n+1} & & \end{array}$$

The standard dimension shifting argument gives us the following:

$$\mathrm{Ext}_R^{n+1}(M,N)\simeq \mathrm{Ext}_R^{n}(M,C_1)\simeq\cdots\simeq \mathrm{Ext}_R^{1}(M,C_n).$$

If $M=R/I$ for some left ideal ${}_RI\le {}_RR$, i. e. M is cyclic then

$$\mathrm{Ext}_R^{1}(R/I,C_n)\simeq \mathrm{Ext}_R^{n+1}(R/I,N)=0,$$

since $pd\,R/I\le n$. On the other hand by our earlier results (in particular, by Proposition 2.16) we get that every homomorphism $f:I\to C_n$ has an extension $\tilde f:R\to C_n$, since the pushout sequence, as an element of $\mathrm{Ex}(R/I,C_n)\simeq \mathrm{Ext}_R^1(R/I,C_n)=0$, must split:

$$\begin{array}{rccccccccc} \mathcal{E}\colon & 0 & \longrightarrow & I & \longrightarrow & R & \longrightarrow & R/I & \longrightarrow & 0 \\ & & & {\scriptstyle f}\big\downarrow & \swarrow_{\tilde f} & \big\downarrow & & \| & & \\ f\mathcal{E}\colon & 0 & \longrightarrow & C_n & \longrightarrow & \tilde K & \longrightarrow & R/I & \longrightarrow & 0 \end{array}$$

By the Injective test lemma this implies that C_n is injective hence for arbitrary M we have $\mathrm{Ext}_R^1(M,C_n)=0$. But as we have seen, this gives $\mathrm{Ext}_R^{n+1}(M,N)$ for arbitrary modules M and N. Thus $lgl\,dim\,R=n$. $\square$

It is clear that computing the projective dimension of syclic modules may still be a considerable task. However, for Artinian rings we have an even smaller set of "test cases".

Corollary 3.13. *If R is left Artinian then $lgl\,dim\,R=\max\{pd\,S\mid S \text{ is simple}\}$.*

Proof. The Artinian assumption implies that for any cyclic module R/I the composition length $\ell(R/I)$ is finite, i. e. every cyclic modules has a finite composition series. But using Proposition 3.8.(5) and induction on the composition length we get that for any module M with $\ell(M)<\infty$ we have $pd\,M\le\max\{pd\,S\mid S \text{ is simple}\}$. In particular this gives that the projective dimension of cyclic modules is bounded above by the projective dimension of simple modules. $\square$

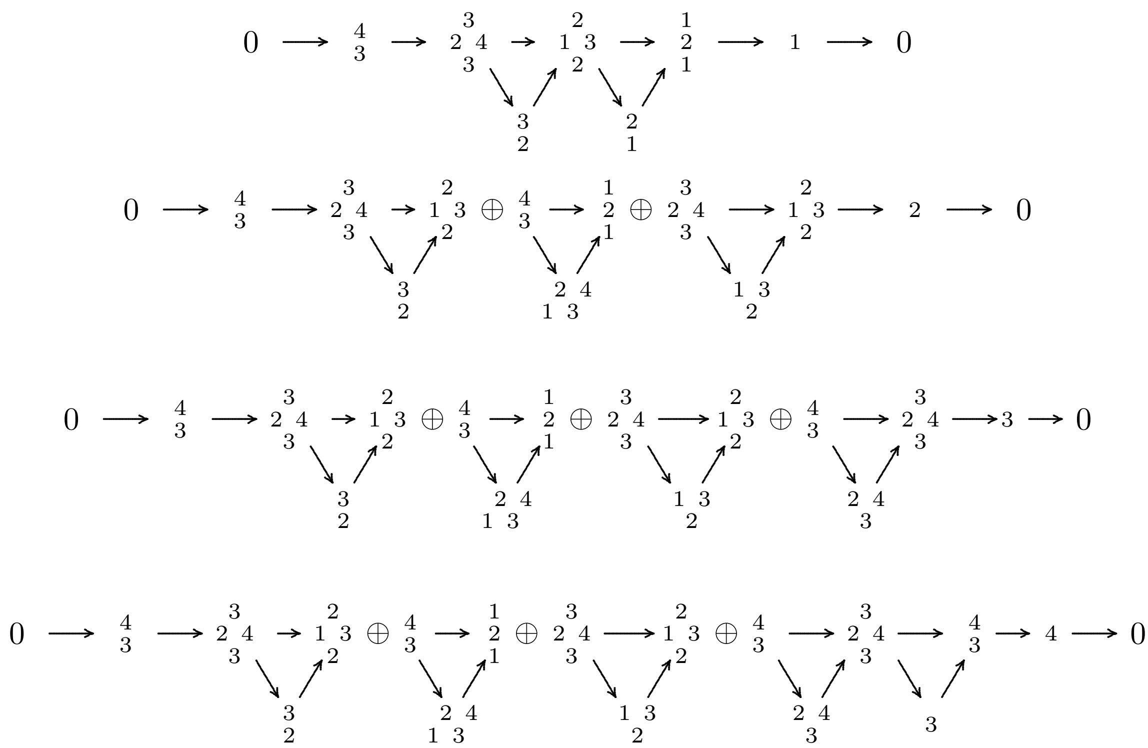
Example 3.14. 1) Let A be the path agebra whose right regular structure can be described by the following decomposition:

$${}_AA=\begin{smallmatrix}1\\2\\1\end{smallmatrix}\oplus\begin{smallmatrix}&2&\\1&&3\\&2&\end{smallmatrix}\oplus\begin{smallmatrix}&3&\\2&&4\\&3&\end{smallmatrix}\oplus\begin{smallmatrix}4\\3\end{smallmatrix}$$

