1. (Hereditary rings revisited:) Prove that the following statements are equivalent for a ring R:

(i) every submodule of every projective left R-module is projective;
(ii) every left ideal of R is projective;
(iii) every homomorphic image of every injective left R-module is injective (i. e. R is left hereditary);
(iv) the left global dimension of R is at most 1.

Solution. Condition (i) trivially implies condition (ii). We will now show that condition (ii) implies condition (iii). Let I be an injective module and $\alpha : I \to M$ a surjective map. We will show that for every ${}_RL \le {}_RR$ left ideal and every map $f : L \to M$ there is an extension of this map to $\varphi : R \to M$. Namely, let us consider the following diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & L & \longrightarrow & R \\ & & f\big\downarrow & \searrow^{\tilde f} & \big\downarrow \bar f \\ 0 & \longleftarrow & M & \xleftarrow[\alpha]{} & I \end{array}$$

Here $\tilde f$ exists since L is projective, while $\bar f$ exists since I is injective. Thus $\varphi = \alpha\bar f : R \to M$ is an extension of f. By the injective test lemma we get that M is injective. This proves (ii) $\Rightarrow$ (iii). Clearly, if R is left hereditary, then the left global dimension of R is at most 2 since every left R module has an injective coresolution of length at most 1 hence the $\mathrm{Ext}^2_R(-, N) = 0$ for every $N \in R$-Mod. Finally, if (iv) holds then the first syzygy of every $M \in R$-Mod is projective hence (i) holds.

2. a) Let Γ be a graph for which the path algebra $K\Gamma$ is finite dimensional. Prove that $K\Gamma$ is (left and right) hereditary.
b*) Prove the same statement without the assumption on the dimension of $K\Gamma$.

Solution. a) Since $K\Gamma$ is finite dimensional, it is enough to show that for every vertex i of Γ the simple module $S(i)$ corresponding to i has a projective resolution of length at most 1. But if e_i is the corresponding idempotent element of $K\Gamma$ then the radical of the projective cover of $S(i)$ is $\mathrm{Rad}(K\Gamma e_i) = \bigoplus_\alpha K\Gamma\alpha$ where the summation is for all arrows ending in i and it is also clear that for $j \xrightarrow{\alpha} i$ we have $K\Gamma\alpha \simeq K\Gamma e_j$. Hence the radical of $K\Gamma e_i$ is projective, thus $pd\, S(i) \le 1$ for each i. – b) One can show that for each left $A = K\Gamma$-module M there is a projective resolution of the following type:

$$0 \to \bigoplus_\alpha Ae_{s(\alpha)} \underset{K}{\otimes} e_{t(\alpha)}M \xrightarrow{f} \bigoplus_{i=1}^{n} Ae_i \underset{K}{\otimes} e_iM \xrightarrow{g} M \to 0$$

where the first summation is for all arrows α in Γ and $s(\alpha)$ and $t(\alpha)$ stand for the starting and terminating vertex of α, furthermore the maps are given by $f(a \otimes m) = a\alpha \otimes m - a \otimes \alpha m$ for each $a \otimes m$ in the component corresponding to α, while $g(a \otimes m) = am$ for each $a \otimes m$ in the component corresponding to the vertex i. For a detailed proof see for example W. Crawley-Boevey: Lectures on representations of quivers (`http://www1.maths.leeds.ac.uk/~pmtwc/quivlecs.pdf`).

3. Consider the Ext^3-spaces of problems #2/6 and #2/7. Represent a non-zero element of these spaces by an exact sequence of length 3.

Solution. We can obtain the corresponding exact sequences from the projective resolution. In case of the first algebra we get the following exact sequence $\mathcal{E}_1$:

$$\begin{array}{ccccccccccccccc} & \cdots & \longrightarrow & \begin{smallmatrix} 1 \\ 2 \\ 1\;3 \end{smallmatrix} & \longrightarrow & \begin{smallmatrix} 4 \\ 1 \end{smallmatrix} & \longrightarrow & \begin{smallmatrix} 2 \\ 1\;3 \\ 4 \end{smallmatrix} & \longrightarrow & \begin{smallmatrix} 1 \\ 2 \\ 1\;3 \end{smallmatrix} & \longrightarrow & 1 & \longrightarrow & 0 \\ & & & \big\downarrow & & \| & & \| & & \| & & \| & & \\ \mathcal{E}_1 : & 0 & \longrightarrow & 1 & \longrightarrow & \begin{smallmatrix} 4 \\ 1 \end{smallmatrix} & \longrightarrow & \begin{smallmatrix} 2 \\ 1\;3 \\ 4 \end{smallmatrix} & \longrightarrow & \begin{smallmatrix} 1 \\ 2 \\ 1\;3 \end{smallmatrix} & \longrightarrow & 1 & \longrightarrow & 0 \end{array}$$

