1. Let $\mathcal{P}^\infty$ denote the subcategory of those left R-modules which have finite projective dimension. Suppose $\mathcal{P}^\infty$ contains all injective modules. Prove that:

$$\big\{X \in R\text{-Mod} \,|\, \mathrm{Ext}^1_R(X,Y) = 0 \ \forall Y \in \mathcal{P}^\infty\big\} = \big\{X \in R\text{-Mod} \,|\, \mathrm{Ext}^i_R(X,Y) = 0 \ \forall Y \in \mathcal{P}^\infty, \forall i > 0\big\}.$$

2. Let A be a finite dimensional algebra, and A^* stand for the Yoneda extension algebra of A (cf. Problem #4/5).

a) Prove that if $gl\,dim\,A = n < \infty$ then $(J(A^*))^{n+1} = 0$.

b) For each $n \in \mathbb{N}$ construct an algebra A such that $gl\,dim\,A = n$ and $(J(A^*))^n \neq 0$.

c) For each $n \in \mathbb{N}$ construct an algebra A such that $gl\,dim\,A = n$ and $(J(A^*))^2 = 0$.

3. For arbitrary $T \in A$-Mod let us define $Gen\,T = \{X \in A\text{-Mod} \,|\, \exists \oplus T \twoheadrightarrow X \text{ an epimorphism}\}$. Suppose now that $pd\,T \leq 1$ and $\mathrm{Ext}^1(T,T) = 0$.

a) Prove that $Gen\,T$ is closed under extensions, direct sums and epimorphic images.

b) Show that $\mathrm{Ext}^1_A(T,Y) = 0$ for each $Y \in Gen\,T$.

4. Suppose there is a ring homomorphism $R \to S$. In this way every S-module becomes an R-module. Prove that for arbitrary $M \in S$-Mod we have: $pd\,_R M \leq pd\,_R S + pd\,_S M$. (Cf. Theorem 3.18 from the lecture.)

5. Show that $l\,gl\,dim\,A \leq 2$ if and only if for each $f \in \mathrm{Hom}(P_1, P_2)$ with P_i projective we have that $\mathrm{Ker}\,f$ is also projective.

6. Consider the following algebras, given by the structure of their projective decomposition:
a) $A_A = \begin{smallmatrix} 1 \\ 1 \end{smallmatrix}$; b) $A_A = \begin{smallmatrix} 1 \\ 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1 \\ 2 \end{smallmatrix}$; c) $A_A = \begin{smallmatrix} & 1 & \\ 1 & & 2 \\ & & 1 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}$.
Check the global dimension of these algebras and for each of them find a corresponding algebra B of finite global dimension so that $A \simeq eBe$ for a suitable idempotent element $e \in B$.

7. Let A be a finite dimensional algebra over K and for arbitrary K-space X denote by X^* the K-dual of X, i. e. $X^* = \mathrm{Hom}_K(X,K)$. Note that if N is a left A-module then N^* carries a natural right A-module structure. Prove that for arbitrary right A-module M and arbitary left A-module N we have $\mathrm{Ext}^n_A(M, N^*) \simeq (\mathrm{Tor}^A_n(M,N))^*$. (Thus computation of Tor can be reduced to the case of computing Ext.

8. Let A be a finite dimensional K-algebra of finite representation type, and let M be the direct sum of all isotypes of indecomposable modules in A-Mod. Denote by B the endomorphism algebra of M (acting on the opposite side to the scalars): $B = \mathrm{End}_A(M)$. (In this case B is called the *Auslander-algebra* of A.)

a) Prove that M_B is projective.

b) The functors $F : N \to \mathrm{Hom}_A(M,N)$ and $G : P \to P \otimes_B M$ give an equivalence between the category A-Mod and the category of projective right B-modules.