# CONSTRUCTION OF CPS-STRATIFIED ALGEBRAS

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*Dedicated to Prof. Vlastimil Dlab on the occasion of his 80th birthday*

Abstract. The results of [DR] and [ADL2] gave a recursive construction for all quasi-hereditary and standardly stratified algebras starting with local algebras and suitable bimodules. Using the notion of stratifying pairs of subcategories, introduced in [AL], we generalize these earlier results to construct recursively all CPS-stratified algebras.

## 1. Introduction

Ever since their introduction by Cline, Parshall and Scott in the late 1980's quasi-hereditary algebras have drawn a lot of attention and they keep playing an important role. One of the key defining features of these algebras is the way how they are put together from simpler algebras (cf. the notions of *recollement* and *partial recollement*). Much of the homological properties and of the structure theory developed for quasi-hereditary algebras carry over to the class of so called *standardly stratified algebras* which is the most straightforward generalization of the original concept. On the other hand for so-called CPS-stratified algebras, which rely on the notion of *stratifying ideals*, defined by Cline, Parshall and Scott in [CPS] (but also investigated earlier by Auslander, Platzeck and Todorov in [APT]) and which seem to be the most general class definable in terms of stratification, no such generalization is known. In particular the lack of adequate structure theory makes it more difficult to handle some general questions concerning these algebras.

In an attempt to provide a basis for such a structure theory, the notion of *stratifying pairs* of module subcategories was introduced in [AL]. This notion was modelled on the subcategories of modules with standard and costandard filtration over standardly stratified algebras. Their homological behaviour is to a large extent determined by the fact that modules in such a pair of subcategories are perpendicular to each other (actually, the individual strata of these modules also have this property), moreover the strata will also satisfy some further homological conditions.

In the present paper we use the language of stratifying subcategories and stratifying pairs to extend earlier results of [DR] and of [ADL2]. Namely, each

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of these classes, i. e. quasi-hereditary algebras, standardly stratified algebras and CPS-stratified algebras – when defined for an algebra together with a complete ordering on a complete set of orthogonal idempotents – come with two sequences of algebras: one is a sequence of quotient algebras and the second a sequence of centralizers (i. e. endomorphism algebras of projective modules). While it is more common to deal with these classes via the recursive approach which uses the sequence of consecutive factors, the papers mentioned earlier ([DR] and [ADL2]) deal with the sequence of centralizers and bimodules with appropriate filtration, and give an explicit construction for quasi-hereditary and standardly stratified algebras. In this way one can obtain each quasi-hereditary and standardly stratified algebra, starting with local algebras. We extend this result to the class of CPS-stratified algebras.

In section 2 of the paper we establish a few results about the functorial connection between stratifying subcategories for the original algebra and stratifying subcategories for the centralizer algebras. Then in section 3 first we give precise conditions in terms of the Peirce decompositions of the algebra to be CPS-stratified. Finally we show how these conditions can be applied to construct from suitable algebras and bimodules a CPS-stratified algebra. We also show that this construction is universal in the sense that every CPS-stratified algebra can be obtained this way, starting with local algebras. We conclude with examples.

For background and unexplained notions concerning quasi-hereditary and standardly stratified algebras we refer for example to [DR], [ADL2] and the literature quoted there, however we shall not need them in this paper.

## 2. Stratifying subcategories and centralizers

$A$ will always denote a basic finite dimensional algebra over a field $K$. Modules – unless otherwise stated – will be right modules and mod-$A$ (or $A$-mod) will stand for the category of finitely generated right $A$-modules (left $A$-modules, respectively).

Let us recall some of the basic characterizations of so-called stratifying ideals.

Definition. (Cf. [CPS], [APT], [ADL2]) An idempotent ideal $AeA$ of the algebra $A$ (with $e^2 = e \in A$) is called *stratifying* if it satisfies any of the following equivalent conditions (S1), (S1′), (S2), (S3):

(S1) (*i*) the multiplication map induces a bijection $Ae \underset{eAe}{\otimes} eA \to AeA$, and

(*ii*) $\mathrm{Tor}_t^{eAe}(Ae, eA) = 0$ for all $t > 0$;

(S1′) (*i*) the multiplication map induces a bijection $Ae \underset{eAe}{\otimes} eA \to AeA$, and

(*ii*) $\mathrm{Ext}^t_{eAe}(Ae, D(eA)) = 0$ for all $t > 0$, where $D$ denotes $K$-duality;

(S2) $\mathrm{Ext}^t_{A/AeA}(X, Y) = \mathrm{Ext}^t_A(X, Y)$ for all $t \geq 0$ and $X, Y \in$ mod-$A/AeA$;

(S3) Each term in the minimal projective resolution of $AeA_A$ is in $\mathrm{add}(eA)$.

The last condition is of particular interest to us.

Definition. Let $e \in A$ be an idempotent element. The subcategory $\mathcal{P}(e)$ consists of all those $A$-modules for which there is a projective resolution with all projective terms in $\mathrm{add}(eA)$. In particular, $AeA$ is a stratifying ideal if and only if $AeA \in \mathcal{P}(e)$. Dually, $\mathcal{Q}(e)$ consists of all those $A$ modules for wich there is an injective resolution with all injective terms in $\mathrm{add}(D(Ae))$.

It is easy to see that $M \in \mathcal{P}(e)$ if and only if $\mathrm{Ext}^t_A(M,N) = 0$ for $t \geq 0$ and all modules $N$ with $Ne = 0$. This implies that $\mathcal{P}(e)$ is closed under extensions, direct summnads and kernels of epimorphisms.

For $\mathcal{C} \subseteq \text{mod-}A$ we use the notation $\mathcal{F}(\mathcal{C})$ for the class of modules filtered by elements of $\mathcal{C}$. Furthermore, take

$$\mathcal{C}^{\perp} = \mathcal{C}^{\perp}_A = \left\{\, N \in \text{mod-}A \,|\, \mathrm{Ext}^t_A(M,N) = 0 \ \forall\ t > 0 \text{ and } M \in \mathcal{C} \,\right\},$$
$$^{\perp}\mathcal{C} = {}^{\perp}\mathcal{C}_A = \left\{\, M \in \text{mod-}A \,|\, \mathrm{Ext}^t_A(M,N) = 0 \ \forall\ t > 0 \text{ and } N \in \mathcal{C} \,\right\}.$$

It is obvious that $^{\perp}\mathcal{C}$ is always a *resolving* subcategory in mod-$A$, i. e. it is closed under extensions, direct summands and kernels of epimorphisms, and it contains the projective modules, and similarly, $\mathcal{C}^{\perp}$ is necessarily a *coresolving* subcategory in mod-$A$, i. e. it is closed under extensions, direct summands and cokernels of monomorphisms, and it contains the injective modules.

Note that if $\mathcal{C}$ is resolving (coresolving, respectively) then in the above definitions it is enough to require that $\mathrm{Ext}^1(M,N) = 0$.

We list here some further homological properties of modules in $\mathcal{P}(e)$. (A version of) the next statment can be found in [APT]. For the convenience of the readers we include a proof.

Lemma 2.1. *Let $AeA$ be a stratifying ideal and $X \in \mathcal{P}(e)$. Then:*
*(a)* $\mathrm{Tor}^{eAe}_t(Xe, eA) = 0$ *for all* $t > 0$*;*
*(b)* $X \simeq Xe \underset{eAe}{\otimes} eA$.

*Proof.* Let us take the projective cover of $X$:

$$0 \to \Omega \to P \to X \to 0.$$

We can apply to this sequence the exact functor $\mathrm{Hom}_A(eA, -)$ to obtain

$$0 \longrightarrow \Omega e \overset{\kappa}{\longrightarrow} Pe \longrightarrow Xe \longrightarrow 0 \tag{1}$$

and then the functor $- \underset{eAe}{\otimes} eA$ to get the exact sequence:

$$0 \to K \to \Omega e \underset{eAe}{\otimes} eA \to Pe \underset{eAe}{\otimes} eA \to Xe \underset{eAe}{\otimes} eA \to 0 \tag{2}$$

where $K = \mathrm{Tor}^{eAe}_1(Xe, eA)$. Note that the exactness follows from the fact that $\mathrm{Tor}^{eAe}_1(Pe, eA) = 0$ by (S1)$(ii)$. We can also observe that $Ke = 0$ since by applying $\mathrm{Hom}_A(eA, -)$ to (2) we get the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Ke & \longrightarrow & \Omega e \underset{eAe}{\otimes} eAe & \overset{\kappa \otimes \mathrm{id}}{\longrightarrow} & Pe \underset{eAe}{\otimes} eAe & \longrightarrow & Xe \underset{eAe}{\otimes} eAe & \longrightarrow & 0 \\
 & & & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \\
 & & & & \Omega e & \overset{\kappa}{\longrightarrow} & Pe & \longrightarrow & Xe & \longrightarrow & 0
\end{array}$$

Here $\kappa$ is a monomorphism by (1) hence so is $\kappa \otimes \mathrm{id}$.

