1. A reminder: a ring R is *(left) hereditary* if and only if every homomorphic image of every injective (left) module is injective (for example, $\mathbb{Z}$ is hereditary). (Note that it can be shown that this is equivalent to the fact that submodules of projective modules are projective.) Prove that if R is hereditary then for every $M, N \in R$-Mod left R-module $\mathrm{Ext}^k_R(M,N) = 0$ for all $k > 1$.

Solution. If R is left hereditary, then M has a projective resolution of length 1. Thus the terms of higher index are all 0. When we apply the functor $\mathrm{Hom}_R(-,N)$ to this complex, the higher terms (and hence the higher cohomologies), starting with index 2 will be 0.

2. a) Let A, B be cyclic abelian groups. Determine the groups $\mathrm{Ext}^1_{\mathbb{Z}}(A,B)$.
b) Do the same when A and B are arbitrary finite abelian groups.

Solution. a) If $A \simeq \mathbb{Z}$, that is A is the infinite cyclic group, then A is projective, hence $\mathrm{Ext}^1_{\mathbb{Z}}(A,B) = 0$ for each B. Suppose now that $A \simeq \mathbb{Z}_n$, and let us consider the projective resolution of A: we have $0 \to \mathbb{Z} \xrightarrow{\varphi} \mathbb{Z} \to \mathbb{Z}_n \to 0$ where φ is the multiplication by n. We can apply the functor $\mathrm{Hom}_{\mathbb{Z}}(-,B)$ to obtain the following long exact sequence: $0 \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_n, B) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, B) \xrightarrow{\mathrm{Hom}(\varphi, 1_B)} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, B) \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_n, B) \to 0$. Observe that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, B) \simeq B$ and when we apply the canonical isomorphism, the action of the morphism $\mathrm{Hom}(\varphi, 1_B)$ is multiplication by n. Hence $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_n, B) \simeq B/nB$. In case $B \simeq \mathbb{Z}$ this is isomorphic to $\mathbb{Z}_n$. When B is a finite cyclic group, i.e. $B \simeq \mathbb{Z}_m$, then $B/nB \simeq \mathbb{Z}_m/n\mathbb{Z}_m \simeq \mathbb{Z}_{(m,n)}$. Summing up:

$$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}, B) \simeq 0$$
$$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_n, \mathbb{Z}) \simeq \mathbb{Z}_n$$
$$Ext^1_{\mathbb{Z}}(\mathbb{Z}_n, \mathbb{Z}_m) \simeq \mathbb{Z}_{(m,n)}$$

– b) For finite abelian groups use the fundamental theorem of abelian groups, the previous result and the fact that the Ext functors – similarly to Hom – preserve finite direct sums.

3. Let $0 \to K_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ be exact and suppose that P_i is projective for all i. Prove that $\mathrm{Ext}^k_R(M,N) \simeq \mathrm{Ext}^{k-n}_R(K_n, N)$ for every $k > n$.

Solution. Let us break the projective resolution and break it to short exact sequences, i.e. write it as the Yoneda product of the following seuqnces: $0 \to K_{i-1} \to P_i \to K_i \to 0$, ahol $0 \le i \le n-1$ s $K_0 = M$. We can apply the functor $\mathrm{Hom}_R(-,N)$ to these sequences. The corresponding long exact sequences will contain the following segments:

$$\cdots \to \mathrm{Ext}^{\ell}_R(P_i, N) \to \mathrm{Ext}^{\ell}_R(K_i, N) \to \mathrm{Ext}^{\ell+1}(K_{i-1}, N) \to \mathrm{Ext}^{\ell+1}_R(P_i, N) \to \cdots$$

Since for $\ell > 0$ we have 0's at the two ends of the segment above, the middle morphism must be an isomorphism. Thus: $\mathrm{Ext}^k_R(M,N) = \mathrm{Ext}^k_R(K_0, N) \simeq \mathrm{Ext}^{k-1}_R(K_1, N) \simeq \cdots \simeq \mathrm{Ext}^{k-n}_R(K_n, N)$ for each $k > n$. (*Remark:* The method described above where the computation of higher Ext's is reduced to the computation of lower ones is called *dimension shifting*.)

4. Suppose that the resolution in the previous problem is minimal and N is simple. Prove that $\mathrm{Ext}^n_R(M,N) \simeq \mathrm{Hom}_R(K_n, N)$.

