1. Consider the setup of the 3×3 lemma, i.e. a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & X' & \to & Y' & \to & Z' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & X & \to & Y & \to & Z & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & X'' & \to & Y'' & \to & Z'' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with exact rows. Show that the exactness of the first and third column does not imply the exactness of the middle column.

Solution. Let us consider the following diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & X' & \xrightarrow{\varphi'} & Y' & \xrightarrow{\psi'} & Z' & \to & 0 \\
 & & \downarrow \alpha' & & \downarrow \beta' & & \downarrow \gamma' & & \\
0 & \to & X & \xrightarrow{\varphi} & Y & \xrightarrow{\psi} & Z & \to & 0 \\
 & & \downarrow \alpha'' & & \downarrow \beta'' & & \downarrow \gamma'' & & \\
0 & \to & X'' & \xrightarrow{\varphi''} & Y'' & \xrightarrow{\psi''} & Z'' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

A simple diagram chasing shows that under the assumptions β' is injective and β'' is surjective. For example, if $\beta'(y') = 0$ for some $y' \in Y$ then $\psi\beta'(y') = \gamma'\psi'(y') = 0$. Since γ' is injective, this gives that $\psi'(y') = 0$, hence $y' \in \operatorname{Ker} \Psi' = \operatorname{Im} \varphi'$. Thus there exists $x' \in X'$ such that $y' = \varphi'(x')$. But then $\varphi\alpha'(x') = \beta'\varphi'(x') = \beta'(y') = 0$ and since both φ and α' are injective, we get that $x' = 0$. But this implies that $y' = \varphi(x') = 0$, giving that $\beta'$is injective. (The proof that β'' is surjective is similar.) One should also observe that if the middle column would be half exact at Y then we would have a short exact sequence of complexes (the vertical columns) where the homologies would be everywhere 0 with one possible exception, the homology at Y. But taking the long exact sequence of homologies would imply that the homology at Y is also 0. Thus, if we want to construct a counterexample, we really need a diagram where $\operatorname{Im} \beta' \not\subseteq \operatorname{Ker} \beta''$. The following diagram provides such a counterexample in the category of K vector spaces:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & 0 & \xrightarrow{0} & K & \xrightarrow{[1]} & K & \to & 0 \\
 & & \downarrow 0 & & \downarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} & & \downarrow [1] & & \\
0 & \to & K & \xrightarrow{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} & K \oplus K & \xrightarrow{[0\ 1]} & K & \to & 0 \\
 & & \downarrow [1] & & \downarrow [1\ 1] & & \downarrow 0 & & \\
0 & \to & K & \xrightarrow{[1]} & K & \xrightarrow{0} & 0 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

2. Consider the chain map $f_\bullet : X_\bullet \longrightarrow Y_\bullet$:

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}_4 & \xrightarrow{1 \mapsto 4} & \mathbb{Z}_8 & \xrightarrow{1 \mapsto 1} & \mathbb{Z}_2 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}_8 & \xrightarrow{1 \mapsto 2} & \mathbb{Z}_8 & \xrightarrow{1 \mapsto 4} & \mathbb{Z}_8 & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

Here the vertical maps are the natural embeddings. Compute the long exact sequence of homologies obtained from the short exact sequence of complexes

$$0_\bullet \longrightarrow X_\bullet \xrightarrow{f_\bullet} Y_\bullet \longrightarrow \operatorname{Coker} f_\bullet \longrightarrow 0_\bullet.$$

Solution. Here is the complete diagram, completed with the cokernel terms.

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}_4 & \xrightarrow{1\mapsto 4} & \mathbb{Z}_8 & \xrightarrow{1\mapsto 1} & \mathbb{Z}_2 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & & & \downarrow f_2 & & \downarrow f_1 & & \downarrow f_0 & & & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}_8 & \xrightarrow{1\mapsto 2} & \mathbb{Z}_8 & \xrightarrow{1\mapsto 4} & \mathbb{Z}_8 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & & & \downarrow g_2 & & \downarrow g_1 & & \downarrow g_0 & & & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

Here the columns form short exact sequences (the endterms are omitted) and the other vertical maps are defined by $f_2 : 1 \mapsto 2,\ \ f_1 : 1 \mapsto 1$ and $f_0 : 1 \mapsto 4$, furthermore $g_i : 1 \mapsto 1$ $(i = 2, 1, 0)$. The homologies are the following:

$$\begin{array}{ccccc}
0 & 2\mathbb{Z}_4 & 2\mathbb{Z}_8/4\mathbb{Z}_8 & \mathbb{Z}_2/\mathbb{Z}_2 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & 4\mathbb{Z}_8 & 2\mathbb{Z}_8/2\mathbb{Z}_8 & \mathbb{Z}_8/4\mathbb{Z}_8 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \mathbb{Z}_2 & 0 & \mathbb{Z}_2/0 & 0
\end{array}$$

