**1.** Give an example showing that the assumption in Gabriel's theorem on the base field being algebraically closed is indeed needed (i. e. for any field $K$ which is not algebraicaly closed find a finite dimensional basic algebra over $K$ which is not a path algebra modulo relations).

**2.** Find the Gabriel quiver $\Gamma(A)$ and a suitable admissible relation ideal $I$ for the following algebras $A$ so that $A \simeq K\Gamma(A)/I$:

a) $A = \begin{pmatrix} K & 0 & 0 & 0 \\ K & K & 0 & 0 \\ K & 0 & K & 0 \\ K & 0 & 0 & K \end{pmatrix}$; b) $A = \left\{ \begin{pmatrix} a & b & c \\ 0 & d & f \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d, f \in K \right\}$.

**3.** Recall the algebras from Problem #6/6:

a) $A_A = \begin{smallmatrix} 1 \\ 1 \end{smallmatrix}$; b) $A_A = \begin{smallmatrix} 1 \\ 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1 \\ 2 \end{smallmatrix}$; c) $A_A = \begin{smallmatrix} & 1 & \\ 1 & & 2 \\ & 1 & \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}$.

Find the structure (i. e. the graph and relations) of the algebras $B$ arising in Auslander's construction (i. e. where $B = \operatorname{End}(\oplus_{i=0}^{n} A/J(A)^i)$).

**4.** Find the structure of the Yoneda extension algebras of the following algebras, given by the structure of their regular module:

a) $A_A = \begin{smallmatrix} 1 \\ 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 1 & & 3 \\ & 2 & \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 2 \end{smallmatrix}$;

b) $A_A = 1 \oplus \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} & 4 & \\ 3 & & \\ 2 & & 5 \\ & 1 & \end{smallmatrix} \oplus \begin{smallmatrix} 5 \\ 1 \end{smallmatrix}$;

c) $A_A = \begin{smallmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 3 \\ 1 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 1 \\ 2 \end{smallmatrix}$;

d) $A_A = \begin{smallmatrix} 1 \\ 3 \\ 4 \end{smallmatrix} \oplus 2 \oplus \begin{smallmatrix} 3 \\ 4 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 4 \\ 2 \end{smallmatrix}$.

**5.**
a) Prove that if $R/J(R)$ is semisimple, then for any $R$-module $\operatorname{Rad} M = J(R)M$.
b) Define the *radical series* of a module $M$ by $M_0 = M$ and $M_i = \operatorname{Rad} M_{i-1}$ for each $i \geq 1$. Similarly define the *socle series* of $M$ by $M^0 = 0$ and define $M^i$ as the full preimage of $\operatorname{Soc}(M/M^{i-1})$. The length of the radical series is $n$ if $M_{n-1} \neq 0$ and $M_n = 0$. Similarly one may define the length of the socle series. Prove that if $R$ is semiprimary (i. e. $R/J(R)$ is semisimple and $J(R)$ nilpotent) then the lengths of the radical series and of the socle series are both finite and they coincide.
c) Define the *Loewy length* of a module as $L\ell\, M = \min\left\{k \in \mathbb{N} \mid J(R)^k M = 0\right\}$. Show that for semiprimary rings the Loewy length of a module coincides with the radical length and the socle length of the module.

**6.**
a) An algebra $A$ is *left (right) uniserial* if all indecomposable projective left (resp. right) modules are uniserial (i. e. they have a unique composition series). Give an example of an algebra which is left uniserial but not right uniserial.
b) Describe the quiver of those algebras which are both left and right uniserial (these are called *Nakayama algebras*).

**7*.** Prove that a finite dimensional Nakayama algebra is representation finite and describe its indecomposable modules.

**8*.** Let $A$ be a finite dimensional algebra with finite global dimension. Let $C \in M_n(\mathbb{Z})$ be the Cartan matrix of $A$. It is well known (cf. Problem #4/10) that $C$ is invertible in $M_n(\mathbb{Z})$. Define the following (non-symmetric) bilinear form on $\mathbb{Z}^n \times \mathbb{Z}^n$:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \left(C^T\right)^{-1} \mathbf{y}.$$

(This is the so called *Euler-characteristic of A*.) Prove that if for a module $X$ the *dimension vector* $\underline{\dim} X$ stands for the vector consisting of composition multiplicites of simple modules in $X$ (i. e. $(\underline{\dim} X)_i = [X : S(i)]$) then the following equality holds for arbitrary $X, Y \in A$-Mod:

$$\langle \underline{\dim} X, \underline{\dim} Y \rangle = \textstyle\sum_{i \geq 0} (-1)^i \dim \operatorname{Ext}_A^i(X, Y).$$