Let us factor out $K$ from the first two non-zero terms of (2) and apply the natural multiplication maps $\beta$ and $\gamma$ to the last two terms to get the following diagram:

$$\begin{array}{ccccccccc} 0 & \to & (\Omega e \underset{eAe}{\otimes} eA)/K & \to & Pe \underset{eAe}{\otimes} eA & \to & Xe \underset{eAe}{\otimes} eA & \to & 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\ 0 & \to & \Omega & \to & P & \to & X & \to & 0 \end{array}$$

$X \in \mathcal{P}(e)$ implies that $PeA = P$ and $XeA = X$, hence $\gamma$ is surjective and by (S1)($i$) the map $\beta$ is an isomorphism. Thus, by the Snake lemma $\operatorname{Ker}\gamma \simeq \operatorname{Coker}\alpha$ and $\operatorname{Ker}\alpha = 0$. Since $(Xe \underset{eAe}{\otimes} eA)e \simeq Xe$, we get that $(\operatorname{Ker}\gamma)e = 0$. On the other hand, $X \in \mathcal{P}(e)$ clearly implies $\Omega \in \mathcal{P}(e)$ and this gives $\Omega = \Omega eA$. So $\operatorname{Coker}\alpha = (\operatorname{Coker}\alpha)eA \simeq (\operatorname{Ker}\gamma)eA = 0$. Thus $\gamma$ and $\alpha$ are also isomorphisms. This proves ($b$).

$\Omega \in \mathcal{P}(e)$ and $Ke = 0$ gives that $\operatorname{Ext}^1(\Omega, K) = 0$. Thus the sequence $0 \to K \to \Omega e \underset{eAe}{\otimes} eA \to \Omega \to 0$ splits, giving that $K$ is a direct summand of $\Omega e \underset{eAe}{\otimes} eA$. This implies that $K = KeA = 0$, i.e. $\operatorname{Tor}_1(Xe, eA) = 0$. Applying the result for the syzygies of $X$, we get the same statment for higher Tor's, hence we proved ($a$). □

For an arbitrary idempotent element $e \in A$ we denote by $C = eAe$ the corresponding centralizer algebra in $A$. Throughout the paper we shall also make use of the following functors:

$$\begin{array}{rcccl} \Phi & = & \operatorname{Hom}_A(eA, -) & : & \text{mod-}A \to \text{mod-}C \\ \Gamma & = & - \underset{C}{\otimes} eA & : & \text{mod-}C \to \text{mod-}A \\ \Theta & = & \operatorname{Hom}_C(Ae, -) & : & \text{mod-}C \to \text{mod-}A \end{array}$$

Using the well-known natural isomorphism we shall identify $\Phi(X)$ with $Xe$. Observe also that the functor $\Theta$ is naturally equivalent to $D\Gamma^\circ D$, where $D = \operatorname{Hom}_K(-, K)$ is the standard $K$-duality functor, and $\Gamma^\circ = Ae \underset{C}{\otimes} - : C\text{-mod} \to A\text{-mod}$. Indeed, we have $\operatorname{Hom}_K(Ae \underset{C}{\otimes} D(X), K) \simeq \operatorname{Hom}_C(Ae, \operatorname{Hom}_K(D(X), K)) = \Theta(DD(X)) \simeq \Theta(X)$. It is also easy to see that both $\Phi\Gamma$ and $\Phi\Theta$ are naturally equivalent to $\mathrm{id}_{\text{mod-}C}$, so $\Phi$ and $\Gamma$ (or $\Phi$ and $\Theta$) give an equivalence between mod-$C$ and the image of $\Gamma$ (or the image of $\Theta$, respectively). We shall also use the following adjointness relations between these functors:

$$\operatorname{Hom}_A(\Gamma(X), Y) \simeq \operatorname{Hom}_C(X, \Phi(Y)) \quad \text{for } X \in \text{mod-}C,\ Y \in \text{mod-}A;$$

$$\operatorname{Hom}_C(\Phi(X), Y) \simeq \operatorname{Hom}_A(X, \Theta(Y)) \quad \text{for } X \in \text{mod-}A,\ Y \in \text{mod-}C.$$

Note that if $AeA$ is a stratifying ideal, the functors $\Phi$, $\Gamma$ and $\Theta$ are the functors of the so-called *recollement* on the module category level (cf. [CPS] (1.3.1), (2.1.2.2), and (2.1.2.3)). However we shall only use the properties listed above.

We shall adopt the following convention: when $\mathcal{C}$ is an isomorphism-closed subcategory of mod-$A$ or mod-$C$, respectively, then $\Phi(\mathcal{C})$, $\Gamma(\mathcal{C})$ and $\Theta(\mathcal{C})$ will stand for the isomorphism-closed subcategory of mod-$C$ or mod-$A$, respectively, which is generated by modules of the form $\Phi(X_A)$, $\Gamma(Y_C)$ or $\Theta(Z_C)$.

LEMMA 2.2. *Suppose that for an idempotent element $e \in A$ the ideal $AeA$ is a stratifying ideal. Then $\mathcal{P}(e) \underset{\Gamma}{\overset{\Phi}{\rightleftarrows}} \Phi(\mathcal{P}(e))$ and $\mathcal{Q}(e) \underset{\Theta}{\overset{\Phi}{\rightleftarrows}} \Phi(\mathcal{Q}(e))$ are equivalences between the given subcategories of* mod-$A$ *and* mod-$C$*, with $\Phi$, $\Gamma$ and $\Theta$ being exact.*

*Proof.* We have already seen that $\Phi\Gamma \simeq \mathrm{id}_{\text{mod-}C}$, hence the same natural isomorphism applies to the restriction of $\Gamma$ to $\Phi(\mathcal{P}(e))$. Next, Lemma 2.1($b$) implies that $\Gamma\Phi(X) \simeq X$ for every $X \in \mathcal{P}(e)$ and it is easy to see that the isomorphism is natural. Thus, $\Phi$ and $\Gamma$ are inverse equivalences when restricted to $\mathcal{P}(e)$ and $\Phi(\mathcal{P}(e))$. The exactness of $\Phi$ is obvious, while the exactness of $\Gamma$ when restricted to $\Phi(\mathcal{P}(e))$ follows from Lemma 2.1($a$).

The statement about the equivalence of $\mathcal{Q}(e)$ and $\Phi(\mathcal{Q}(e))$ follows by $K$-duality from the previous part, since $D(\mathcal{Q}(e)) = \mathcal{P}^\circ(e)$, where $\mathcal{P}^\circ(e)$ consists of all left $A$-modules for which there is a projective resolution with all projective terms in add$(Ae)$. □

LEMMA 2.3. *Suppose $AeA$ is a stratifying ideal. Then*
($a$) *$\Phi(\mathcal{P}(e))$ is a resolving and $\Phi(\mathcal{Q}(e))$ is a coresolving subcategory of* mod-$C$*;*
($b$) *If $\mathcal{P}' \subseteq$ mod-$C$ is a resolving subcategory, and $D(eA) \in (\mathcal{P}')^\perp$ then $\Gamma(\mathcal{P}') \subseteq \mathcal{P}(e)$. Dually, if $\mathcal{Q}' \subseteq$ mod-$C$ is a coresolving subcategory, and $Ae \in {}^\perp(\mathcal{Q}')$ then $\Theta(\mathcal{Q}') \subseteq \mathcal{Q}(e)$.*

*Proof.* To prove ($a$), we shall use the fact that $\mathcal{P}(e)$ is closed under extensions, direct summands and kernels of epimorphisms.

Let us take an exact sequence in mod-$C$:

$$0 \to X \to Y \to Z \to 0. \tag{3}$$

We can apply the right exact functor $\Gamma = - \underset{C}{\otimes} eA$ to get the following (not necessarily exact) sequence:

$$0 \to \Gamma(X) \to \Gamma(Y) \to \Gamma(Z) \to 0. \tag{4}$$

If $Z \in \Phi(\mathcal{P}(e))$, i. e. $Z \simeq Z'e$ for some $Z' \in \mathcal{P}(e)$ then by Lemma 2.1 ($a$) the sequence (4) is exact. Furthermore, by Lemma 2.1 ($b$), if $M_A \in \mathcal{P}(e)$, then $\Gamma\Phi(M) \simeq M$, hence for any $N \in \Phi(\mathcal{P}(e))$ we get that $\Gamma(N) \in \mathcal{P}(e)$.

Thus if $X, Z \in \Phi(\mathcal{P}(e))$, then (4) is exact, and $\Gamma(X), \Gamma(Z) \in \mathcal{P}(e)$, giving that $\Gamma(Y) \in \mathcal{P}(e)$ and $Y \simeq \Phi\Gamma(Y) \in \Phi(\mathcal{P}(e))$. Similarly, if $Y, Z \in \Phi(\mathcal{P}(e))$ then $X \in \Phi(\mathcal{P}(e))$. Thus $\Phi(\mathcal{P}(e))$ is closed under extensions and kernels of epimorphisms.

