Finitistic dimension of standardly stratified algebras

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Abstract. We prove that the projectively and the injectively defined finitistic dimensions of a standardly stratified algebra are always finite by giving the optimal bound for these numbers in terms of the number of simple modules.

The notion of standardly stratified algebras is a natural generalization of quasi-hereditary algebras, introduced earlier by Cline, Parshall and Scott (cf. [CPS1]); quasi-hereditary algebras are precisely those standardly stratified algebras which have finite global dimension ([D], [W]). For further generalizations and broader context of this class we refer to [CPS2]; see also [APT].

In [DR1], Dlab and Ringel showed that the global dimension of a quasi-hereditary algebra is always bounded by $2n-2$, where n denotes the number of simple modules for the given algebra, and this bound is the best possible.

The aim of the present paper is to show that the same bound holds for the projectively and the injectively defined finitistic dimensions of standardly stratified algebras (Theorem 2.1 and Theorem 3.1). Observe that — unlike the notion of a quasi-hereditary algebra — the concept of a standardly stratified algebra is not two-sided thus the two different finitistic dimensions should be examined separately. For the historical background of the finitistic dimension conjecture we refer to [ZH].

We should note that the finiteness of the finitistic dimension of standardly stratified algebras can also be deduced from [H], however neither the sharp bounds nor the left-right symmetry follow from this argument.

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1. Preliminaries

In what follows, A will stand for a finite dimensional associative algebra over a field K, and mod-A will denote the category of finitely generated right A-modules. For a subclass $\mathcal{S} \subseteq$ mod-A the projective dimension of $\mathcal{S}$ is defined to be $pd\,\mathcal{S} = \sup\{\,pd\,M \mid M \in \mathcal{S}\,\}$. We will use similar notation for the injective dimension, too. Let $fin.dim\,\mathcal{S} = \sup\{\,pd\,M \mid pd\,M < \infty,\ M \in \mathcal{S}\,\}$. The (projectively defined) finitistic dimension of an algebra A is defined as $fin.dim\,A = fin.dim\,\text{mod-}A$, that is:

$$fin.dim\,A = \sup\{\,pd\,M \mid M \in \text{mod-}A,\ pd\,M < \infty\,\}\ .$$

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Observe that by taking the supremum of the injective dimensions of those modules for which this number is finite, we get the finitistic dimension of A^{opp}.

For simplicity, we shall also assume that A is basic. Let $\mathbf{e} = (e_1, e_2, \dots, e_n)$ be a fixed ordered sequence of primitive orthogonal idempotents so that $e_1+e_2+\cdots+e_n = 1$. To this sequence we shall associate another sequence of idempotents, defined by $\varepsilon_i = e_i + e_{i+1} + \cdots + e_n$ for $1 \le i \le n$. For convenience we take $\varepsilon_{n+1} = 0$.

By fixing the order of the idempotents, we also get an ordering of the indecomposable projective A-modules $P(i) \simeq e_iA$ and the simple modules $S(i) \simeq P(i)/\operatorname{rad} P(i)$. For $1 \le i \le n$, the *i-th standard module* $\Delta(i)$ is defined to be the largest quotient of $P(i)$ which contains no composition factors $S(j)$ for $j > i$. Thus, $\Delta(i) \simeq e_iA/e_iA\varepsilon_{i+1}A$. The *proper standard module* $\overline{\Delta}(i)$ is the largest quotient of $\Delta(i)$ with semisimple endomorphism ring, i.e. $\overline{\Delta}(i) \simeq e_iA/e_i \operatorname{rad} A\, \varepsilon_iA$. Dually, we get the notion of *costandard modules* $\nabla(i)$ and *proper costandard modules* $\overline{\nabla}(i)$, by taking the appropriate submodules of the injective hull $I(i)$ of $S(i)$. Thus, $\nabla(i) \simeq D\Delta^\circ(i)$ and $\overline{\nabla}(i) \simeq D\overline{\Delta}^\circ(i)$, where D denotes the K-dual of a module, and $\Delta^\circ(i)$ and $\overline{\Delta}^\circ(i)$ stands for the similarly defined left standard and left proper standard modules. By Δ (resp. $\overline{\Delta}$, ∇ and $\overline{\nabla}$) we shall denote the set of all standard (resp. proper standard, costandard and proper costandard modules). Similar notation will be used for the corresponding left modules, too.

