1. Give an example of two chain complexes $A_\bullet$ s $B_\bullet$ and chain maps $f_\bullet$ and $g_\bullet$ which are homologic but not homotopic.

Solution. Take a short exact sequence $E_\bullet$: $0 \to A \to B \to C \to 0$ which is non-split and consider two chain maps from the complex to itself: the first one is the 0 map, while the second one the identity map of $E_\bullet$. These two chain maps are clearly homologic since the homology modules are all 0, thus both maps act on them trivially. On the other hand a homotopy between the zero map and the identity map would give a splitting of the sequence.

2. Let R be a ring, $X_\bullet$ and $Y_\bullet \in C(R)$ chain complexes. Show that the set of nullhomotopic chain maps $\{f_\bullet \in \mathrm{Hom}_{C(R)}(X_\bullet, Y_\bullet) \mid f_\bullet \sim 0_\bullet\}$ is closed under addition hence it is a subgroup. Conclude from this that $K(R)$, the homotopy category of chain complexes over R, where $\mathrm{Ob}\, K(R) = \mathrm{Ob}\, C(R)$ and $\mathrm{Hom}_{K(R)}(X_\bullet, Y_\bullet) = \mathrm{Hom}_{C(R)}(X_\bullet, Y_\bullet)/\sim$, is a preadditive category.

Solution. If two pairs of chain maps are homotopic then so are their sums; the homotopy is provided by the sum of the corresponding homotopy maps. This shows that taking the homotopy classes of chain maps gives an Abelian group. Since the composition of maps is distributive with respect to the addition, the homotopy category is preadditive.

3. Prove the "delicate" version of the Five lemma. Suppose that the following diagram is commutative and has exact rows:

$$\begin{array}{ccccccccc}
A_1 & \longrightarrow & B_1 & \longrightarrow & C_1 & \longrightarrow & D_1 & \longrightarrow & E_1 \\
\alpha\downarrow & & \beta\downarrow & & \gamma\downarrow & & \delta\downarrow & & \varepsilon\downarrow \\
A_2 & \longrightarrow & B_2 & \longrightarrow & C_2 & \longrightarrow & D_2 & \longrightarrow & E_2
\end{array}$$

(i) If α is surjective, while β and δ are injective then γ is injective.

(ii) If ε is injective, while β and δ are surjective then γ is surjective.

Solution. Diagram chasing.

4. Let $0 \to A \to B \to C \to 0$ and $0 \to A' \to B' \to C' \to 0$ be short exact sequences. Determine for which of the following diagrams will the existence of morphisms denoted by dashed arrows automatically follow, making the whole diagram commutative?

a)
$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, dashed] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

b)
$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d, dashed] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

c)
$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

Solution. The morphism denoted with a dashed arrow exists for the first and third diagram. For example, the restriction of the map $B \to B'$ on the first diagram will map A (more precisely, the image of A in B) into the kernel of the map $B' \to C'$ since continuing in the other direction, i. e. via the mapping $B \to C$ it is mapped to 0. Since the kernel of $B' \to C'$ is equal to A' (i. e. to its image in B'), the restriction of $B \to B'$ maps A to A$'$. A similar argument settles the case of the third diagram. On the other hand the second diagram cannot always be completed. If $A = \mathbb{Z}_3$, $B = \mathbb{Z}_9$ and $C = \mathbb{Z}_3$, furthermore $A' = \mathbb{Z}_3$, $B' = \mathbb{Z}_3 \oplus \mathbb{Z}_3$ and $C' = \mathbb{Z}_3$, then if we choose the maps $A \to A'$ and $C \to C'$ to be isomorphisms, then the middle morphism would have to be an isomorphism (by the 5-lemma), which is impossible since $N \not\simeq B'$.

5. a) (The existence of pullback.) Prove the following statement:

For every
$$\begin{tikzcd} & C \arrow[d] \\ A \arrow[r] & B \end{tikzcd}$$
there exists
$$\begin{tikzcd} D \arrow[r] \arrow[d] & C \arrow[d] \\ A \arrow[r] & B \end{tikzcd}$$
comm., so that if
$$\begin{tikzcd} D' \arrow[r] \arrow[d] & C \arrow[d] \\ A \arrow[r] & B, \end{tikzcd}$$
then there is
$$\begin{tikzcd} D' \arrow[dr, dashed] \arrow[drr] \arrow[ddr] & & \\ & D \arrow[r] \arrow[d] & C \arrow[d] \\ & A \arrow[r] & B. \end{tikzcd}$$

Construct D together with the morphisms. Show that if $A \to B$ is a monomorphism (epimorphism), then so is $D \to C$. Prove that these two maps have isomorphic kernels.

b) (The existence of pushout.) Formulate the dual of the previous statement.

