Semigroups for Delay equations

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February 12, 2004

Contents

Preface

Partial differential equations with delay arise in many areas of applied mathematics. An understanding of the linear case plays a crucial role in the analysis of such equations, particularly when the asymptotic behaviour is concerned. In these notes we develop a general theory of the linear situation using methods that, in particular, allow a systematic discussion of the asymptotic behaviour of solutions.

In order to explain how delay equations arise, we present an example arising in control theory. Consider a hot bar of length one that is insulated at its endpoints $s = 0, 1$. We assume that the bar can be heated around the point $s_0 \in (0, 1)$. Denote by $x(t, s)$ its temperature at position $s \in [0, 1]$ and time $t \geq 0$ and by $x_0(\cdot)$ the initial temperature profile. Then $x(t, s)$ satisfies

$$\begin{cases} \dfrac{\partial x(t,s)}{\partial t} = \dfrac{\partial^2 x(t,s)}{\partial s^2} + b(s)u(t) & \text{for } t \geq 0 \text{ and } s \in [0,1], \\ \dfrac{\partial x(t,0)}{\partial s} = \dfrac{\partial x(t,1)}{\partial s} = 0 & \text{for } t \geq 0, \\ x(0,s) = x_0(s) & \text{for } s \in [0,1]. \end{cases} \tag{1}$$

Here b represents the “shaping” function around the control point s_0, for example

$$b(s) := \frac{1}{2\varepsilon}\chi_{[s_0-\varepsilon, s_0+\varepsilon]}$$

for some $\varepsilon > 0$, where χ_J denotes the characteristic function of a set J. The control $u(\cdot)$ is a function $u : [0, \infty) \to \mathbb{R}$ and represents the temperature around the point s_0.

Assume now that the control temperature $u(t)$ at time t is not arbitrary but is computed by averaging the temperature profile $x(t, \cdot)$ as follows

$$u(t) := \int_0^1 x(t,s)\, ds. \tag{2}$$

Then (1) becomes a *feedback* system as studied, for example, in Curtain, Zwart [16], Lasiecka, Triggiani [54].

However, such a feedback does not appear to be very realistic since it assumes knowledge of $\int_0^1 x(t, s)\, ds$ at time t itself. A more realistic feedback signal would be $x(t - \tau)$ instead of $x(t)$, and then (1) becomes a *retarded feedback* system and hence a *partial differential equation with delay* of the form

$$\begin{cases} \dfrac{\partial x(t,s)}{\partial t} = \dfrac{\partial^2 x(t,s)}{\partial s^2} + b(s)\displaystyle\int_0^1 x(t-\tau,s)\,ds & \text{for } t \geq 0 \text{ and } s \in [0,1], \\ \dfrac{\partial x(t,0)}{\partial s} = \dfrac{\partial x(t,1)}{\partial s} = 0 & \text{for } t \geq 0, \\ x(0,s) = x_0(s) & \text{for } s \in [0,1]. \end{cases} \tag{3}$$

The systematic investigation of such equations began at the beginning of the 20th century with the work of V. Volterra [78]. However, the asymptotic behaviour of the solutions was not well understood at that time because the usual approach was based on fixed-point arguments.

It was J. Hale who in 1971 [40] showed that these solutions (which he still obtained by fixed-point methods) give rise to an operator semigroup on an appropriate function space, called *history* or *phase space*. He subsequently obtained with S. Verduyn Lunel [42] a complete asymptotic description of the solution in the finite dimensional case using semigroup theory. Later, G. Webb [79] inverted this procedure by first constructing a semigroup and then showing that this semigroup yields the solutions of the delay equation. This is the idea we will systematically employ throughout these notes.

The present lectures are based on the material presented by the authors at the Internet-seminar 2002/2003 and on the lectures held by Susanna Piazzera at the International Minicourse-Workshop “Interplay between (C_0)-semigroups and PDEs: theory and applications”, September 22-27, 2003 in Bari.

