1. Consider the setup of the 3×3 lemma, i.e. a commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & X' & \to & Y' & \to & Z' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & X & \to & Y & \to & Z & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & X'' & \to & Y'' & \to & Z'' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

with exact rows. Show that the exactness of the first and third column does not imply the exactness of the middle column.

2. Consider the chain map $f_\bullet : X_\bullet \longrightarrow Y_\bullet$:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \mathbb{Z}_4 & \overset{1\mapsto 4}{\longrightarrow} & \mathbb{Z}_8 & \overset{1\mapsto 1}{\longrightarrow} & \mathbb{Z}_2 & 0 \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \mathbb{Z}_8 & \overset{1\mapsto 2}{\longrightarrow} & \mathbb{Z}_8 & \overset{1\mapsto 4}{\longrightarrow} & \mathbb{Z}_8 & 0 \longrightarrow & \cdots \end{array}$$

Here the vertical maps are the natural embeddings. Compute the long exact sequence of homologies obtained from the short exact sequence of complexes

$$0_\bullet \longrightarrow X_\bullet \xrightarrow{f_\bullet} Y_\bullet \longrightarrow \operatorname{Coker} f_\bullet \longrightarrow 0_\bullet.$$

3. Let A be a K-algebra, M and N A-modules, and let $\mathcal{E}$ and $\mathcal{E}'$ be two non-split extensions in the K-space $\operatorname{Ext}^1_A(M,N)$. Suppose that $\mathcal{E}$ and $\mathcal{E}'$ are linearly dependent. Show that the middle terms in $\mathcal{E}$ and $\mathcal{E}'$ are isomorphic.

4. a) Consider the algebra given by $A_A = \begin{smallmatrix}1\\2\\1\end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 1 & & 3\end{smallmatrix} \oplus \begin{smallmatrix}3\end{smallmatrix}$. Find the projective resolutions of the modules $X = \begin{smallmatrix}1\\2\end{smallmatrix}$ and of $Y = \begin{smallmatrix}2\end{smallmatrix}$.

b) Consider the algebra given by $B_B = \begin{smallmatrix}1\\2\\2\end{smallmatrix} \oplus \begin{smallmatrix}2\\2\\2\end{smallmatrix}$. Compute the projective resolution of the module $Z = \begin{smallmatrix} & & & 1\\ 1 & & 2 & \\ & 2 & & \end{smallmatrix}$.

5. Take the deleted projective resolution of the module X from the previous problem and apply to it the contravariant functor $\operatorname{Hom}(-, \begin{smallmatrix}1\\2\end{smallmatrix})$. Compute the dimension of the cohomologies of this new complex.

6. Consider the algebra $A_A = \begin{smallmatrix}1\end{smallmatrix} \oplus \begin{smallmatrix}2\end{smallmatrix} \oplus \begin{smallmatrix} & 3 & \\ 1 & & 2\end{smallmatrix}$ and take the short exact sequence $\mathcal{E} : 0 \longrightarrow \begin{smallmatrix}1\end{smallmatrix} \oplus \begin{smallmatrix}2\end{smallmatrix} \longrightarrow \begin{smallmatrix} & 3 & \\ 1 & & 2\end{smallmatrix} \longrightarrow \begin{smallmatrix}3\end{smallmatrix} \longrightarrow 0$. Find the (pullback) sequence $E\mu$ if $\mu : \begin{smallmatrix}3\\2\end{smallmatrix} \longrightarrow \begin{smallmatrix}3\end{smallmatrix}$ is the natural epimorphism.