Here the first vertical map corresponds to a nonzero homology in the third homology space since it cannot be factored via the module $\begin{smallmatrix} 4 \\ 1 \end{smallmatrix}$. – The projective resolution is the same for the second algebra but the exact sequence $\mathcal{E}_2$ obtained in this case is different since we have a different first term of the long exact sequence:

$$\begin{array}{cccccccccccccc}
 & \cdots & \longrightarrow & \begin{smallmatrix}1\\2\\1\;3\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}4\\1\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\;3\\4\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}1\\2\\1\;3\end{smallmatrix} & \longrightarrow & 1 & \longrightarrow & 0\\
 & & & \downarrow & & \downarrow & & \| & & \| & & \| & & \\
\mathcal{E}_2: & 0 & \longrightarrow & \begin{smallmatrix}2\\1\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\;4\\1\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}2\\1\;3\\4\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}1\\2\\1\;3\end{smallmatrix} & \longrightarrow & 1 & \longrightarrow & 0
\end{array}$$

The fact that the sequence corresponds to a non-zero homology follows similarly.

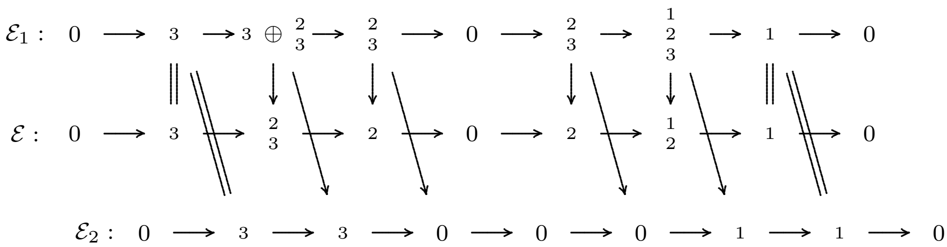
4. Decide whether the following exact sequence in $\mathrm{Ex}^2_A(1, 3)$ is equivalent to the 0 element

$$0 \to 3 \to \begin{smallmatrix}2\\3\end{smallmatrix} \to \begin{smallmatrix}1\\2\end{smallmatrix} \to 1 \to 0$$

when the regular representation of the algebra can be described as follows:

(i) ${}_AA = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus 3$; (ii) ${}_AA = \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \oplus 3$.

Solution. a) From the projective resolution of the simple module 1 we can see that the functor $\mathrm{Ext}^2_A(1, -)$ is 0, since the second term of this resolution (more precisely, the term with index 2) is 0. Thus the given long exact sequence is equivalent to the zero element. (By the way, the given algebra is hereditary.) – One can check that any element of $\mathrm{Ex}^n(M, N)$ for $n > 1$ is equivalent to the zero element if it is equivalent to the exact sequence $0 \to N \to N \to 0 \to \cdots \to 0 \to M \to M \to 0$ since the latter sequence can be obtained as the product of split short exact sequences hence it indeed has to be zero. (Note that the converse is also true.) In the case of this concrete sequence we will show this equivalence. Let us write the exact sequence $\mathcal{E}$ as the product of two short exact sequences since the situation this way will become more transparent. Let us now lift the second exact sequence to the projective resolution of the module 1 and then take the pullback of the first sequence along the morphism obtained on the kernel term of the second sequence. In this way we get the sequence $\mathcal{E}_1$. Now, it is easy to see that one can get a morphism from $\mathcal{E}_1$ to the sequence $\mathcal{E}_2$ of the prescribed form. This shows directly that $\mathcal{E}$ is equivalent to the zero element of $\mathrm{Ex}^2_A(1, 3)$:



b) By dimension shifting we get that $\mathrm{Ext}^2_A(1, 3) \simeq \mathrm{Hom}_A(3, 3) \neq 0$. Thus we know that the corresponding extension space has a nontrivial element. Let us try to find such an element in the corresponding homology. We may observe that the projective resolution of 1 gives us the initial sequence, hence we may choose the identity maps for the corresponding lifting maps:

$$\begin{array}{cccccccccccc}
P_\bullet: & 0 & \longrightarrow & 3 & \longrightarrow & \begin{smallmatrix}2\\3\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}1\\2\end{smallmatrix} & \longrightarrow & 1 & \longrightarrow & 0\\
 & & & \| & & \| & & \| & & \| & & \\
\mathcal{E}: & 0 & \longrightarrow & 3 & \longrightarrow & \begin{smallmatrix}2\\3\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}1\\2\end{smallmatrix} & \longrightarrow & 1 & \longrightarrow & 0
\end{array}$$

It is easy to see that the first vertical map, i. e. the identity map $3 \to 3$ cannot be lifted to a map $\begin{smallmatrix}2\\3\end{smallmatrix} \to 3$, hence the obtained element of the corresponding homology space is non-zero.

5. Let A be a finite dimensional (left) hereditary algebra and A^* its Yoneda-extension algebra: this means that if $\hat{S}$ is a semisimple module which is the direct sum of all isomorphism types of simple modules over A, then $A^* = \bigoplus_{i\geq 0} \mathrm{Ext}^i_A(\hat{S}, \hat{S})$ as a vector space and the multiplication is defined via the Yoneda product. Show that in this case $J(A^*)^2 = 0$.