For a subclass $\mathcal{S}$ of right (or left) A-modules we denote by $\mathcal{F}(\mathcal{S})$ the class of those right (resp. left) A-modules for which there exists a filtration where the corresponding factor modules all belong to $\mathcal{S}$. Recall that $(A, \mathbf{e})$ is *quasi-hereditary* if $A_A \in \mathcal{F}(\Delta)$ and $\operatorname{End}_A\big(\Delta(i)\big)$ is semisimple for $1 \le i \le n$ (cf. [DR1] or [DR2]). Observe that the semisimplicity condition on the endomorphism rings of standard modules is equivalent to the condition that $\Delta(i) = \overline{\Delta}(i)$ for $1 \le i \le n$ (cf. [D], [ADL]). The algebra $(A, \mathbf{e})$ is *standardly stratified* if $A_A \in \mathcal{F}(\Delta)$ (see [CPS2], [D] or [ADL]). In view of the obvious fact that $\operatorname{Ext}^1_A\big(\Delta(i), \Delta(j)\big) = 0$ for $i > j$ (see, for example, [DR2]; similar condition holds also for $\overline{\Delta}$), the condition that $M \in \mathcal{F}(\Delta)$ (or $M \in \mathcal{F}(\overline{\Delta})$, resp.) is equivalent to the condition that the *trace filtration* $M = M\varepsilon_1A \supseteq M\varepsilon_2 \supseteq \cdots \supseteq M\varepsilon_nA \supseteq 0$ can be refined to a filtration with factors in Δ (or $\overline{\Delta}$, resp.). Note that for a module M the multiplicities of $\Delta(i)$-s (resp. $\overline{\Delta}(i)$-s) in any Δ-filtration (resp. $\overline{\Delta}$-filtration) of M must be unique.

Thus, if A is standardly stratified, the trace ideal Ae_nA is projective as a right A-module, and the factor algebra $\bar{A} = A/Ae_nA$ is also standardly stratified. We shall frequently make use of the following simple fact (see [CPS1], [DR1], [APT]):

PROPOSITION 1.1. *Let $I \triangleleft A$ be an idempotent ideal which is projective as a right A-module and let $\bar{A} = A/I$. Then for any $X, Y \in$ mod-$\bar{A}$ and any $i \ge 0$ we have:* $\operatorname{Ext}^i_{\bar{A}}(X, Y) = \operatorname{Ext}^i_A(X, Y)$.

Idempotent ideals with the above property will be called *stratifying ideals* (cf. [CPS2]). Observe that we have identified the $\bar{A}$-modules with their images under the canonical inclusion functor mod-$\bar{A} \to$ mod-A.

In contrast to the quasi-hereditary situation, $A_A \in \mathcal{F}(\Delta)$ does not imply, in general, that ${}_AA \in \mathcal{F}(\Delta^\circ)$. In fact, one has the following proposition (see [D]):

PROPOSITION 1.2. *The following are equivalent for an algebra $(A, \mathbf{e})$:*
(*i*) $A_A \in \mathcal{F}(\Delta)$;
(*ii*) $D(A_A) \in \mathcal{F}(\nabla^\circ)$;
(*iii*) $D({}_AA) \in \mathcal{F}(\overline{\nabla})$;
(*iv*) ${}_AA \in \mathcal{F}(\overline{\Delta}^\circ)$.

We shall also need the following consequence of Theorem 3.1 from [ADL].

PROPOSITION 1.3. *Let* $(A, \mathbf{e})$ *be given and assume that* $A_A \in \mathcal{F}(\overline{\Delta})$. *Then* $\mathcal{F}(\overline{\Delta}) = \{ X \mid \mathrm{Ext}^1_A(X, \nabla(j)) = 0 \}$ *for all* $1 \le j \le n$.

Finally, we shall also need the following result from [ADL].

LEMMA 1.4. *For given* $(A, \mathbf{e})$ *the class* $\mathcal{F}(\overline{\Delta})$ *is closed under kernels of epimorphisms.*

2. Standardly stratified algebras

The main result of this section is the following theorem:

THEOREM 2.1. *Let* $(A, \mathbf{e})$ *be given and assume that* $A_A \in \mathcal{F}(\Delta)$, *i. e.* A *is standardly stratified. Then* $fin.dim\, A \le 2n - 2$.