(For the notation used above where numbers denote composition factors corresponding to the vertices of the graph and the diagram reflects the Loewy structure of the indecomposable projective modules we refer to the problem seets and their solutions.) One can check that the simple modules have the following projective dimensions:

$$pd\ 1 = 3, \quad pd\ 2 = 4, \quad pd\ 3 = 5, \quad pd\ 4 = 6 \quad \Rightarrow \quad lgl\,dim\, A = 6$$

Namely, we can write the following projective resolutions:



2) Let Γ be a graph, and assume that the path algebra $A = K\Gamma$ is finite dimensional (i. e. Γ is finite and has no oriented cycles). Then A is (left) hereditary, i. e. the global dimension of A is at most 1.

Proof. Let e_i be the idempotent corresponding to vertex i and suppose that the arrows ending at i are $j_1 \xrightarrow{\alpha_1} i$, $j_2 \xrightarrow{\alpha_2} i$, $\ldots$, $j_k \xrightarrow{\alpha_k} j_k$. Then $\mathrm{rad}(Ae_i)$ can be written as

$$\mathrm{rad}\, Ae_i = A\alpha_1 \oplus A\alpha_2 \oplus \cdots \oplus A\alpha_k$$

Please, note that $A\alpha_\ell \simeq Ae_{j_\ell}$ is projective for every ℓ: the isomorphism from Ae_{j_ℓ} to $A\alpha_\ell$ can be given as the right multiplication by $\alpha_e ll$. This means that a projctive resolution of the simple module $S(i) = Ae_i/\,\mathrm{rad}\, Ae_i$ can be written as:

$$0 \to \mathrm{rad}\, Ae_i \to Ae_i \to S(i) \to 0$$

Thus $pd\, S(i) \leq 1$ for every i.

3) $lgl\,dim\, \mathbb{Z}_4 = \infty$ since $pd\, \mathbb{Z}_2 = \infty$.

4) It can be shown that if R is a commutative integral domain then R is hereditary iff R is a Dededkind domain.

Remark 3.15. 1) In defining the left global dimension we referred to left modules. It is easy to see that $lgldim\, R = 0 \Leftrightarrow rgldim\, R = 0$ fr any ring R (since both conditions are equivalent to requiering that R is semisimple. On the other hand the two global dimensions need not coincide in general: in 1967 Jategaonkar showed that for every $1 \leq m, n \leq \infty$ there exists a ring $R_{m,n}$ such that

$$lgldim\, R_{m,n} = m \qquad \text{and} \qquad rgldim\, R_{m,n} = n.$$

2) For a finite dimensional algebra A over a field the two global dimensions coincide: $lgldim\, A = rgldim\, A$, and this common value will be called the *global dimension* of A and will be denoted by $gldim\, A$. (For the proof of this fact see the problem sheets.) in general it can be shown that if R is both left and right Noetherian then $lgldim\, R = rgldim\, R$.

As one can see from the examples, many rings have infinite global dimension. This measure usually can be seen from an easy example of a module having infinite projective dimension. On the other hand a question may be asked about the existence of modules with arbitrarily large but finite projective dimension. This motivates the following definition.

Definition 3.16. For a ring R we define the following numbers:

$$pFindim\, R = \sup\{pd\, M \mid M \in R\text{-Mod},\ pd\, M < \infty\}$$
$$pfindim\, R = \sup\{pd\, M \mid M \in \text{Mod-}R,\ pd\, M < \infty\}$$
$$iFindim\, R = \sup\{id\, M \mid M \in R\text{-Mod},\ id\, M < \infty\}$$
$$ifindim\, R = \sup\{id\, M \mid M \in \text{Mod-}R,\ id\, M < \infty\}$$

(Actually one should also take into the account that here we consider left modules only but we shall refrain from defining four more dimensions.) Here $pFindim\, R$ is called the *projectively defined big finitistic dimension of* R, while $pfindim\, R$ is the *projectively defined little finitistic dimension of* R. Please, note that in the definition of the little finitistic dimensions we look at the projective dimensions of finitely generated modules only.

One of the most important long-standing conjectures in the theory of finite dimensional algebras is the so called *finitistic dimension conjecture*. They first appeared in 1960 in a paper by Bass (and were originally formulated as problems).

Conjecture 3.17. (Finitistic dimension conjectures) *Let A be a finite dimensional algebra. Then:*
(1) $pfindim\, A = pFindim\, A$.
(2) (FDC) $pfindim\, A < \infty$.

Originally the problem was posed in a wider context, however it soon became apparent that the conjectures are false for many classes of rings. But the questions remained open for many decades with partial results for some special cases. Let us just list some important known results in connection of these conjectures.

1) The first conjecture was refuted by Zimmermann-Huisgen in 1991. Later on many examples were constructed, showing that the gap between the little and big finitistic dimension can be arbitrarily large. On the other hand the second conjecture (denoted by (FDC)) is still unsolved.

2) An example of Kirkman and Kuzmanovich shows that the second conjeture (about the finiteness of the little finitistic dimension) is not true for a slightly more general class, namely the authors constructed a semiprimary ring (i. e. a ring with semisimple top and nilpotent radical) so that the FDC fails for this ring.

3) A result by Green and Zimmermann-Huisgen shows that FDC holds for algebras with $J^3(A) = 0$.

4) Green, Kirkman and Kuzmanovich verified the conjecture for the class of monomial algebras (i. e. path algebras modulo relations, where the relations are generated by monomials (paths) only).