When (3) is a split sequence then (4) is also split exact. In this case, if $Y \in \Phi(\mathcal{P}(e))$ then $\Gamma(Y) \in \mathcal{P}(e)$, so we get that $\Gamma(X), \Gamma(Z) \in \mathcal{P}(e)$ and $X \simeq \Phi\Gamma(X)$ and $Z \simeq \Phi\Gamma(Z)$ belong to $\Phi(\mathcal{P}(e))$, i. e. $\Phi(\mathcal{P}(e))$ is closed under taking direct summands.

Finally, since $eA \in \mathcal{P}(e)$ and $\Phi(eA) = eAe = C \in \Phi(\mathcal{P}(e))$, we get that $\Phi(\mathcal{P}(e))$ contains the projectives in mod-$C$, hence $\Phi(\mathcal{P}(e))$ is a resolving subcategory.

To prove ($b$), let $X$ be a module in $\mathcal{P}'$ and let us take a minimal projective resolution of $X$ in mod-$C$. We know that $\mathcal{P}'$ is resolving, hence each syzygy is in

$\mathcal{P}'$. Since $D(eA) \in (\mathcal{P}')^{\perp}$ implies that $\mathrm{Tor}_t^C(M, eA) \simeq D(\mathrm{Ext}_C^t(M, D(eA)) = 0$ for $M \in \mathcal{P}'$, we get that $\Gamma$ maps the projective resolution of $X$ into an exact sequence. Clearly, projective $C$-modules are mapped to projective $A$-modules from $\mathrm{add}(eA)$, hence $\Gamma(X) \in \mathcal{P}(e)$. This shows that $\Gamma(\mathcal{P}') \subseteq \mathcal{P}(e)$.

The dual statements about $\mathcal{Q}(e)$ and $\mathcal{Q}'$ can be proved by applying the statements about $\mathcal{P}(e)$ and $\mathcal{P}'$ to left modules and taking $K$-duals, using the natural equivalence between $\Theta$ and $D\Gamma^{\circ}D$ □

From now on we shall fix a complete order $\mathbf{e} = (e_1, \dots, e_n)$ of primitive orthogonal idempotents and define the idempotents $\varepsilon_i = e_i + \dots + e_n$. Let $e = \varepsilon_i$ for a fixed $i \geq 2$; then the corresponding centralizer algebra is $C = \varepsilon_i A \varepsilon_i$, with a fixed order of primitive orthogonal idempotents $\mathbf{e}' = (e_i, \dots, e_n)$. The functors defined earlier are $\Phi = \mathrm{Hom}_A(\varepsilon_i A, -)$, $\Gamma = - \underset{C}{\otimes} \varepsilon_i A$, $\Theta = \mathrm{Hom}_C(A\varepsilon_i, -)$. We shall also use the notation $B = A/A\varepsilon_i A$ and $\mathbf{e}'' = (e_1, \dots, e_{i-1})$.

We need to recall a few concepts from [AL].

Definition. For $(A, \mathbf{e})$ we define the subcategories

$$\mathcal{P}_i(\mathbf{e}) = \left\{ M \in \text{mod-}A \,|\, M\varepsilon_{i+1} = 0 \text{ and } \mathrm{Ext}^t(M, S(j)) = 0 \ \forall j < i, \ \forall t \geq 0 \right\},$$

where $S(j)$ denotes the simple top of the projective module $e_j A$, and

$$\mathcal{Q}_i(\mathbf{e}) = \left\{ N \in \text{mod-}A \,|\, N\varepsilon_{i+1} = 0 \text{ and } \mathrm{Ext}^t(S(j), N) = 0 \ \forall j < i, \ \forall t \geq 0 \right\}.$$

Observe that if $M \in \mathcal{P}_i(\mathbf{e})$ then the top of $M$ is in $\mathrm{add}(S(i))$. Dually, if $N \in \mathcal{Q}_i(\mathbf{e})$ then the socle of $N$ is in $\mathrm{add}(S(i))$. Let us also note that $\mathcal{P}_j(\mathbf{e}) \subseteq \mathcal{P}(\varepsilon_i)$ and $\mathcal{Q}_j(\mathbf{e}) \subseteq \mathcal{Q}(\varepsilon_i)$ for $j \geq i$, and in the case when $A\varepsilon_{i+1}A$ is a stratifying ideal, $\mathcal{P}_i(\mathbf{e}) = \mathcal{P}_{A/A\varepsilon_{i+1}A}(e_i)$ and $\mathcal{Q}_i(\mathbf{e}) = \mathcal{Q}_{A/A\varepsilon_{i+1}A}(e_i)$. Finally, let

$$\mathcal{P}(\mathbf{e}) = \mathcal{F}(\mathcal{P}_1(\mathbf{e}), \dots, \mathcal{P}_n(\mathbf{e})) \text{ and}$$

$$\mathcal{Q}(\mathbf{e}) = \mathcal{F}(\mathcal{Q}_1(\mathbf{e}), \dots, \mathcal{Q}_n(\mathbf{e})).$$

Example. For an algebra $A$ with a fixed order $\mathbf{e}$ we define the *standard modules* $\Delta(i) = e_i A / e_i A \varepsilon_{i+1} A$ and *proper standard modules* $\bar{\Delta}(i) = e_i A / e_i \operatorname{rad} A \varepsilon_i A$. The algebra $(A, \mathbf{e})$ is said to be *$\Delta$-filtered* if $A_A \in \mathcal{F}(\Delta)$ and *$\bar{\Delta}$-filtered* if $A_A \in \mathcal{F}(\bar{\Delta})$ (with $\Delta = \{ \Delta(1), \dots, \Delta(n) \}$ and $\bar{\Delta} = \{ \bar{\Delta}(1), \dots, \bar{\Delta}(n) \}$). We call $(A, \mathbf{e})$ *standardly stratified* if it is either $\Delta$-filtered or $\bar{\Delta}$-filtered. By a result of Dlab ([D]) $(A, \mathbf{e})$ is $\Delta$-filtered if and only if $(A^{opp}, \mathbf{e})$ is $\bar{\Delta}$-filtered. Thus being standardly stratified is a two-sided concept. Observe also that both the notion of standard modules and that of standardly stratified algebras depend on the given order of idempotents. By reverse induction on $i$ it is easy to show (see for example [ADL1]) that if $(A, \mathbf{e})$ is $\Delta$-filtered then $\Delta(i) \in \mathcal{P}_i(\mathbf{e})$ for $1 \leq i \leq n$ and thus $A_A \in \mathcal{P}(\mathbf{e})$. Similarly, if $(A, \mathbf{e})$ is $\bar{\Delta}$-filtered then $\bar{\Delta}(i) \in \mathcal{P}_i(\mathbf{e})$ for $1 \leq i \leq n$, so $A_A \in \mathcal{P}(\mathbf{e})$.

Next we give the definition of the so-called CPS-stratified algebras, originating in the work of Cline, Parshall and Scott. In view of what was said in the previous example, this is a generalization of standardly stratified algebras.

Definition. (Cf. [AL].) The algebra $(A, \mathbf{e})$ is *CPS-stratified* if $A_A \in \mathcal{P}(\mathbf{e})$, or equivalently, if $D({}_AA) \in \mathcal{Q}(\mathbf{e})$. (We should emphasize that the concept of a CPS-stratified algebra, similarly to the notion of standard modules and standardly stratified algebras, depends on the order of idempotents.)

In the original definition of CPS-stratified algebras (see [CPS] and [ADL1]) $(A, \mathbf{e})$ is called CPS-stratified if $Ae_nA$ is a stratifying ideal in $A$ and $(A/Ae_nA, \mathbf{e}'')$ is CPS-stratified with $\mathbf{e}'' = (e_1, \dots, e_{n-1})$ (the trivial one-element algebra being considered as CPS-stratified). However, since by Lemma 2.2 of [AL] a module $X$ is in $\mathcal{P}(\mathbf{e})$ if and ony if $Xe_nA \in \mathcal{P}_n(\mathbf{e})$ and $X/Xe_nA \in \mathcal{F}(\mathcal{P}_1(\mathbf{e}), \dots, \mathcal{P}_{n-1}(\mathbf{e}))$, this recursive definition is equivalent to the one given above. Furthermore, using the recursive version of the definition and condition (S2) from the definition of stratifying ideals, it is easy to see that $(A, \mathbf{e})$ is CPS-stratified if and only if the ideals $A\varepsilon_iA$ are stratifying ideals in $A$ for $1 \leq i \leq n$.

By [AL], $\mathcal{P}(\mathbf{e})$ is a resolving and $\mathcal{Q}(\mathbf{e})$ is a coresolving subcategory for $(A, \mathbf{e})$ if $(A, \mathbf{e})$ is CPS-stratified.