Solution. From the solution of the previous problem we know that $\mathrm{Ext}^n_R(M,N) \simeq \mathrm{Ext}^1_R(K_{n-1}, N)$. Let us now look at the short exact sequence $0 \to K_n \to P_{n-1} \to K_{n-1} \to 0$ and apply the functor $\mathrm{Hom}_R(-,N)$. We get the following exact sequence: $0 \to \mathrm{Hom}_R(K_{n-1}, N) \to \mathrm{Hom}_R(P_{n-1}, N) \to \mathrm{Hom}_R(K_n, N) \to \mathrm{Ext}^1_R(K_{n-1}, N) \to \mathrm{Ext}^1_R(P_{n-1}, N) = 0$. We can now use that the projective resolution was minimal and that N is simple: from these facts it follows that $P_{n-1}/\operatorname{Rad} P_{n-1} \simeq K_n/\operatorname{Rad} K_n$, hence the map $\mathrm{Hom}_R(K_{n-1}, N) \to \mathrm{Hom}_R(P_{n-1}, N)$ is not only injective but it is also surjective. This implies that the map $\mathrm{Hom}_R(K_n, N) \to \mathrm{Ext}^1_R(K_{n-1}, N)$ is an isomorphism, giving the required isomorphism $\mathrm{Ext}^n_R(M,N) \simeq \mathrm{Hom}_R(K_n, N)$.

5. Let $A_A = \begin{smallmatrix} & 1 & \\ & 2 & \\ 1 & & 3 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 1 & & 3 \\ & 4 & \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 4 \end{smallmatrix} \oplus \begin{smallmatrix} 4 \\ 1 \end{smallmatrix}$.

a) Draw a graph Γ and admissible ideal I of relations, so that $A \simeq K\Gamma/I$.

b) Determine the dimension of $\mathrm{Ext}^3_A(\,1\,,\,1\,)$.

Solution. a) The vertices of the graph are given by the isomorphism types of simples, while the arrows can be read from the structure of indecomposable projectives: when in the semisimple module $\mathrm{Rad}(e_iA)/\mathrm{Rad}^2(e_iA)$ (i.e. in the "second layer" of the projective module e_iA) there are k basis elements of type j (i.e. composition factors of type j) then we draw k arrows from i to j. (The graph obtained this way from the algebra A is called the *Gabriel quiver* of A.) The relations can be also read from the structure of projectives. Thus the graph of the given algebra is:

$$\Gamma: \qquad \begin{tikzcd} 4 \arrow[d, "\varepsilon"'] & 3 \arrow[l, "\delta"'] \\ 1 \arrow[r, shift right, "\alpha"'] & 2 \arrow[l, shift right, "\beta"'] \arrow[u, "\gamma"'] \end{tikzcd}$$

and the relations are $I = (\alpha\gamma\delta, \beta\alpha, \delta\varepsilon, \varepsilon\alpha)$. With these data we have $A \simeq K\Gamma/I$. – b) Take the projective resolution of the module 1 and use the isomorphisms given in the earlier problems #2/3 and #2/4:

$$\begin{tikzcd}[column sep=small] \cdots \arrow[rr] \arrow[dr] & & \begin{smallmatrix} & 2 & \\ 1 & & 3 \\ & 4 & \end{smallmatrix} \arrow[rr] \arrow[dr] & & \begin{smallmatrix} & 1 & \\ & 2 & \\ 1 & & 3 \end{smallmatrix} \arrow[rr] \arrow[dr] & & \begin{smallmatrix} 4 \\ 1 \end{smallmatrix} \arrow[rr] \arrow[dr] & & \begin{smallmatrix} & 2 & \\ 1 & & 3 \\ & 4 & \end{smallmatrix} \arrow[rr] \arrow[dr] & & \begin{smallmatrix} & 1 & \\ & 2 & \\ 1 & & 3 \end{smallmatrix} \arrow[r] & 1 \arrow[r] & 0 \\ & 4 \arrow[ur] & & \begin{smallmatrix} & 2 & \\ 1 & & 3 \end{smallmatrix} \arrow[ur] & & 1 \arrow[ur] & & 4 \arrow[ur] & & \begin{smallmatrix} & 2 & \\ 1 & & 3 \end{smallmatrix} \arrow[ur] & & & \end{tikzcd}$$

The method of dimension shifting gives $\mathrm{Ext}^3_R(\,1\,,\,1\,) \simeq \mathrm{Ext}^2_R(\begin{smallmatrix} & 2 & \\ 1 & & 3 \end{smallmatrix},\,1\,) \simeq \mathrm{Ext}^1_R(\,4\,,\,1\,) \simeq \mathrm{Hom}_R(\,1\,,\,1\,)$. Thus the required dimension is 1.