Then we have: $H_2(f_\bullet) : [2] \mapsto [4]$, that is the generating element of $2\mathbb{Z}_4 \cong \mathbb{Z}_2$ maps to the generating element of $4\mathbb{Z}_8 \cong \mathbb{Z}_2$, $\ H_2(g_\bullet) : [4] \mapsto [4] = [0]$, $\ \partial_2 : [1] \leftarrow 1 \to 2 \leftarrow [2]$, $\ H_0(g_\bullet) : [1] \mapsto [1]$, and the rest of the maps of the long exact sequence is obvioulsy 0. Thus the long exact sequence of homologies is as follows:

$$\begin{array}{ccccccccccc}
 & 0 & & \mathbb{Z}_2 & & \mathbb{Z}_2 & & 0 & & 0 & \\
 & \downarrow & \nearrow & \downarrow \cong & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \\
\cdots & 0 & & \mathbb{Z}_2 & \cong & 0 & & \mathbb{Z}_4 & & 0 & \cdots \\
 & \downarrow & \nearrow & 0 \downarrow & \nearrow & \downarrow & \nearrow & \cong \downarrow & \nearrow & \downarrow & \\
 & 0 & & \mathbb{Z}_2 & & 0 & & \mathbb{Z}_4 & & 0 &
\end{array}$$

3. Let A be a K-algebra, M and N A-modules, and let $\mathcal{E}$ and $\mathcal{E}'$ be two non-split extensions in the K-space $\operatorname{Ext}^1_A(M, N)$. Suppose that $\mathcal{E}$ and $\mathcal{E}'$ are linearly dependent. Show that the middle terms in $\mathcal{E}$ and $\mathcal{E}'$ are isomorphic.

Solution. We may assume that $\mathcal{E}' = \lambda \cdot \mathcal{E} = (\lambda\, \mathrm{id}_N)\mathcal{E}$ with $\lambda \in K$ nonzero. Then we get:

$$\begin{array}{ccccccccc}
0 & \to & N & \to & K & \to & M & \to & 0 \\
 & & \downarrow \lambda\, \mathrm{id}_N & & \downarrow \kappa & & \downarrow \mathrm{id}_M & & \\
0 & \to & N & \to & \tilde{K} & \to & M & \to & 0
\end{array}$$

Since the first and third vertical map is an isomorphism, so is κ. Thus $K \simeq \tilde{K}$.

4. a) Consider the algebra given by $A_A = \begin{smallmatrix} 1 \\ 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 1 & & 3 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \end{smallmatrix}$. Find the projective resolutions of the modules $X = \begin{smallmatrix} 1 \\ 2 \end{smallmatrix}$ and of $Y = \begin{smallmatrix} 2 \end{smallmatrix}$.

b) Consider the algebra given by $B_B = \begin{smallmatrix} 1 \\ 2 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 2 \\ 2 \end{smallmatrix}$. Compute the projective resolution of the module $Z = \begin{smallmatrix} & & 1 \\ 1 & 2 & \\ & 2 & \end{smallmatrix}$.

Solution. a)

$$0 \to \begin{smallmatrix} 3 \end{smallmatrix} \to \begin{smallmatrix} & 2 & \\ 1 & & \mathbf{3} \end{smallmatrix} \to \begin{smallmatrix} 1 \\ \mathbf{2} \\ 1 \end{smallmatrix} \to \begin{smallmatrix} 1 \\ 2 \\ \mathbf{1} \end{smallmatrix} \to \begin{smallmatrix} \mathbf{1} \\ 2 \end{smallmatrix} \to 0$$

$$0 \to \begin{smallmatrix} 3 \end{smallmatrix} \to \begin{smallmatrix} & 2 & \\ 1 & & \mathbf{3} \end{smallmatrix} \to \begin{smallmatrix} 1 \\ \mathbf{2} \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \end{smallmatrix} \to \begin{smallmatrix} & 2 & \\ \mathbf{1} & & \mathbf{3} \end{smallmatrix} \to \begin{smallmatrix} \mathbf{2} \end{smallmatrix} \to 0,$$

and here the image of a generating element of a projective module is inidicated by a bold face number in the next term.

b)

$$\cdots \to \begin{smallmatrix} 2 \\ 2 \\ \mathbf{2} \end{smallmatrix} \to \begin{smallmatrix} 2 \\ \mathbf{2} \\ 2 \end{smallmatrix} \to \begin{smallmatrix} 2 \\ 2 \\ \mathbf{2} \end{smallmatrix} \to \begin{smallmatrix} 1 \\ \mathbf{2} \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 2 \\ \mathbf{2} \end{smallmatrix} \to \begin{smallmatrix} & & 1 \\ 1 & 2 & \\ & 2 & \end{smallmatrix} \to 0$$

The map $P_1 \to P_0$ will take the generating element of $\begin{smallmatrix} 2 \\ 2 \\ 2 \end{smallmatrix}$ into the difference of the two basis elements, indicated by bold face letters.