Definition. $\mathcal{P} \subseteq$ mod-$A$ is a *stratifying subcategory for* $(A, \mathbf{e})$ if it is resolving, and $\mathcal{P} = \mathcal{F}(\mathcal{P}_1, \dots, \mathcal{P}_n)$ for some $\mathcal{P}_i \subseteq \mathcal{P}_i(\mathbf{e})$. Similarly, $\mathcal{Q} \subseteq$ mod-$A$ is a *costratifying subcategory for* $(A, \mathbf{e})$ if it is coresolving, and $\mathcal{Q} = \mathcal{F}(\mathcal{Q}_1, \dots, \mathcal{Q}_n)$ for some $\mathcal{Q}_i \subseteq \mathcal{Q}_i(\mathbf{e})$. A stratifying subcategory $\mathcal{P}$ and a costratifying subcategory $\mathcal{Q}$ for $(A, \mathbf{e})$ form a *stratifying pair for* $(A, \mathbf{e})$ if $\mathcal{Q} = \mathcal{P}^{\perp}$ and $\mathcal{P} = {}^{\perp}\mathcal{Q}$.

It was shown in [AL] that if $\mathcal{P}$ is a stratifying subcategory for $(A, \mathbf{e})$ then $\mathcal{P}^{\perp}$ is a costratifying subcategory and similarly, if $\mathcal{Q}$ is costratifying then ${}^{\perp}\mathcal{Q}$ is stratifying. Every CPS-stratified algebra has (at least one) stratifying pair: in fact, if $(A, \mathbf{e})$ is CPS-stratified then $\mathcal{P}(\mathbf{e})$ and $\mathcal{P}(\mathbf{e})^{\perp}$ form such a pair.

Lemma 2.4. *Let $A\varepsilon_iA$ be a stratifying ideal in $(A, \mathbf{e})$, and $j \geq i$. Then the pairs of functors $\mathcal{P}_j(\mathbf{e}) \underset{\Gamma}{\overset{\Phi}{\rightleftarrows}} \Phi(\mathcal{P}(\varepsilon_i)) \cap \mathcal{P}_j(\mathbf{e}')$ and $\mathcal{Q}_j(\mathbf{e}) \underset{\Theta}{\overset{\Phi}{\rightleftarrows}} \Phi(\mathcal{Q}(\varepsilon_i)) \cap \mathcal{Q}_j(\mathbf{e}')$ define equivalences between the corresponding subcategories of* mod-$A$ *and* mod-$C$.

*Proof.* Since $\mathcal{P}_j(\mathbf{e}) \subseteq \mathcal{P}(\varepsilon_i)$, we can apply Lemma 2.2. So it suffices to prove that $\Phi(\mathcal{P}_j(\mathbf{e})) = \Phi(\mathcal{P}(\varepsilon_i)) \cap \mathcal{P}_j(\mathbf{e}')$.

Suppose that $X \in \mathcal{P}_j(\mathbf{e})$. Then $X$ has a projective resolution in $\mathcal{P}(\varepsilon_i)$, whose projective terms belong to $\mathrm{add}(\varepsilon_jA)$, so $\Phi(X)$ has a projective resolution with projective terms in $\mathrm{add}(\varepsilon_jA\varepsilon_i)$. Furthermore, if $X \in$ mod-$A/A\varepsilon_{j+1}A$, i. e. $X\varepsilon_{j+1} = 0$, then $(X\varepsilon_i)\varepsilon_{j+1} = 0$. Thus for $X \in \mathcal{P}_j(\mathbf{e})$, we have $\Phi(X) \in \mathcal{P}_j(\mathbf{e}')$.

Conversely, let $X$ be in $\Phi(\mathcal{P}(\varepsilon_i)) \cap \mathcal{P}_j(\mathbf{e}')$, and consider a minimal projective resolution of $X$. By Lemma 2.3 $(a)$ the syzygies of this resolution are in $\Phi(\mathcal{P}(\varepsilon_i))$. Lemma 2.1 yields that by applying the functor $\Gamma$ to this resolution we get a projective resolution of $\Gamma(X)$ with projective terms in $\Gamma(\mathrm{add}(\varepsilon_jA\varepsilon_i)) = \mathrm{add}(\varepsilon_jA\varepsilon_i \underset{C}{\otimes} \varepsilon_iA) = \mathrm{add}(\varepsilon_jA)$. Furthermore, if $X\varepsilon_{j+1} = 0$, then $\Gamma(X)\varepsilon_{j+1} = X \underset{C}{\otimes} \varepsilon_iA\varepsilon_{j+1} = X\varepsilon_iA\varepsilon_{j+1} \underset{C}{\otimes} \varepsilon_{j+1} = X\varepsilon_{j+1} \underset{C}{\otimes} \varepsilon_{j+1} = 0$. So $\Gamma(X) \in \mathcal{P}_j(\mathbf{e})$, and $X \simeq \Phi\Gamma(X) \in \Phi(\mathcal{P}_j(\mathbf{e}))$.

The second statement follows from the first by $K$-duality. $\square$

The following two propositions give a connection between stratifying subcategories of $(A, \mathbf{e})$ and those of $(C, \mathbf{e}')$ and $(B, \mathbf{e}'')$.

Proposition 2.5. *Let $\mathcal{P}$ be a stratifying subcategory for $(A, \mathbf{e})$ and let $\mathcal{P}'$ be the image of $\mathcal{P}$ under the functor $\Phi$, i. e. $\mathcal{P}' = \Phi(\mathcal{P}) = P\varepsilon_i$. Then $\mathcal{P}'$ is a stratifying subcategory for $(C, \mathbf{e}')$ and $\mathcal{P}'' = \mathcal{P} \cap (\text{mod-}B)$ is a stratifying subcategory for $(B, \mathbf{e}'')$.*

*Proof.* Let us observe that $\Phi(\mathcal{P}) = \Phi(\mathcal{F}(\mathcal{P}_1, \dots, \mathcal{P}_n)) = \Phi(\mathcal{F}(\mathcal{P}_i, \dots, \mathcal{P}_n))$, where $\mathcal{F}(\mathcal{P}_i, \dots, \mathcal{P}_n) = \mathcal{P} \cap \mathcal{P}(\varepsilon_i)$ since for any $X \in \mathcal{P} \cap \mathcal{P}(\varepsilon_i)$ we have $X = X\varepsilon_i A$ and $X\varepsilon_j A / X\varepsilon_{j+1} A \in \mathcal{F}(\mathcal{P}_j)$ by Lemma 2.2 of [AL]. Since $\mathcal{P} \cap \mathcal{P}(\varepsilon_i)$ is closed under extensions, kernels of epimorphisms and direct summands, Lemma 2.2 and Lemma 2.3 $(a)$ give that $\Phi(\mathcal{P}) = \Phi(\mathcal{P} \cap \mathcal{P}(\varepsilon_i))$ is also closed under these operations. Furthermore, $\varepsilon_i A \varepsilon_i = \Phi(\varepsilon_i A) \in \Phi(\mathcal{P})$ also holds, thus $\Phi(\mathcal{P})$ is a resolving subcategory. By Lemma 2.4, $\Phi(\mathcal{P}_j) \subseteq \mathcal{P}_j(\mathbf{e}')$ for all $j \geq i$, so $\mathcal{P}' = \Phi(\mathcal{P}) = \mathcal{F}(\Phi(\mathcal{P}_i) \cdots, \Phi(\mathcal{P}_n))$ is a stratifying subcategory for $(C, \mathbf{e}')$.

To show that $\mathcal{P}'' = \mathcal{P} \cap \text{mod-}B = \mathcal{F}(\mathcal{P}_1, \dots, \mathcal{P}_{i-1})$ is a stratifying subcategory for $(B, \mathbf{e}'')$, observe first that it is clearly resolving. Thus we have to show only that $\mathcal{P}_j \subseteq \mathcal{P}_j(\mathbf{e}'')$ for $j < i$. But this follows from the fact that $A\varepsilon_i A$ is a stratifying ideal, hence by (S2) the extensions of $B$-modules over $B$ and $A$ are the same, thus the subcategories $\mathcal{P}_j(\mathbf{e}'')$ and $\mathcal{P}_j(\mathbf{e})$ are identical for $j < i$. □

Proposition 2.6. *For a given $(A, \mathbf{e})$ suppose that*

*$(i)$ $\mathcal{P}'$ is a stratifying subcategory for $(C, \mathbf{e}')$, and $\mathcal{P}''$ is a stratifying subcategory for $(B, \mathbf{e}'')$;*

*$(ii)$ $A\varepsilon_i \underset{C}{\otimes} \varepsilon_i A \simeq A\varepsilon_i A$;*

*$(iii)$ $A\varepsilon_i \in \mathcal{P}'$, $D(\varepsilon_i A) \in (\mathcal{P}')^{\perp}$.*

*Then $\mathcal{P} = \mathcal{F}(\mathcal{P}'', \Gamma(\mathcal{P}'))$ is a stratifying subcategory for $(A, \mathbf{e})$ such that $\Phi(\mathcal{P}) = \mathcal{P}'$.*

*Proof.* Note first that conditions $(ii)$ and $(iii)$ imply by (S1$'$) that $A\varepsilon_i A$ is a stratifying ideal. It is also clear that $A_A \in \mathcal{P}$, since $A\varepsilon_i A \simeq A\varepsilon_i \underset{C}{\otimes} \varepsilon_i A \in \Gamma(\mathcal{P}')$ and $A/A\varepsilon_i A \in \mathcal{P}''$.