6. Let $A_A = \begin{smallmatrix} & 1 & \\ & 2 & \\ 1 & & 3 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 1 & & 3 \\ & 4 & \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 4 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 4 \\ 1 \end{smallmatrix}$.

a) Describe A as a path algebra modulo some relations.

b) Compute $\dim \mathrm{Ext}^3_A(\,1\,,\begin{smallmatrix} 2 \\ 1 \end{smallmatrix})$.

c) Show that for every $n \in \mathbb{N}$ there exists $k > n$, such that $\dim \mathrm{Ext}^k_A(\,1\,,\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}) \neq 0$.

Solution. a) The graph Γ is the same as the graph of the previous problem (problem #2/5) while the relations are: $I = (\alpha\gamma\delta, \beta\alpha, \varepsilon\alpha)$. With these we have $A \simeq K\Gamma/I$. – b) We shall use the method of dimension shifting. To this end we shall write the projective resolution of the module 1 which "formally" agrees with the resolution in the previous problem. Now, however, we do not get the last isomorphism; we know only that $\mathrm{Ext}^3_R(\,1\,,\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}) \simeq \mathrm{Ext}^1_R(\,4\,,\begin{smallmatrix} 2 \\ 1 \end{smallmatrix})$. The dimension of the last extension space can be computed from the exact sequence $0 \to 1 \to \begin{smallmatrix} 4 \\ 1 \end{smallmatrix} \to 4 \to 0$. Namely, the beginning of the corresponding long exact sequence is as follows:

$$0 \to \mathrm{Hom}_R(\,4\,,\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}) \to \mathrm{Hom}_R(\begin{smallmatrix} 4 \\ 1 \end{smallmatrix},\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}) \to \mathrm{Hom}_R(\,1\,,\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}) \to \mathrm{Ext}^1_R(\,4\,,\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}) \to 0.$$

Here the first two nontrivial terms are 0, while the third one is 1-dimensional. The exactness of the sequence gives that the dimension of the required extension space is also 1. – c) The periodicity of the projective resolution shows that $\mathrm{Ext}^k_R(\,1\,, M) \simeq \mathrm{Ext}^{k+3}(\,1\,, M)$ for each k and M. Thus from part b) we get that for each $3 \mid k$ we have $\mathrm{Ext}^k(\,1\,,\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}) \neq 0$.

7. Give a new proof that every projective module is flat.

Solution. A projective module has a projective resolution where all the terms of the resolution are 0 with the exception of the 0'th term. After applying the tensor functor to this resolution we still get 0's almost everywhere thus the i-th homology is 0 for each $i > 0$. Thus for any projective module P_R and any module ${}_RM$ we have $\mathrm{Tor}_i^R(P, M) = 0$ for $i > 0$. Now, if we apply the functor $P \underset{R}{\otimes} -$ to an arbitrary sequence $0 \to N \to K \to M \to 0$ we get that $\mathrm{Tor}_1^R(P, M) \to P \underset{R}{\otimes} N \to P \underset{R}{\otimes} K \to P \underset{R}{\otimes} M \to 0$ is exact and since the first term in this sequence is 0, we get that the functor $P \underset{R}{\otimes} -$ is exact, hence P is flat.

8. Give a short exact sequence of Abelian groups $0 \to \mathbb{Z}_4 \to M \to \mathbb{Z}_4 \to 0$ which is not split, but the middle term M is not indecomposable.

Solution. Take the sequence $0 \to \mathbb{Z}_4 \xrightarrow{\alpha} \mathbb{Z}_2 \oplus \mathbb{Z}_8 \xrightarrow{\beta} \mathbb{Z}_4 \to 0$ where the morphisms are defined as follows. If the first copy of $\mathbb{Z}_4$ is generated by an element a, while $\mathbb{Z}_2 = \langle b \rangle$ and $\mathbb{Z}_8 = \langle c \rangle$ then take $\alpha(a) = (b, c^2)$ and define β as the natural quotient map. Clearly this sequences is non-split since the decomposition of the middle term is unique.