Solution. a) Let us denote the maps by $\alpha : A \to B$ and $\gamma : C \to B$. We can construct D as follows: $D = \{(a,c) \in A \oplus C \mid \alpha(a) = \gamma(c)\}$, and let the maps $\bar{\alpha} : D \to A$ and $\bar{\gamma} : D \to C$ be the corresponding projections. The definition of D gives that the second diagram is commutative. If the maps of the third diagram are denoted by $\alpha' : D' \to A$ and $\gamma' : D' \to B$ then $\alpha\alpha' = \gamma\gamma'$ implies that for arbitrary $d \in D'$ we have $(\alpha'(d), \gamma'(d)) \in D$. Hence we can define the required map $D' \to D$ making the third diagram commutative. If $(a,c) \in \operatorname{Ker}\bar{\gamma} \subseteq D$, that is $\bar{\gamma}(a,c) = c = 0$, then $\alpha(a) = \gamma(c) = \gamma(0) = 0$ implies that when α is injective, $a = 0$, hence $\bar{\gamma}$ is injective. When α is surjective then for every $c \in C$ there exists an $a \in A$, such that $\alpha(a) = \gamma(c)$, hence $(a,c) \in D$ and $\gamma'(a,c) = c$. Thus $\bar{\gamma}$ is surjective, too. Finally, the kernel of $\bar{\gamma} : D \to C$ consists of pairs $(a,0) \in A \oplus C$ for which $\alpha(a) = 0$, i. e. $\operatorname{Ker}\bar{\gamma} \simeq \operatorname{Ker}\alpha$. – b) We shall give only the construction of D: let $D = A \oplus C / \{(\alpha(b), -\gamma(b)) \mid b \in B\}$. One can check easily that the dual properties, defining the pushout diagram, hold.

6. Suppose that the module ${}_RM$ has a finite finitely generated free resolution, i. e. :

$$0 \to F_n \to \cdots \to F_0 \to M \to 0,$$

with F_i finitely generated and free. Prove that if M is projective then there exists an $m \in \mathbb{N}$, so that $M \oplus R^m$ is free (M is called in this case *"stably free"*).

Solution. We prove by induction on n. If $n = 0$ then M is free (hence stably free). If $n > 0$ then the projectivity of M gives us that the short exact sequence $0 \to K \to F_0 \to M \to 0$ is split hence $F_0 = R^k = K \oplus M$ for some $K \leq F_0$ and $k \in \mathbb{N}$. But here K is projective, too, and it has a finitely generated free resolution of length $n-1$. Hence by induction we get that there exists an $\ell \in \mathbb{N}$, for which $K \oplus R^\ell = R^m$ is free. But then $M \oplus K \oplus R^\ell = M \oplus R^m$, at the same time $M \oplus K \oplus R^\ell = R^k \oplus R^\ell$ is free as required.

7. Let $0 \to P'_\bullet \to P_\bullet \to P''_\bullet \to 0$ be a short exact sequence of chain complexes of projective R-modules. Show that if F is an additive functor from R-Mod to S-Mod, then applying the extension $\bar{F}$ to the sequence above, we get a short exact sequence of chain complexes over S, i.e. the sequence $0 \to \bar{F}(P')_\bullet \to \bar{F}(P)_\bullet \to \bar{F}(P'')_\bullet \to 0$ is also exact.

Solution. Since each short exact of projective modules splits and additive functors take split exact sequences to split exact sequences, we get the statement.

8. Let M be an arbitrary R-module, and P_1 and P_2 projective modules for which $0 \to K_1 \to P_1 \to M \to 0$ and $0 \to K_2 \to P_2 \to M \to 0$ are exact. Prove that $K_1 \oplus P_2 \simeq K_2 \oplus P_1$. (*Hint:* Take the pullback of the diagram $\begin{tikzcd} & P_1 \arrow[d] \\ P_2 \arrow[r] & M \end{tikzcd}$.)

Solution. Let us denote by P the pullback of the diagram given in the hint. Taking into the account the fact that each map is surjective (cf. also Problem #1/4) and completing the diagram with the kernel terms, we get the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & K_1' \arrow[r] \arrow[d] & K_1 \arrow[d] & \\
0 \arrow[r] & K_2' \arrow[r] \arrow[d] & P \arrow[r, "\alpha'"] \arrow[d, "\beta'"'] & P_1 \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & K_2 \arrow[r] & P_2 \arrow[r, "\alpha"'] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Here the maps $K_1' \to K_1$ and $K_2' \to K_2$ exist by Problem #1/3 and are isomorphisms by Problem #1/4. If we now consider the horizontal and vertical exact sequences going through P we get that both are split exact since the cokernel terms are P_1 and P_2 which are projective. This gives that $P = K_1 \oplus P_2 = K_2 \oplus P_1$. (*Note:* In the literature this statement is known as Schanuel's lemma.)

