Throughout the problem sheet A will denote a finite dimensional K-algebra, and gl dim R will stand for the left global dimension of the ring R.

1. Determine *gl dim* $\mathbb{Z}_n$ for each n.

2. Prove that for finite dimensional algebras the left and right global dimensions coincide.

3. Suppose that $\dim J(A) = \dim A - 1$. Show that for $M \in A$-Mod we have: $pd\, M < \infty \Leftrightarrow pd\, M = 0$ hence the projectively defined finitistic dimension of A is 0.

4. Suppose $(J(A))^2 = 0$ and *gl dim* $A = n$. Prove that $\dim A \geq 2n + 1$.

5. Suppose $(J(A))^2 = 0$. Prove that the finitistic dimension of A is finite.

6. Suppose *gl dim* $A = n \neq 0, \infty$. Show that there exists a simple module ${}_AS$ for which $pd\, S = n - 1$.

7. For every $n \in \mathbb{N}$ construct a graph Γ_n and a set of relations I_n such that:

a) *gl dim* $K\Gamma_n/I_n = n$;

b) *gl dim* $K\Gamma_n/I_n = n$, and Γ_n has two vertices.

8. Let $I \lhd A$ and $B = A/I$. Suppose that ${}_AI$ is projective. Prove that for arbitrary $M \in B$-Mod we have $pd\, {}_AM \leq pd\, {}_BM + 1$.

9. Let $I \lhd A$ and $B = A/I$. Suppose that ${}_AI$ is projective and $I^2 = I$. Prove that for $M, N \in B$-Mod and arbitrary $i \geq 0$ we have

$$\mathrm{Ext}^i_B(M, N) \simeq \mathrm{Ext}^i_A(M, N).$$

10. Let A be basic and write ${}_AA = P(1) \oplus \cdots \oplus P(n)$ where the modules $P(i)$ are indecomposable projective modules, nonisomorphic for $i \neq j$. Denote by $S(i)$ the simple module corresponding to $P(i)$, i. e. $S(i) = P(i)/\operatorname{Rad} P(i)$. For modules M with a composition series let us denote by $[M : S]$ the multiplicity of a simple module S in this composition series. The *Cartan matrix* $C(A)$ of the algebra A is defined as the $n \times n$ integer matrix where the i-th column consists of the composition multiplicities of the projective module $P(i)$, i. e. $c_{ji} = [P(i) : S(j)]$. Prove that if *gl dim* $A < \infty$ then $\det C(A) = \pm 1$. (A famous and longstanding conjecture – the so-called *Cartan determinant conjecture* – states that this determinant is always $+1$.)

11*. Prove that if *gl dim* $A \leq 2$ then $\det C(A) = +1$.

12*. Let $T \in A$-Mod be a module with $pd\, T \leq 1$ and $\mathrm{Ext}^1_A(T, T) = 0$. Prove that there is a module $X \in A$-Mod satisfying the following conditions: i) $pd\, X \leq 1$; ii) $\mathrm{Ext}^1_A(T \oplus X, T \oplus X) = 0$; iii) there exists an exact sequence $0 \to A \to Y_0 \to Y_1 \to 0$ with Y_i being direct sums of direct summands of T and X. (In such a situation the module $T \oplus X$ is called *a tilting module*.)