Lemma 2.3 $(b)$ implies that $\Gamma(\mathcal{P}') \subseteq \mathcal{P}(\varepsilon_i)$, and from the equivalence given by the (exact) functors $\Gamma(\mathcal{P}') \underset{\Gamma}{\overset{\Phi}{\rightleftarrows}} \mathcal{P}' = \Phi\Gamma(\mathcal{P}')$ (see Lemma 2.2) it follows that $\Gamma(\mathcal{P}')$ is closed under extensions, kernels of epimorphisms and direct summands.

Lemma 2.4 proves that $\Gamma(\mathcal{P}'_j) \subseteq \mathcal{P}_j(\mathbf{e})$ for $j \geq i$, and the elements of $\mathcal{P}$ are filtered by $\mathcal{P}''_1, \dots, \mathcal{P}''_{i-1}, \Gamma(\mathcal{P}'_i), \dots, \Gamma(\mathcal{P}'_n)$. Since the latter satisfy the closure properties of Proposition 2.6 in [AL], $\mathcal{P}$ is a stratifying subcategory for $(A, \mathbf{e})$.

Finally, $\Phi(\mathcal{P}) = \Phi(\mathcal{F}(\mathcal{P}'', \Gamma(\mathcal{P}'))) = \Phi\Gamma(\mathcal{P}') = \mathcal{P}'$. □

To establish a similar connection between stratifying pairs of subcategories for a CPS-stratified algebra $(A, \mathbf{e})$ and its centralizer algebra $(C, \mathbf{e}')$, we need first the following lemma.

LEMMA 2.7. *Let $(A, \mathbf{e})$ be a CPS-stratified algebra, $X \in \mathcal{F}(\mathcal{P}_i(\mathbf{e}), \ldots, \mathcal{P}_n(\mathbf{e}))$ and $Y \in \mathcal{F}(\mathcal{Q}_i(\mathbf{e}), \ldots, \mathcal{Q}_n(\mathbf{e}))$. Then for arbitrary $t > 0$*

$$\mathrm{Ext}^t_A(X, Y) = 0 \iff \mathrm{Ext}^t_C(\Phi(X), \Phi(Y)) = 0.$$

*Proof.* We prove the statement only for $t = 1$; then the general statement will follow by a usual dimension shifting argument, using the fact that the syzygies of $X \in \mathcal{F}(\mathcal{P}_i(\mathbf{e}), \ldots, \mathcal{P}_n(\mathbf{e})) = \mathcal{P} \cap \mathcal{P}(\varepsilon_i)$ also belong to $\mathcal{F}(\mathcal{P}_i(\mathbf{e}), \ldots, \mathcal{P}_n(\mathbf{e}))$, and $\Phi$ maps a projective resolution into a projective resolution.

Let us assume first that $\mathrm{Ext}^1_A(X, Y) = 0$. This is equivalent to saying that for the projective cover of $X$ in mod-$A$:

$$0 \longrightarrow \Omega \stackrel{\alpha}{\longrightarrow} P \longrightarrow X \longrightarrow 0$$

the map $\mathrm{Hom}_A(P, Y) \to \mathrm{Hom}_A(\Omega, Y)$ is surjective. That is to say, for every $\beta \in \mathrm{Hom}_A(\Omega, Y)$ there is $\gamma \in \mathrm{Hom}_A(P, Y)$ making the following diagram commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega & \stackrel{\alpha}{\longrightarrow} & P & \longrightarrow & X & \longrightarrow & 0 \\ & & {\scriptstyle\beta}\big\downarrow & \swarrow{\scriptstyle\gamma} & & & & & \\ & & Y & & & & & & \end{array}$$

By applying the functor $\Phi$ to this diagram we get the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Phi(\Omega) & \stackrel{\Phi(\alpha)}{\longrightarrow} & \Phi(P) & \longrightarrow & \Phi(X) & \longrightarrow & 0 \\ & & {\scriptstyle\Phi(\beta)}\big\downarrow & \swarrow{\scriptstyle\Phi(\gamma)} & & & & & \\ & & \Phi(Y) & & & & & & \end{array}$$

where $\Phi(P)$ is projective. Note that $X \in \mathcal{F}(\mathcal{P}_i(\mathbf{e}), \ldots, \mathcal{P}_n(\mathbf{e})) \subseteq \mathcal{P}(\varepsilon_i)$ implies that $\Omega \in \mathcal{P}(\varepsilon_i)$ so Lemma 2.1 $(b)$ and the adjointness of the functors $\Gamma$ and $\Phi$ give

$$\mathrm{Hom}_A(\Omega, Y) \simeq \mathrm{Hom}_A(\Gamma\Phi(\Omega), Y) \simeq \mathrm{Hom}_C(\Phi(\Omega), \Phi(Y)).$$

This shows that $\mathrm{Hom}_C(\Phi(P), \Phi(Y)) \to \mathrm{Hom}_C(\Phi(\Omega), \Phi(Y))$ is also surjective, hence $\mathrm{Ext}^1_C(\Phi(X), \Phi(Y)) = 0$.

Conversely, let us now assume that $\mathrm{Ext}^1_C(\Phi(X), \Phi(Y)) = 0$. This means that if we take the projective cover of $\Phi(X)$ in mod-$C$:

$$0 \longrightarrow \Omega' \stackrel{\alpha}{\longrightarrow} P' \longrightarrow \Phi(X) \longrightarrow 0,$$

the map $\mathrm{Hom}_C(\mathcal{P}', \Phi(Y)) \to \mathrm{Hom}_C(\Omega', \Phi(Y))$ is surjective:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega' & \stackrel{\alpha}{\longrightarrow} & P' & \longrightarrow & \Phi(X) & \longrightarrow & 0. \\ & & {\scriptstyle\beta}\big\downarrow & \swarrow{\scriptstyle\gamma} & & & & & \\ & & \Phi(Y) & & & & & & \end{array}$$

This gives rise to the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma(\Omega') & \stackrel{\Gamma(\alpha)}{\longrightarrow} & \Gamma(P') & \longrightarrow & \Gamma\Phi(X) & \longrightarrow & 0. \\
 & & {\scriptstyle\beta'}\big\downarrow & \swarrow{\scriptstyle\gamma'} & & & & & \\
 & & \Theta\Phi(Y) & & & & & &
\end{array}$$

Here the maps $\beta'$ and $\gamma'$ are obtained from $\beta$ and $\gamma$ using the natural isomorphisms, coming from the adjointness of $\Phi$ and $\Theta$:

$$\mathrm{Hom}_C(M,N) \simeq \mathrm{Hom}_C(\Phi\Gamma(M),N) \simeq \mathrm{Hom}_A(\Gamma(M),\Theta(N)).$$

As in the previous part, we get that $\mathrm{Hom}_A(\Gamma(P'),\Theta\Phi(Y)) \to \mathrm{Hom}_A(\Gamma(\Omega'),\Theta\Phi(Y))$ is surjective. Since $\Gamma(P')$ is projective, this proves that $\mathrm{Ext}^1_A(\Gamma\Phi(X),\Theta\Phi(Y)) = 0$. But $X \in \mathcal{P}(\varepsilon_i)$ implies $\Gamma\Phi(X) \simeq X$ and $Y \in \mathcal{Q}(\varepsilon_i)$ implies $\Theta\Phi(Y) \simeq Y$ by Lemma 2.1 (*b*) and its dual. So $\mathrm{Ext}^1_A(X,Y) = 0$. □

Proposition 2.8. *Let $(A,\mathbf{e})$ be a CPS-stratified algebra. Then the following are equivalent for a pair $(\mathcal{P}',\mathcal{Q}')$ of subcategories of* mod-$C$.

*(1) $(\mathcal{P}',\mathcal{Q}')$ is a stratifying pair over $(C,\mathbf{e}')$ with $A\varepsilon_i \in \mathcal{P}'$ and $D(\varepsilon_i A) \in \mathcal{Q}'$.*

*(2) There is a stratifying pair $(\mathcal{P},\mathcal{Q})$ over $(A,\mathbf{e})$ such that $(\mathcal{P}',\mathcal{Q}') = (\Phi(\mathcal{P}),\Phi(\mathcal{Q}))$.*

*Proof.* Let us fix a stratifying pair $(\mathcal{P}'',\mathcal{Q}'')$ for $(B,\mathbf{e}'')$.