9. Consider the graph $\Gamma : \ 1 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} 2 \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} 3 \underset{\beta_3}{\overset{\alpha_3}{\rightleftarrows}} 4$ together with the ideal of defining relations $I = (\alpha_1\alpha_2, \alpha_2\alpha_3, \beta_2\beta_1, \beta_3\beta_2, \beta_1\alpha_1 - \alpha_2\beta_2, \beta_2\alpha_2 - \alpha_3\beta_3, \beta_3\alpha_3)$ and the path algebra $A = K\Gamma/I$ modulo these relations. Describe the minimal projective resolution of all simple modules.

Solution. With the usual notation for modules we get the following: (please, note that in class for simplicity we dealt with right A-modules, however in this case A is left-right symmetric):

$$A_A \simeq \begin{smallmatrix}1\\2\\1\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\1&&3\\&2&\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\2&&4\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}4\\3\end{smallmatrix}.$$

Then the minimal projective resolutions of simple right A-modules are the following:

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & \begin{smallmatrix}4\\3\end{smallmatrix} \arrow[r] & \begin{smallmatrix}&3&\\2&&4\\&3&\end{smallmatrix} \arrow[rr] \arrow[dr] & & \begin{smallmatrix}&2&\\1&&3\\&2&\end{smallmatrix} \arrow[rr] \arrow[dr] & & \begin{smallmatrix}1\\2\\1\end{smallmatrix} \arrow[r] & 1 \arrow[r] & 0 \\
 & & & \begin{smallmatrix}3\\2\end{smallmatrix} \arrow[ur] & & \begin{smallmatrix}2\\1\end{smallmatrix} \arrow[ur] & & &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & \begin{smallmatrix}4\\3\end{smallmatrix} \arrow[r] & \begin{smallmatrix}&3&\\2&&4\\&3&\end{smallmatrix} \arrow[rr] \arrow[dr] & & \begin{smallmatrix}&2&\\1&&3\\&2&\end{smallmatrix} \oplus \begin{smallmatrix}4\\3\end{smallmatrix} \arrow[rr] \arrow[dr] & & \begin{smallmatrix}1\\2\\1\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\2&&4\\&3&\end{smallmatrix} \arrow[rr] \arrow[dr] & & \begin{smallmatrix}&2&\\1&&3\\&2&\end{smallmatrix} \arrow[r] & 2 \arrow[r] & 0 \\
 & & & \begin{smallmatrix}3\\2\end{smallmatrix} \arrow[ur] & & \begin{smallmatrix}&2&&4\\1&&3&\end{smallmatrix} \arrow[ur] & & \begin{smallmatrix}1&&3\\&2&\end{smallmatrix} \arrow[ur] & & &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & \begin{smallmatrix}4\\3\end{smallmatrix} \arrow[r] & \begin{smallmatrix}&3&\\2&&4\\&3&\end{smallmatrix} \arrow[rr] \arrow[dr] & & \begin{smallmatrix}&2&\\1&&3\\&2&\end{smallmatrix} \oplus \begin{smallmatrix}4\\3\end{smallmatrix} \arrow[rr] \arrow[dr] & & \begin{smallmatrix}1\\2\\1\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\2&&4\\&3&\end{smallmatrix} \arrow[rr] \arrow[dr] & & \begin{smallmatrix}&2&\\1&&3\\&2&\end{smallmatrix} \oplus \begin{smallmatrix}4\\3\end{smallmatrix} \arrow[rr] \arrow[dr] & & \begin{smallmatrix}&3&\\2&&4\\&3&\end{smallmatrix} \arrow[r] & 3 \arrow[r] & 0 \\
 & & & \begin{smallmatrix}3\\2\end{smallmatrix} \arrow[ur] & & \begin{smallmatrix}&2&&4\\1&&3&\end{smallmatrix} \arrow[ur] & & \begin{smallmatrix}1&&3\\&2&\end{smallmatrix} \arrow[ur] & & \begin{smallmatrix}2&&4\\&3&\end{smallmatrix} \arrow[ur] & & &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & \begin{smallmatrix}4\\3\end{smallmatrix} \arrow[r] & \begin{smallmatrix}&3&\\2&&4\\&3&\end{smallmatrix} \arrow[rr] \arrow[dr] & & \begin{smallmatrix}&2&\\1&&3\\&2&\end{smallmatrix} \oplus \begin{smallmatrix}4\\3\end{smallmatrix} \arrow[rr] \arrow[dr] & & \begin{smallmatrix}1\\2\\1\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\2&&4\\&3&\end{smallmatrix} \arrow[rr] \arrow[dr] & & \begin{smallmatrix}&2&\\1&&3\\&2&\end{smallmatrix} \oplus \begin{smallmatrix}4\\3\end{smallmatrix} \arrow[rr] \arrow[dr] & & \begin{smallmatrix}&3&\\2&&4\\&3&\end{smallmatrix} \arrow[rr] \arrow[dr] & & \begin{smallmatrix}4\\3\end{smallmatrix} \arrow[r] & 4 \arrow[r] & 0 \\
 & & & \begin{smallmatrix}3\\2\end{smallmatrix} \arrow[ur] & & \begin{smallmatrix}&2&&4\\1&&3&\end{smallmatrix} \arrow[ur] & & \begin{smallmatrix}1&&3\\&2&\end{smallmatrix} \arrow[ur] & & \begin{smallmatrix}2&&4\\&3&\end{smallmatrix} \arrow[ur] & & 3 \arrow[ur] & & &
\end{tikzcd}$$