9. Let us take the graph $\Gamma: \quad 1 \to 2 \leftarrow 3 \to 4$ and the path algebra $A = K\Gamma$.

a) Find an indecomposable (right) A module M of composition length 4 (there is only one such module).

b) Let N be the simple module corresponding to vertex 3. Show that $\dim(\mathrm{Ext}^1_A(N, M)) = 1$, and describe the middle term of the nonzero elements in this extension space.

c) Find as many indecomposable A-modules as you can.

Solution. Let us determine first the structure of the indecomposable projective modules over A:

$$A_A = \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}2\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\2&&4\end{smallmatrix} \oplus \begin{smallmatrix}4\end{smallmatrix}$$

a) There is an indecomposable module whose structure can be described by $\begin{smallmatrix}1&&3&\\&2&&4\end{smallmatrix}$. The fact that it is indecomposable can be shown by observing that if the basis element of the vector space corresponding to the first vertex is in, say, the first summand then all the other basis elements have to belong to the same summand. – b) Take the projective resolution of the simple module $\begin{smallmatrix}3\end{smallmatrix}$:

$$0 \to \begin{smallmatrix}2\end{smallmatrix} \oplus \begin{smallmatrix}4\end{smallmatrix} \to \begin{smallmatrix}&3&\\2&&4\end{smallmatrix} \to \begin{smallmatrix}3\end{smallmatrix} \to 0.$$

Apply the functor $\mathrm{Hom}(-, M)$ to this sequence. The corresponding long exact sequence is:

$$0 \to \mathrm{Hom}(\begin{smallmatrix}3\end{smallmatrix}, \begin{smallmatrix}1&&3&\\&2&&4\end{smallmatrix}) \to \mathrm{Hom}(\begin{smallmatrix}&3&\\2&&4\end{smallmatrix}, \begin{smallmatrix}1&&3&\\&2&&4\end{smallmatrix}) \to \mathrm{Hom}(\begin{smallmatrix}2\end{smallmatrix} \oplus \begin{smallmatrix}4\end{smallmatrix}, \begin{smallmatrix}1&&3&\\&2&&4\end{smallmatrix}) \to \mathrm{Ext}^1(\begin{smallmatrix}3\end{smallmatrix}, \begin{smallmatrix}1&&3&\\&2&&4\end{smallmatrix}) \to 0.$$

Here the dimension of the first Hom-space is 0, the dimension of the second is 1, while the dimension of the third is 2. The additivity of dimensions on exact sequences implies that the dimension of the last Ext-space is 1. It can be shown (see problem #3/3) that the middle terms of non-zero elements of this one-dimensional Ext-space are all isomorphic, hence we have to give only one non-split extension of N by M:

$$0 \to \begin{smallmatrix}1&&3&\\&2&&4\end{smallmatrix} \to \begin{smallmatrix}3\\4\end{smallmatrix} \oplus \begin{smallmatrix}1&&3\\&2&\end{smallmatrix} \to \begin{smallmatrix}3\end{smallmatrix} \to 0.$$

Clearly the direct summands of the middle term are indecomposable. – c) Here is a complete list of indecomposable modules:

$$\begin{smallmatrix}1\end{smallmatrix}, \begin{smallmatrix}2\end{smallmatrix}, \begin{smallmatrix}3\end{smallmatrix}, \begin{smallmatrix}4\end{smallmatrix}, \begin{smallmatrix}1\\2\end{smallmatrix}, \begin{smallmatrix}3\\2\end{smallmatrix}, \begin{smallmatrix}3\\4\end{smallmatrix}, \begin{smallmatrix}&3&\\2&&4\end{smallmatrix}, \begin{smallmatrix}1&&3\\&2&\end{smallmatrix}, \begin{smallmatrix}1&&3&\\&2&&4\end{smallmatrix}.$$

10. Give a characterization of the fact that a short exact sequence is split in terms of homomorphisms, hence conclude that any additive functor preserves split exact sequences.

Solution. Prove that $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is split exact if and only if there exist maps $f' : B \to A$ and $g' : C \to B$ such that: 1) $f'f = 1_A$; 2) $gg' = 1_C$; 3) $gf = 0$; 4) $f'g' = 0$; 5) $ff' + g'g = 1_B$. Observe that the above properties are far from being independent.