Let $\mathcal{H}$ denote the set of all those pairs $(\mathcal{P},\mathcal{Q})$ of stratifying and costratifying subcategories for $(A,\mathbf{e})$ for which that $\mathcal{P} \cap (\text{mod-}B) = \mathcal{P}''$, $\mathcal{Q} \cap \text{mod-}B = \mathcal{Q}''$ and $\mathcal{Q} \subseteq \mathcal{P}^{\perp}$.

Let $\mathcal{H}'$ denote the set of all those pairs $(\mathcal{P}',\mathcal{Q}')$ of stratifying and costratifying subcategories for $(C,\mathbf{e}')$ for which $A\varepsilon_i \in \mathcal{P}'$, $D(\varepsilon_i A) \in \mathcal{Q}'$, and $\mathcal{Q}' \subseteq \mathcal{P}'^{\perp}$.

Consider the following maps:

$$\begin{array}{lcll}
\mu & : & (\mathcal{P},\mathcal{Q}) \mapsto (\Phi(\mathcal{P}),\Phi(\mathcal{Q})) & \text{for each } (\mathcal{P},\mathcal{Q}) \in \mathcal{H} \\
\nu & : & (\mathcal{P}',\mathcal{Q}') \mapsto (\mathcal{F}(\mathcal{P}'',\Gamma(\mathcal{P}')),\mathcal{F}(\mathcal{Q}'',\Theta(\mathcal{Q}'))) & \text{for each } (\mathcal{P}',\mathcal{Q}') \in \mathcal{H}'
\end{array}$$

Then $\mu$ maps every pair $(\mathcal{P},\mathcal{Q}) \in \mathcal{H}$ to a pair $(\mathcal{P}',\mathcal{Q}') \in \mathcal{H}'$, since Proposition 2.5 and its dual imply that $\mathcal{P}' = \Phi(\mathcal{P})$ is a stratifying and $\mathcal{Q}' = \Phi(\mathcal{Q})$ is a costratifying subcategory for $(C,\mathbf{e}')$; $A_A \in \mathcal{P}$ gives that $A\varepsilon_i \in \mathcal{P}'$, and similarly, $D({}_AA) \in \mathcal{Q}$ gives $D(\varepsilon_i A) = D(A)\varepsilon_i \in \mathcal{Q}'$; finally, $\mathcal{Q}' \subseteq \mathcal{P}'^{\perp}$ follows from Lemma 2.7, since $\Phi(\mathcal{P}) = \Phi(\mathcal{F}(\mathcal{P}_i,\ldots,\mathcal{P}_n))$, $\Phi(\mathcal{Q}) = \Phi(\mathcal{F}(\mathcal{Q}_i,\ldots,\mathcal{Q}_n))$, and $\mathcal{F}(\mathcal{Q}_i,\ldots,\mathcal{Q}_n) \subseteq \mathcal{F}(\mathcal{P}_i,\ldots,\mathcal{P}_n)^{\perp}$.

Next we show that the map $\nu$ maps every pair $(\mathcal{P}',\mathcal{Q}') \in \mathcal{H}'$ to a pair $(\mathcal{P},\mathcal{Q}) \in \mathcal{H}$. Proposition 2.6 and its dual imply that $\mathcal{P} = \mathcal{F}(\mathcal{P}'',\Gamma(\mathcal{P}'))$ is a stratifying and $\mathcal{Q} = \mathcal{F}(\mathcal{Q}'',\Theta(\mathcal{Q}'))$ is a costratifying subcategory for $(A,\mathbf{e})$. Furthermore, $\mathcal{P} \cap (\text{mod-}B) = \mathcal{F}(\mathcal{P}'',\Gamma(\mathcal{P}')) \cap (\text{mod-}B) = \mathcal{P}''$, similarly, $\mathcal{Q} \cap (\text{mod-}B) = \mathcal{Q}''$. We still have to prove that $\mathcal{Q} \subseteq \mathcal{P}^{\perp}$. First, $\mathrm{Ext}^t_A(\mathcal{P}'',\mathcal{Q}'') = \mathrm{Ext}^t_B(\mathcal{P}'',\mathcal{Q}'') = 0$ for $t > 0$, since $(\mathcal{P}'',\mathcal{Q}'')$ is a stratifying pair for $(B,\mathbf{e}'')$ and $A\varepsilon_i A$ is a stratifying ideal. Next, $\mathrm{Ext}^t_A(\mathcal{P}'',\Theta(\mathcal{Q}')) = \mathrm{Ext}^t_A(\Gamma(\mathcal{P}'),\mathcal{Q}'') = 0$ for $t > 0$, since $\Theta(\mathcal{Q}') \in \mathcal{Q}(\varepsilon_i)$ and $\Gamma(\mathcal{P}') \in \mathcal{P}(\varepsilon_i)$ by Lemma 2.3 (*b*). Finally, $\Theta(\mathcal{Q}') = \mathcal{F}(\mathcal{Q}_i,\ldots,\mathcal{Q}_n)$,

$\Gamma(\mathcal{P}') = \mathcal{F}(\mathcal{P}_i, \ldots, \mathcal{P}_n)$ and $\mathrm{Ext}^t_C(\mathcal{Q}', \mathcal{P}') = 0$ for $t > 0$, so Lemma 2.7 gives that $\mathrm{Ext}^t_A(\Gamma(\mathcal{P}'), \Theta(\mathcal{Q}') = 0$.

It is clear that $\mu\nu = \mathrm{id}_{\mathcal{H}'}$. On the other hand, for any pair $(\mathcal{P}, \mathcal{Q}) \in \mathcal{H}$, $\mathcal{F}(\mathcal{P}'', \Gamma\Phi(\mathcal{P})) = \mathcal{P}$, since $\Gamma\Phi(\mathcal{P}) = \Gamma\Phi(\mathcal{F}(\mathcal{P}_i, \ldots, \mathcal{P}_n))$, which is equal to $\mathcal{F}(\mathcal{P}_i, \ldots, \mathcal{P}_n)$ by Lemma 2.2, and dually, $\mathcal{F}(\mathcal{Q}'', \Theta\Phi(\mathcal{Q})) = \mathcal{Q}$. So $\nu\mu = \mathrm{id}_{\mathcal{H}}$.

Now let us assume that $(\mathcal{P}, \mathcal{Q})$ is a stratifying pair over $(A, \mathbf{e})$, and let $\mathcal{P}'' = \mathcal{P} \cap (\text{mod-}B)$ and $\mathcal{Q}'' = \mathcal{Q} \cap (\text{mod-}B)$. Consider the classes $\mathcal{H}$ and $\mathcal{H}'$ and the maps $\mu$ and $\nu$ with this fixed pair $(\mathcal{P}'', \mathcal{Q}'')$. Since $\mathcal{Q} = \mathcal{P}^\perp$, $\mathcal{Q}$ is the largest costratifying subcategory such that $(\mathcal{P}, \mathcal{Q}) \in \mathcal{H}$. Thus, for $(\mathcal{P}', \mathcal{Q}') = \mu(\mathcal{P}, \mathcal{Q})$, the subcategory $\mathcal{Q}'$ is the largest costratifying subcategory for $(C, \mathbf{e}')$ such that $(\mathcal{P}', \mathcal{Q}') \in \mathcal{H}'$. But $(\mathcal{P}', (\mathcal{P}')^\perp)$ is also in $\mathcal{H}'$, so $(\mathcal{P}')^\perp \subseteq \mathcal{Q}' \subseteq (\mathcal{P}')^\perp$, i. e. $(\mathcal{P}')^\perp = \mathcal{Q}'$, and similarly, ${}^\perp(\mathcal{Q}') = \mathcal{P}'$. So $(\mathcal{P}', \mathcal{Q}')$ is a stratifying pair with $A\varepsilon_i \in \mathcal{P}'$ and $D(\varepsilon_i A) \in \mathcal{Q}'$. This proves that (2) implies (1).