10. Let A be a finite dimensional K-algebra, M an arbitrary left A-module and let $DM = \mathrm{Hom}_K(M, K)$ denote the K-dual of M (which is automatically a right A-module). Show that ${}_AM$ is projective (injective) if and only if $(DM)_A$ is injective (projective, resp.).

Solution. We will show first that idecomposable projective modules are mapped to indecomposeble injectives. We know that indecomposable projective modules are of the form $P = Ae$ with e a primitive idempotent element in A. To check the injectivity of DP we have to check the existence of an extension of $f : I_A \to DP$ to $\bar{f} : A_A \to DP$ for arbitrary right ideal $I_A \leq A_A$. Apply the duality functor to this diagram:

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d, "f"'] & A \arrow[dl, dashed, "\bar{f}\ (?)"] \\ & DP & \end{tikzcd}$$

Since $DDP \simeq P$ and P is projective, we get that there is a lifting of Df to a φ:

$$\begin{tikzcd} & DDP = P \arrow[dl, dashed, "\varphi"'] \arrow[d, "Df"] & \\ DA \arrow[r] & DI \arrow[r] & 0 \end{tikzcd}$$

Applying the duality functor once more and observing that $DDA \simeq A$ and $DDI \simeq I$ we get that $\bar{f} = D\varphi$ is an extension of $DDf = f$. Hence DP is injective. Since the duality obviously preserves indecomposability and isomorphism types, we get that duality takes indecomposable projective modules to indesomposable injectives. Actually, we get all of them since the number of indecomposable projective and indecomposable injective modules is the same (and equals the number of isotypes of simple modules). Thus, by applying the same argument to A^{opp} we also obtain that indecomposable injective modules are mapped to indecomposable projectives. Now, finite dimensional injective (projective) modules are finite direct sums of indecomposable injectives (projectives, resp.) and the duality functor D preserves finite direct sums, we get that the statement is true for finite dimensional modules. Finally, for arbitrary injective or projective modules the statement will follow from the following facts: 1) over a finite dimensional algebra each injective module is a direct sum of indecomposable injective modules which are all finite dimensional; 2) over a finite dimensional algebra each projective module is a direct sum of indecomposable projective modules which are all finite dimensional; 3) the duality functor D takes direct sums to direct products; 4) the direct product of injectives is always injective; 5) the direct product of projectives is projective for left artinian rings.

11. Let $A = K\Gamma/I$ be a finite dimensional path algebra modulo relations and let e_i denote the idempotent corresponding to the i-th vertex of Γ. Prove that $D(Ae_i)$ is an indecomposable injective right A-module and $\operatorname{Soc}(DAe_i) \simeq (e_iA/\operatorname{Rad}(e_iA))$.

Solution. In Problem #1/10 we proved that the functor D maps indecomposable projective modules to indecomposable injectives. Indecomposable projectives have simple top (quotient modulo the radical) while indecomposable injectives have simple socle. (The duality functor obviously preserves the submodule lattice, just turns it upside down.) In recognizing the simple module types, it is worth observing that $e_iAe_j \subseteq e_i \operatorname{Rad} A$ for $i \neq j$; hence e_j annihilates the simple modules corresponding to different vertices, while this is not the case for e_jA, since $e_j \in e_jAe_j$ and $e_j \notin \operatorname{Rad} e_jA$. Thus we can recognize a simple isotype by simply multiplying it with the idempotents. Now it is a routine calculation (using the definition of the module structure on Hom-sets) that the (simple) socle of DAe_i is not annihilated by e_i hence it is isomorhic to the top of e_iA.