Actually, the result will follow by an easy induction from the following, more general result:

THEOREM 2.2. *Let* $e \in A$ *be a primitive idempotent element. Suppose that the ideal* AeA_A *is projective and for the algebra* $\bar{A} = A/AeA$ *the finitistic dimension* $fin.dim\, \bar{A} = k < \infty$. *Then* $fin.dim\, A \le k + 2$.

We shall need some preliminary lemmas.

LEMMA 2.3. *Let* A *be a local algebra. If* $pd\, M_A < \infty$ *then* M_A *is projective.*

Proof. Observe that each projective module has the same Loewy-length, thus for P and P' projective there is no embedding $P \to \mathrm{rad}\, P'$. Thus no finite minimal projective resolution of length larger than 0 exists. □

LEMMA 2.4. *Let* $e \in A$ *be a primitive idempotent and assume that* AeA_A *is projective. If* $M \in \mathrm{mod}\text{-}A$ *has finite projective dimension then* Me *is a projective* eAe*-module.*

Proof. Consider a finite projective resolution of M_A:

$$0 \to P_t \to \cdots \to P_0 \to M \to 0\,.$$

Then applying the exact functor $\mathrm{Hom}_A(eA, -)$ we get the following exact sequence of right eAe-modules:

$$0 \to P_t e \to \cdots \to P_0 e \to Me \to 0\,.$$

Here the projectivity of AeA_A implies that the modules $P_i eA$ are projective as A-modules, hence the modules $P_i e$ are projective as eAe-modules for $i = 0, \ldots, t$. Thus $pd\, Me_{eAe} < \infty$. Then Lemma 2.3 implies that Me is projective. □

LEMMA 2.5. *Let* $e \in A$ *be a primitive idempotent and assume that* AeA_A *is projective. Assume that for a module* $M \in \mathrm{mod}\text{-}A$ *the module* Me_{eAe} *is projective. Let*

$$0 \to \Omega \to P \to MeA \to 0$$

be an exact sequence of right A*-modules with* $P \to MeA$ *a projective cover. Then* $\Omega e = 0$.

Proof. From the sequence above we get the following short exact sequence of eAe-modules:

$$0 \to \Omega e \to Pe \to MeAe = Me \to 0\,.$$

Then the projectivity of Me implies that the sequence splits. On the other hand, using that $P = PeA$, we get that Ωe is mapped into $\operatorname{rad} Pe$. Hence $\Omega e = 0$. □

LEMMA 2.6. *Let $e \in A$ be a primitive idempotent and assume that AeA_A is projective. Suppose $pd\, M_A < \infty$ and let*

$$0 \to \Omega \to P \to M \to 0$$

be an exact sequence of right A-modules with $P \to M$ a projective cover. Then $\Omega \cap PeA \simeq X \oplus \big(\oplus eA\big)$ for some X with $Xe = 0$.

Proof. The exact sequence above gives rise to the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \to & \Omega \cap PeA & \to & PeA & \to & MeA & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \Omega & \to & P & \to & M & \to & 0\,. \end{array}$$

Here the projectivity of AeA gives that $PeA \simeq \oplus eA$ and $pd\, M < \infty$ implies by Lemma 2.4 that Me_{eAe} is projective.

Let $0 \to X \to P' \to MeA \to 0$ be an exact sequence of right A-modules with $P' \to MeA$ a projective cover. Then, clearly, $P' \simeq \oplus eA$ and the projectivity of $MeAe_{eAe} = Me_{eAe}$ implies, by Lemma 2.5, that $Xe = 0$. By Schanuel's Lemma, $P' \oplus (\Omega \cap PeA) \simeq PeA \oplus X$. Hence, by the Krull-Schmidt Theorem we get that $\Omega \cap PeA$ is of the required form. □

LEMMA 2.7. *Let $e \in A$ be an idempotent and assume that AeA_A is projective. Let $\bar{A} = A/AeA$. If $M \in \text{mod-}\bar{A}$ has finite projective dimension as an $\bar{A}$-module, then $pd\, M_A \leq pd\, M_{\bar{A}} + 1$.*