With an analogous argument we get that for any stratifying pair $(\mathcal{P}', \mathcal{Q}') \in \mathcal{H}'$ the pair $(\mathcal{P}, \mathcal{Q}) = \nu(\mathcal{P}', \mathcal{Q}')$ is a stratifying pair for $(A, \mathbf{e})$ with $\mathcal{P}' = \Phi(\mathcal{P})$ and $\mathcal{Q}' = \Phi(\mathcal{Q})$. So (1) implies (2). □

## 3. Recursive construction of CPS-stratified algebras

Theorem 3.1. *For $(A, \mathbf{e})$ let $\varphi_i = e_1 + \ldots + e_{i-1} = 1 - \varepsilon_i$, and $E = \varphi_i A \varepsilon_i$, $F = \varepsilon_i A \varphi_i$, $C = \varepsilon_i A \varepsilon_i$, $B = A/A\varepsilon_i A$. Then $(A, \mathbf{e})$ is CPS-stratified if and only if the following conditions hold:*

*(i) $(C, \mathbf{e}')$ and $(B, \mathbf{e}'')$ are CPS-stratified;*

*(ii) the multiplication map $E \otimes_C F \to \varphi_i A \varphi_i$ is injective;*

*(iii) there is a stratifying pair $(\mathcal{P}', \mathcal{Q}')$ for $(C, \mathbf{e}')$ such that $E \in \mathcal{P}'$ and $D(F) \in \mathcal{Q}'$.*

*Proof.* Let us assume first that $(A, \mathbf{e})$ is CPS-stratified. Then $\mathcal{P} = \mathcal{P}(\mathbf{e})$ is a stratifying subcategory. By Proposition 2.5, $(C, \mathbf{e}')$ and $(B, \mathbf{e}'')$ are CPS-stratified, as stated in condition $(i)$. Since $A\varepsilon_i A$ is a stratifying ideal, the multiplication map $A\varepsilon_i \otimes_C \varepsilon_i A \to A\varepsilon_i A$ is injective by (S1), so $\varphi_i A\varepsilon_i \otimes_C \varepsilon_i A\varphi_i \to \varphi_i A\varepsilon_i A\varphi_i$ is also injective, proving condition $(ii)$. Finally, by Lemma 2.5, $\mathcal{P}' = \Phi(\mathcal{P}(\mathbf{e})) \subseteq \Phi(\mathcal{P}(\varepsilon_i))$ is a stratifying subcategory for $(C, \mathbf{e}')$ with $E = \Phi(\varphi_i A) \in \mathcal{P}'$, and by Lemma 2.1 $(a)$ the costratifying subcategory $\mathcal{Q}' = (\mathcal{P}')^\perp$ contains $D(\varepsilon_i A)$ and its direct summand $D(F) = D(\varepsilon_i A\varphi_i)$ as well. Since ${}^\perp(\mathcal{Q}')$ is a stratifying subcategory for $(C, \mathbf{e}')$, by Proposition 2.6 $\Gamma({}^\perp(\mathcal{Q}'))$ can be included in a stratifying subcategory for $(A, \mathbf{e})$, so it is in $\mathcal{P}(\mathbf{e})$, hence ${}^\perp(\mathcal{Q}') = \Phi\Gamma({}^\perp(\mathcal{Q}')) \subseteq \Phi(\mathcal{P}(\mathbf{e})) = \mathcal{P}'$. But the latter is clearly contained in ${}^\perp(\mathcal{Q}')$, so ${}^\perp(\mathcal{Q}') = \mathcal{P}'$. Thus $(\mathcal{P}', \mathcal{Q}')$ is a stratifying pair for $(C, \mathbf{e}')$, satisfying condition $(iii)$.

Now suppose that conditions $(i)$, $(ii)$ and $(iii)$ hold. Condition $(ii)$, i. e. the injectivity of the map $E \otimes_C F \to \varphi_i A\varphi_i$ implies that the multiplication map $A\varepsilon_i \otimes_C \varepsilon_i A = (E \oplus C) \otimes_C (F \oplus C) \to A$ is also injective, since the injectivity for the other three direct components is obvious. Thus condition $(ii)$ of Proposition 2.6 holds. Condition $(iii)$ of the theorem implies that $A\varepsilon_i = E \oplus C \in \mathcal{P}'$ and $D(\varepsilon_i A) = D(F) \oplus D(C) \in \mathcal{Q}'$, since $C_C$ is projective and $D({}_C C)$ is injective, so condition $(iii)$ of Proposition 2.6 is also satisfied. Finally, we can take any stratifying subcategory for $(B, \mathbf{e}'')$ as $\mathcal{P}''$ to satisfy condition $(i)$ of Proposition 2.6, so there exists a stratifying subcategory for $(A, \mathbf{e})$, i. e. $(A, \mathbf{e})$ is CPS-stratified. □

We should note that condition $(iii)$ of the previous theorem can be checked without fully describing the corresponding stratifying pair. Namely, Proposition 3.7 of [AL] states that a pair of modules $X \in \mathcal{P}(\mathbf{e})$ and $D(Y) \in \mathcal{Q}(\mathbf{e})$ over a CPS-stratified algebra $(A, \mathbf{e})$ is contained in a stratifying pair of subcategories if and only if the corresponding strata of $X$ and $D(Y)$ are perpendicular, that is, if $\mathrm{Ext}^t_A(X_j, D(Y_j)) = 0$ for $1 \le j \le n$ and $t > 0$, where $X_j = X\varepsilon_j A / X\varepsilon_{j+1}A$ and $Y_j = A\varepsilon_j Y / A\varepsilon_{j+1}Y$.

If $i = n$ in the previous theorem then $(C, \mathbf{e}')$ is automatically CPS-stratified since $C$ is local. Furthermore, in this case condition $(iii)$ is equivalent (by the above mentioned Proposition 3.7 of [AL]) to saying that $\mathrm{Ext}^t(E, D(F)) = 0$ for all $t > 0$, so $(ii)$ and $(iii)$ together give, by (S1′), the condition that $A\varepsilon_i A$ is a stratifying ideal. This is the usual recursive definition of a CPS-stratified algebra.

Similarly, if $i = 2$, then $B$ is local, hence the condition on the algebra $B$ can be dropped. Actually, we can use the previous theorem for this situation to construct all CPS-stratified algebras. The construction follows closely the construction of $\Delta$- and $\bar{\Delta}$ filtered (i. e. standardly stratified) algebras in [ADL2], so we only prove what is different in this case.

Let us take a local algebra $L$ with unit element $e_1$, and a CPS-stratified algebra $(C, \mathbf{e}')$, where $\mathbf{e}' = (e_2, \ldots, e_n)$. Furthermore, let ${}_LE_C$ and ${}_CF_L$ be bimodules such that $E_C \in \mathcal{P}'$ and $D({}_CF) \in \mathcal{Q}'$, where $(\mathcal{P}', \mathcal{Q}')$ is a stratifying pair for $(C, \mathbf{e}')$. As we mentioned above, such a stratifying pair exists if and only if $E_C \in \mathcal{P}(\mathbf{e}')$, $D({}_CF) \in \mathcal{Q}(\mathbf{e}')$ and $\mathrm{Ext}^t_C(E\varepsilon_j C / E\varepsilon_{j+1}C, D(C\varepsilon_j F / C\varepsilon_{j+1}F)) = 0$ for all $t > 0$ and $j \ge 2$. We also fix a $C$-$C$ bimodule homomorphism $\mu : F \underset{L}{\otimes} E \to \operatorname{rad} C$.

We extend $L$ to a larger local algebra $\widetilde{L}$ so that

$$\widetilde{L} = L \ltimes (E \underset{C}{\otimes} F)$$

is the split extension of $L$ by the algebra $E \underset{C}{\otimes} F$, where the algebra multiplication on $E \underset{C}{\otimes} F$ is defined by

$$(E \underset{C}{\otimes} F) \underset{L}{\otimes} (E \underset{C}{\otimes} F) \simeq E \underset{C}{\otimes} (F \underset{L}{\otimes} E) \underset{C}{\otimes} F \xrightarrow{\mathrm{id}_E \otimes \mu \otimes \mathrm{id}_F} E \underset{C}{\otimes} C \underset{C}{\otimes} F \simeq E \underset{C}{\otimes} F.$$

In a similar fashion we extend the $L$-module structure on $E$ and $F$ to an $\widetilde{L}$-module structure. Thus $E$ and $F$ become $\widetilde{L}$-$C$ and $C$-$\widetilde{L}$ bimodules. Finally, we define the algebra $\tilde{A}$ as

$$\tilde{A} = \begin{pmatrix} \tilde{L} & E \\ F & C \end{pmatrix}$$

with the natural algebra structure. In an obvious way $\mathbf{e} = (e_1, \ldots, e_n)$ gives a complete ordered set of primitive orthogonal idempotents in $\widetilde{A}$.

Since $\varepsilon_2 \widetilde{A} \varepsilon_2 = C$, $\widetilde{A}/\widetilde{A}\varepsilon_2\widetilde{A} \simeq L$, $(e_1 \widetilde{A} \varepsilon_2)_C = E_C$ and ${}_C(\varepsilon_2 \widetilde{A} e_1) = {}_CF$, conditions $(i)$ and $(iii)$ of Theorem 3.1 are satisfied. The construction of $\widetilde{L} = e_1 \widetilde{A} e_1$ ensures that $E \underset{C}{\otimes} F = e_1 \widetilde{A} \varepsilon_2 \widetilde{A} e_1$, so condition $(ii)$ of Theorem 3.1 also holds. Thus we proved the following theorem.