Solution. Since A is hereditary, we get that $A^* = \mathrm{Hom}_A(\hat{S}, \hat{S}) \oplus \mathrm{Ext}^1_A(\hat{S}, \hat{S})$, since $\mathrm{Ext}^i_A(\hat{S}, \hat{S}) = 0$ for every $i > 1$. Thus A^* is finite dimensional. It is clear that $(\mathrm{Ext}^1_A(\hat{S}, \hat{S}))^2 \subseteq \mathrm{Ext}^2_A(\hat{S}, \hat{S}) = 0$, hence – being a nilpotent ideal – $\mathrm{Ext}^1_A(\hat{S}, \hat{S}) \subseteq J(A^*)$. On the other hand $A^*/\,\mathrm{Ext}^1_A(\hat{S}, \hat{S}) \simeq \mathrm{End}_A(\hat{S})$, which is semisimple by the Schur lemma. This shows that $J(A^*) = \mathrm{Ext}^1_A(\hat{S}, \hat{S})$, and the square of the radical is 0.

6*.** Suppose A is an abelian group for which $\mathrm{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z}) = 0$. Is it true that A is necessarily free?

Solution. This is the so-called *Whitehead-problem* which was solved by S. Shelah in 1974. He showed that the statement is undecidable in the usual ZFC axiom system of set theory. On one hand he proved that ZFC + the axiom of cunstrucibility ($V = L$) implies that the statement is true, i. e. $\mathrm{Ext}^1(A, \mathbb{Z}) = 0$ implies that A is free. On the other hand he showed that Martin's axiom, together with the negation of the CH implies that the implication is false. (For the full proof see for example Eklof, Paul C.: Whitehead's Problem is Undecidable, *The American Mathematical Monthly* **83 (10)** (1976), 775–788.)

7. Prove that if A is a torsion abelian group, then $\mathrm{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z}) \simeq \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{R}/\mathbb{Z})$.

Solution. Apply the functor $\mathrm{Hom}_{\mathbb{Z}}(A, -)$ to the sequence $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0$. We get the following long exact sequence:

$$0 = \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{R}) \to \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{R}/\mathbb{Z}) \to \mathrm{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z}) \to \mathrm{Ext}^1_{\mathbb{Z}}(A, \mathbb{R}) = 0$$

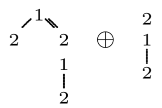
The first term of the sequence above is 0 because there is no non-zero homomorphism from a torsion group to a torsion-free group, while the last term is 0 because $\mathbb{R}$ is divisible, hence injective.

8. Let A be a finite dimensional K-algebra for which ${}_AA$ is injective. (Such an algebra is also called a *quasi-Frobenius algebra.*) Prove that if for a module $M \in A$-Mod we have $pd\, M < \infty$ then $pd\, M = 0$ i. e. M is projective.

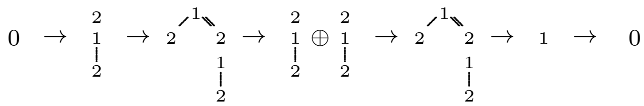
Solution. The condition implies that A is (left) noetherian hence the direct sum of injective modules is always injective. Since ${}_AA$ is injective, this implies that every free and consequently every projective module is injective. If $0 \to P_n \to P_{n-1} \to P_{n-2} \to \cdots \to P_0 \to M \to 0$ is a finite projective resolution of M which is of minimal length $n > 0$ then P_n is injective, and the short exact sequence $0 \to P_n \to P_{n-1} \to K_{n-1} \to 0$ splits. But this would imply that K_{n-1} is also projective hence we would get a strictly shorter projective resolution, a contradiction. Hence only projective modules will have finite projective resolution and finite projective dimension. (Thus the projectively defined *finitistic dimension* of selfinjetive algebras is 0.)

9. Take the graph $1 \;\substack{\xrightarrow{\alpha} \\ \xrightarrow{\beta} \\ \xleftarrow{\gamma}}\; 2$ and take the path algebra modulo relations $K\Gamma/I$ where $I = (\alpha\gamma, \gamma\beta)$. Compute the (left) global dimension of A.

Solution. The structure of the right regular module A_A can be described as follows:

$A_A =$ 

We can check the projective resolution of the simple modules. Observe that the first syzygy of the first simple module has a summand isomorphic to the second simple, so it is enough to compute the projective resolution of the first simple only.



This shows that the global dimension of the algebra A is 3.

10. Prove that for an arbitrary ring R we have $l\ gl\ dim\, R = l\ gl\ dim\, M_n(R)$.

Solution. We know that R and $M_n(R)$ are Morita-equivalent and a categorical equivalence takes exact sequences to exact sequences, projective resolutions to projective resolutions. Thus an equivalence functor preserves the length of projective resolutions hence it preserves the projective dimension of modules and the global dimension of the corresponding rings.