Proof. The projectivity of the ideal AeA implies that if $P_{\bar{A}}$ is projective, then $pd\, P_A \leq 1$. Hence if we take a minimal projective resolution of M as an $\bar{A}$-module, we get that $pd\, M_A \leq t+1$, where t is the length of the resolution. □

Proof of Theorem 2.2. Let $M \in \text{mod-}A$ be a module with $pd\, M < \infty$ and consider the first step of a minimal projective resolution of M:

$$0 \to \Omega \to P \to M \to 0\,.$$

Then Lemma 2.6 implies that $\Omega \cap PeA \simeq X \oplus \big(\oplus eA\big)$ with $Xe = 0$. Since $\Omega eA = (\Omega \cap PeA)eA$, we get that $\Omega eA \simeq \big(X \oplus \big(\oplus eA\big)\big)eA = \oplus eA$, so ΩeA is projective. Hence, from the exact sequence $0 \to \Omega eA \to \Omega \to \bar{\Omega} = \Omega/\Omega eA \to 0$ and from $pd\, \Omega < \infty$, we have $pd\, \bar{\Omega}_A < \infty$. Since AeA is a stratifying ideal, this means that $\bar{\Omega}$, as an $\bar{A} = A/AeA$-module is also of finite projective dimension (cf. Proposition 1.1). Hence $pd\, \bar{\Omega}_{\bar{A}} \leq k$. By Lemma 2.7, $pd\, \bar{\Omega}_A \leq k+1$. The projectivity of ΩeA_A implies now that $pd\, \Omega_A \leq k+1$, hence $pd\, M \leq k+2$.

This finishes the proof. □

3. Algebras with proper standard filtration

The main result of this section is the following theorem:

THEOREM 3.1. *Let* $(A, \mathbf{e})$ *be given and assume that* $A_A \in \mathcal{F}(\overline{\Delta})$, *i. e.* A^{opp} *is standardly stratified. Then* $fin.dim\, A \leq 2n-2$.

Observe that in view of Proposition 1.2 this result implies that the injectively defined finitistic dimension (or equivalently, that the left finitistic dimension) of standardly stratified algebras is also bounded by $2n-2$.

For the proof we will first need a few lemmas.

LEMMA 3.2. *Let* $(A, \mathbf{e})$ *be given and assume that* $A_A \in \mathcal{F}(\overline{\Delta})$. *Then* $id\, \mathcal{F}(\nabla) \leq n-1$.

Proof. In view of Proposition 1.2, using the standard K-duality, we shall prove a dual statement, namely: if $A_A \in \mathcal{F}(\Delta)$, then $pd\, \mathcal{F}(\Delta) \leq n-1$. Clearly, it is enough to show that $pd\, \Delta \leq n-1$.

In the first step, observe that $\Delta(n) = P(n)$, hence $pd\, \Delta(n) = 0$. Next, let us take $\bar{A} = A/Ae_nA$ with the restriction of the order defined by $\mathbf{e}$. Observe that $\Delta(i) = \Delta_A(i) = \Delta_{\bar{A}}(i)$ for $1 \leq i \leq n-1$, and clearly, $\bar{A}_{\bar{A}} \in \mathcal{F}(\Delta_{\bar{A}})$. Hence, by induction, we get that $pd\, \Delta(i)_{\bar{A}} \leq n-2$ for $1 \leq i \leq n-2$. Consequently, by Lemma 2.7, we have $pd\, \Delta(i)_A \leq n-1$ for $1 \leq i \leq n-1$. Thus, we are done.

Let us remark that actually we could have proved a more precise statement, namely that $pd\, \Delta(i) \leq n-i$. □

LEMMA 3.3. *Let* $(A, \mathbf{e})$ *be given and for a module* M_A *let* $\bar{M} = M/Me_nA$. *Let* $0 \to X \to Y \to Z \to 0$ *be an exact sequence of* A*-modules so that* $Ye_nA, Ze_nA \in \mathcal{F}\big(\overline{\Delta}(n)\big)$. *Then the sequence* $0 \to \bar{X} \to \bar{Y} \to \bar{Z} \to 0$ *is also exact.*

Observe here that the filtration condition is satisfied if, for example, $Y, Z \in \mathcal{F}(\Delta)$.