Theorem 3.2. *Let $L$ be a local algebra with unit element $e_1$, $(C, \mathbf{e}')$ a CPS-stratified algebra with $\mathbf{e}' = (e_2, \ldots, e_n)$, ${}_L E_C$ and ${}_C F_L$ bimodules such that $E_C \in \mathcal{P}'$ and $D({}_C F) \in \mathcal{Q}'$, where $(\mathcal{P}', \mathcal{Q}')$ is a stratifying pair for $(C, \mathbf{e}')$, and finally, $\mu : F \underset{L}{\otimes} E \to \operatorname{rad} C$ a bimodule homomorphism. If $\widetilde{A}$ is the algebra constructed above and $\mathbf{e} = (e_1, \ldots, e_n)$, then $(\widetilde{A}, \mathbf{e})$ is CPS-stratified.*

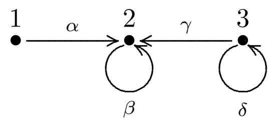
In order to obtain all CPS-stratified algebras, we may have to take quotients of algebras constructed above. In [ADL2] an ideal $H \lhd A$ is called an *auxiliary ideal* of the algebra $(A, \mathbf{e})$ if $H \subseteq e_1 \operatorname{rad} A e_1$ and $H \cap e_1 A \varepsilon_2 A e_1 = 0$. Let us take an auxiliary ideal $H$ in the algebra $\widetilde{A}$ from the previous theorem. Then $A = \widetilde{A}/H$ is also CPS-stratified since $C, E, F$ remain the same and the map $E \underset{C}{\otimes} F \to \widetilde{L}/H$ remains injective.

Theorem 3.3. *Let $(A, \mathbf{e})$ be a CPS-stratified algebra and let us take $L = e_1 A e_1$, $C = \varepsilon_2 A \varepsilon_2$, $E = e_1 A \varepsilon_2$, $F = \varepsilon_2 A e_1$ and the multiplication map $\mu : F \underset{L}{\otimes} E \to \operatorname{rad} C$. Then with the algebra $\widetilde{A}$ and an appropriate auxiliary ideal $H \subseteq e_1 \widetilde{A} e_1$, we have $A \simeq \widetilde{A}/H$.*

*Proof.* Let us define $\nu : E \underset{C}{\otimes} F \to L$ to be the natural multiplication map, and $H = \left\{ u - \nu(u) \mid u \in E \underset{C}{\otimes} F \right\}$. Theorem 3.1 implies that conditions of Theorem 3.2 for $C$, $E$ and $F$ are satisfied, thus we can construct $\widetilde{A}$ in the prescribed way. Furthermore, the proof of Theorem 4.4 of [ADL2] can be applied to show that $H$ is an auxiliary ideal, and $\widetilde{A}/H \simeq A$. □

Let us conclude with two examples.

Example 3.4. We give an example of a situation where a stratifying pair for $(C, \mathbf{e}')$ cannot be extended to a stratifying pair for $(A, \mathbf{e})$ (cf. Proposition 2.8). Let $A = KG/I$, where $G$ is the graph




and $I = (\alpha\beta, \beta^2, \gamma\beta, \delta\gamma, \delta^2)$. So the right regular representation of $A$ is given by

$$A_A = \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}2\\2\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\2&&3\end{smallmatrix}$$

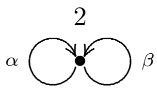
and for $C = \varepsilon_2 A \varepsilon_2$ we have

$$C_C = \begin{smallmatrix}2\\2\end{smallmatrix} \oplus \begin{smallmatrix}&3&\\2&&3\end{smallmatrix}.$$

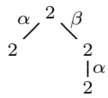
$(C, \mathbf{e}')$ is a $\Delta$-filtered (hence standardly stratified) algebra of finite representation type, with 8 indecomposable modules. It is easy to check that $\mathcal{P}(\mathbf{e}') = \operatorname{add}\left(\begin{smallmatrix}2\\2\end{smallmatrix}, \begin{smallmatrix}&3&\\2&&3\end{smallmatrix}, 2, \begin{smallmatrix}2&3\\&2&3\end{smallmatrix}\right)$, which contains only two stratifying subcategories, giving the stratifying pairs $(\operatorname{add}(C_C) = \mathcal{F}(\Delta), \text{mod-}C)$ and $\left(\mathcal{F}\left(2, \begin{smallmatrix}&3&\\2&&3\end{smallmatrix}\right), \mathcal{F}\left(\begin{smallmatrix}2\\2\end{smallmatrix}, 3\right)\right)$. Here the first stratifying pair cannot be extended to a stratifying pair for $(A, \mathbf{e})$, since $e_1 A \varepsilon_2 \notin \operatorname{add}(C_C)$. On the other hand the second pair can be extended to the pair $\left(\mathcal{F}\left(1, 2, \begin{smallmatrix}&3&\\2&&3\end{smallmatrix}\right), \mathcal{F}\left(1, \begin{smallmatrix}2\\2\end{smallmatrix}, 3\right)\right)$.

Example 3.5. Here we show how Theorem 3.2 can be applied to construct CPS-stratified algebras, starting with two local algebras.

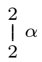
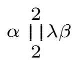
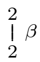
Let $G$ be the graph with one vertex and two loops:




Let us take $C = KG/I$ with $I = (\alpha^2, \beta^2, \alpha\beta)$ and use the notation $e_2 = 1_C$. The regular representation of $C$ can be described by

$C_C =$ 

For $L$ we can take the base field $K$ as a local $K$-algebra. Since $C$ is local, to find suitable bimodules $E$ and $F$ we only have to satisfy the conditions $\mathrm{Ext}^t_C(E, D(F)) = 0$ for $t > 0$: by Propositon 3.7 of [AL] if these conditions are satisfied, we can always find a stratifying pair for $(C, e_2)$, containing the given modules. To this end, let us consider the following modules: $X = C/\beta C$, $Y_\lambda = C/(\alpha - \lambda\beta)C$ for $0 \neq \lambda \in K$ and $Z = C/(\alpha C + \beta\alpha C)$. Thus we have:

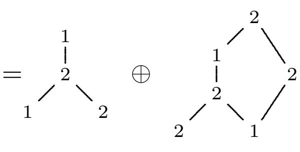
$X_C =$  $(Y_\lambda)_C =$  $Z_C =$ 

One can check easily that the following extension modules are all zero:

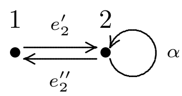
$$\mathrm{Ext}^t_C(X, Y_\lambda) = \mathrm{Ext}^t_C(X, Z) = \mathrm{Ext}^t_C(Y_\lambda, Y_\kappa) = \mathrm{Ext}^t_C(Y_\lambda, Z) = 0$$

for all $t > 0$ and $\lambda \neq \kappa$.

Thus we can start for example with $E_C = X$ and ${}_CF = D(Y_1)$. In order to define the map $\mu : F \underset{L}{\otimes} E \to \mathrm{rad}\, C$, we fix a basis for $C$, $L$, $E$ and $F$. Let $C = \langle e_2, \alpha, \beta, \beta\alpha\rangle$ and $L = \langle e_1\rangle$; and similarly $E = \langle e_2', \alpha'\rangle$ with $e_2'\alpha = \alpha'$ and $e_2'\beta = \alpha'\alpha = \alpha'\beta = 0$ and $F = \langle e_2'', \alpha''\rangle$ with $\alpha e_2'' = \beta e_2'' = \alpha''$ and $\alpha\alpha'' = \beta\alpha'' = 0$. Then we can define the $C$-$C$ bimodule map $\mu$ as follows. Let us take $\mu(e_2'' \otimes e_2') = \beta$; then $\mu(e_2'' \otimes \alpha') = \beta\alpha$, furthermore $\mu(\alpha'' \otimes e_2') = \mu(\alpha'' \otimes \alpha') = 0$. Note that $E \underset{C}{\otimes} F$ is one dimensional and the multiplication on $E \underset{C}{\otimes} F$ becomes zero. Actually we can obtain the complete multiplication table of $\widetilde{A}$. We get that the regular module over $\widetilde{A}$ can be described by

$\widetilde{A}_{\widetilde{A}} =$ 

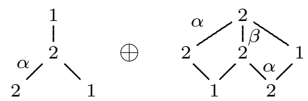
and hence $\widetilde{A} = K\widetilde{G}/I$ where $\widetilde{G}$ is the graph



and $I = (\alpha^2,\ e_2' e_2'' e_2',\ e_2'' e_2' e_2'' - \alpha e_2'')$. In this way we get a CPS-stratified algebra, which is not standardly stratified. Since there are no non-zero auxiliary ideals, this input set $(L, E, F, C, \mu)$ gives only this algebra by our construction.

On the other hand by modifying the map $\mu$ we can obtain a completely different algebra. If $\mu(\mathbf{e}_2'' \otimes e_1') = \beta\alpha$ then we get an algebra $\widetilde{A}$ with regular decomposition

$\widetilde{A}_{\widetilde{A}} =$




Let us note also that by using different perpendicular pairs of bimodules (for example using the modules $X$, $Y_\lambda$ and $Z$ in a different setup) we can get infinitely many CPS-stratified algebras.

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