5. Take the deleted projective resolution of the module X from the previous problem and apply to it the contravariant functor $\mathrm{Hom}(-, \begin{smallmatrix}1\\2\end{smallmatrix})$. Compute the dimension of the cohomologies of this new complex.

Solution. The deleted projective resolution is the following:

$$0 \to 3 \to \begin{smallmatrix} & 2 & \\ 1 & & \mathbf{3}\end{smallmatrix} \to \begin{smallmatrix}1\\ \mathbf{2}\\1\end{smallmatrix} \to \begin{smallmatrix}1\\2\\ \mathbf{1}\end{smallmatrix} \to 0,$$

and after applying the functor $\mathrm{Hom}(-, \begin{smallmatrix}1\\2\end{smallmatrix})$ we get the following sequence:

$$0 \to \mathrm{Hom}(\begin{smallmatrix}1\\2\\1\end{smallmatrix}, \begin{smallmatrix}1\\2\end{smallmatrix}) \xrightarrow{d^0} \mathrm{Hom}(\begin{smallmatrix}1\\2\\1\end{smallmatrix}, \begin{smallmatrix}1\\2\end{smallmatrix}) \xrightarrow{d^1} \mathrm{Hom}(\begin{smallmatrix} & 2 & \\ 1 & & 3\end{smallmatrix}, \begin{smallmatrix}1\\2\end{smallmatrix}) \xrightarrow{d^2} \mathrm{Hom}(3, \begin{smallmatrix}1\\2\end{smallmatrix}) \to 0.$$

Here $\mathrm{Hom}(\begin{smallmatrix}1\\2\\1\end{smallmatrix}, \begin{smallmatrix}1\\2\end{smallmatrix})$ is one dimensional, generated by the natural homomorphism from $\begin{smallmatrix}1\\2\\1\end{smallmatrix}$ to the quotient module obtained by factoring out with the simple socle. The image of this generating element under the action of d^0 is 0 thus d^0 is the zero map. The same element is taken by d^1 into a nontrivial morphism in $\mathrm{Hom}(\begin{smallmatrix} & 2 & \\ 1 & & 3\end{smallmatrix}, \begin{smallmatrix}1\\2\end{smallmatrix})$ which is also one dimensional. Hence it is an isomorphism. Finally $\mathrm{Hom}(3, \begin{smallmatrix}1\\2\end{smallmatrix}) = 0$, hence $d^2 = 0$. This gives us that the 0-th cohomology of the sequence is one dimensional, while the other cohomologies are 0.

6. Consider the algebra $A_A = 1 \oplus 2 \oplus \begin{smallmatrix} & 3 & \\ 1 & & 2\end{smallmatrix}$ and take the short exact sequence

$$\mathcal{E} : 0 \longrightarrow 1 \oplus 2 \longrightarrow \begin{smallmatrix} & 3 & \\ 1 & & 2\end{smallmatrix} \longrightarrow 3 \longrightarrow 0.$$

Find the (pullback) sequence $\mathcal{E}\mu$ if $\mu : \begin{smallmatrix}3\\2\end{smallmatrix} \longrightarrow 3$ is the natural epimorphism.

Solution. The following is the required diagram:

$$\begin{array}{cccccccccc}
\mathcal{E}\mu : & 0 & \longrightarrow & 1 \oplus 2 & \longrightarrow & \begin{smallmatrix} & 3 & \\ 1 & & 2\end{smallmatrix} \oplus 2 & \longrightarrow & \begin{smallmatrix}3\\2\end{smallmatrix} & \longrightarrow & 0 \\
 & & & \| & & \downarrow & & \downarrow{\scriptstyle\mu} & & \\
\mathcal{E} : & 0 & \longrightarrow & 1 \oplus 2 & \longrightarrow & \begin{smallmatrix} & 3 & \\ 1 & & 2\end{smallmatrix} & \longrightarrow & 3 & \longrightarrow & 0
\end{array}$$