Proof. We have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \to & X \cap Ye_nA & \to & Ye_nA & \to & Ze_nA & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & X & \to & Y & \to & Z & \to & 0. \end{array}$$

By Lemma 1.4 we get that $X \cap Ye_nA \in \mathcal{F}(\overline{\Delta})$; moreover, the uniqueness of multiplicities of $\overline{\Delta}(i)$-s in any filtration shows that $X \cap Ye_nA \in \mathcal{F}\big(\overline{\Delta}(n)\big)$. This implies that $X \cap Ye_nA = Xe_nA$, hence we get the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc} 0 & \to & Xe_nA & \to & Ye_nA & \to & Ze_nA & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & X & \to & Y & \to & Z & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \bar{X} & \to & \bar{Y} & \to & \bar{Z} & \to & 0. \end{array}$$

This finishes the proof. □

LEMMA 3.4. *For given* $(A, \mathbf{e})$ *assume that* $A_A \in \mathcal{F}(\overline{\Delta})$. *Then* $fin.dim\, \mathcal{F}(\overline{\Delta}) \leq n-1$.

Proof. Let $M \in \mathcal{F}(\overline{\Delta})$; then we also have $Me_nA \in \mathcal{F}\big(\overline{\Delta}(n)\big)$. Assume that $pd\, M = r < \infty$ and let

$$0 \to P_r \to \cdots \to P_1 \to P_0 \to M \to 0$$

be a minimal projective resolution of M.

Since $Me_nA, P_ie_nA \in \mathcal{F}\big(\Delta(n)\big)$, we get by induction from Lemma 1.4 that every short exact sequence of the minimal projective resolution satisfies the conditions of Lemma 3.3, hence we get the following exact sequence:

$$0 \to \bar{P}_r \to \cdots \to \bar{P}_1 \to \bar{P}_0 \to \bar{M} \to 0.$$

Here $\bar{X} = X/Xe_nA$ for an arbitrary module $X \in$ mod-A.

Since $\bar{P}_i$ is projective over $\bar{A} = A/Ae_nA$, and $\bar{P}_i$ is mapped into $\operatorname{rad} \bar{P}_{i-1}$ for every $1 \leq i \leq r$, we get in this way a minimal projective resolution of $\bar{M}$ as an $\bar{A}$-module. Thus, $pd\, \bar{M}_{\bar{A}} < \infty$, on the other hand, $\bar{M}_{\bar{A}} \in \mathcal{F}(\overline{\Delta}_{\bar{A}})$, hence by induction we get that $pd\, \bar{M}_{\bar{A}} \leq n-2$.

We have obtained that $\bar{P}_j = 0$ for $j \geq n-1$. Thus, for these indices $P_j = \oplus P(n)$. A Loewy-length argument shows that $\oplus_I P(n)$ cannot be embedded into $\operatorname{rad}\big(\oplus_J P(n)\big)$, hence $r \leq n-1$.

Thus, $pd\, M \leq n-1$, as required. □

LEMMA 3.5. *For given* $(A, \mathbf{e})$, *let* $A_A \in \mathcal{F}(\overline{\Delta})$. *Then, for any* $M \in$ mod-A, *the* $n-1$*-th syzygy of* M, *denoted by* $\Omega_{n-1}(M)$ *is in* $\mathcal{F}(\overline{\Delta})$.

Proof. From the projective resolution of M, we get the following equality for arbitrary $X \in$ mod-A: $\operatorname{Ext}^n_A(M, X) \simeq \operatorname{Ext}^1\big(\Omega_{n-1}(M), X\big)$. By taking $X = \nabla(j)$ for $1 \leq j \leq n$, Lemma 3.2 implies that $\operatorname{Ext}^1_A\big(\Omega_{n-1}(M), \nabla(j)\big) = 0$. Hence, by Proposition 1.3 we get that $\Omega_{n-1}(M) \in \mathcal{F}(\overline{\Delta})$, as required. □

We are now ready to prove the main result of the section.

Proof of Theorem 3.1. Let $M \in$ mod-A be such that $pd\, M < \infty$. Then, by Lemma 3.5 we get that $\Omega_{n-1}(M) \in \mathcal{F}(\overline{\Delta})$, and by Lemma 3.4, we have that $pd\, \Omega_{n-1}(M) \leq n-1$. Hence $pd\, M \leq 2n-2$. □

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