There is some overlap between the material presented here and the monograph prepared by the authors on this subject, see [8]. However, we changed considerably the presentation, deleted parts of this manuscript and added some exercises. This is why we omit most general semigroup-theoretic results and many refined results on the asymptotic behaviour of the delay semigroup, such as positivity, critical spectrum or asymptotic almost periodicity.

Throughout the text we assume that the reader is already familiar with basic semigroup theory. This knowledge can be obtained for example from the monographs Belleni-Morante [9], Davies [23], Clément et al. [14], Engel and Nagel [30], Fattorini [31], Goldstein [36], Hille and Phillips [45], Krein [51] and Pazy [64].

However, we have chosen as a general reference Engel and Nagel [30] mainly because the earlier books do not contain two results which are central to our investigations: the perturbation theorem of Miyadera-Voigt and the theorem of Gearhart.

The results presented here are mainly based on work by the authors and their co-authors, P. Binding, A. Dijksma, R. Hryniv, H. Langer and R. Nagel, [1], [2], [3], [5], [7], [6], [13], [61], [65].

We also present related work by L. Maniar and J. Voigt [56] and discuss many new examples.

Finally, we thank Silvia Romanelli, Rosa Maria Mininni and Sandra Lucente for organizing the beatiful International Minicourse-Workshop "Interplay between (C_0)*-semigroups and PDEs: theory and applications", September 22-27, 2003 in Bari and for the opportunity to publish these lectures in this volume.*

Budapest and Tübingen, January 2004

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1 Delay equations and semigroups

In this lecture we present a systematic semigroup approach to linear partial differential equations with delay using operator matrices.

In Section 1.1, we associate an operator $(\mathcal{A}, D(\mathcal{A}))$ on the Banach space $X \times L^p([-1,0],X)$ to the abstract delay equation on a Banach space X. We also show that there is a one-to-one correspondence between the solutions of the abstract delay equation and the solutions of the abstract Cauchy problem corresponding to the associated operator $\mathcal{A}$ (see Theorem 1.11). This reformulation of the delay equation has the advantage that the question of its well-posedness reduces to the question of whether or not the operator $(\mathcal{A}, D(\mathcal{A}))$ generates a strongly continuous semigroup on the Banach space $X \times L^p([-1,0],X)$.

With this in mind, we study in Section 1.2 the generator property of this operator. The main idea here is to write $\mathcal{A}$ as a perturbation of a simpler operator and then apply the Miyadera-Voigt Perturbation Theorem to obtain a sufficient condition for $\mathcal{A}$ to be a generator.

Finally, in Section 1.4 we characterize the spectrum and compute the resolvent of the operator $\mathcal{A}$.

1.1 The semigroup approach

In this section we will see that the appropriate setting for linear delay differential equations is that of an abstract Cauchy problem in an appropriate Banach space.

Consider first the ordinary differential equation

$$u'(t) = A\,u(t), \qquad t \geq 0, \tag{4}$$

for a matrix $A \in \mathcal{L}(\mathbb{C}^n)$. Everybody who has had a basic ODE course knows that the fundamental solution to Equation (4) is given by the exponential function $t \mapsto e^{tA}$. More precisely, for every $x \in \mathbb{C}^n$ the function $u(t) := e^{tA}x$ is the unique solution of Equation (4) with initial value x.

Let us now modify (4) slightly by considering

$$u'(t) = A\,u(t-\tau), \qquad t \geq 0, \tag{5}$$

where $\tau > 0$. Do we still find an exponential function solving this equation? Of course, the function $t \mapsto e^{tA}$ does not work any more and there is no other matrix $B \in \mathcal{L}(\mathbb{C}^m)$ such that $t \mapsto e^{tB}$ is a fundamental solution. Nevertheless, the answer is still "yes" provided we look at it in the right setting.

To do so we have to choose an infinite dimensional setting.

Take a Banach space X and consider a function $u : [-\tau, \infty) \to X$. For each $t \geq 0$ we call the function

$$u_t : [-\tau, 0] \ni \sigma \mapsto u(t + \sigma) \in X \tag{6}$$

history segment with respect to $t \geq 0$.

The *history function* of u is then the function

$$h_u : t \mapsto u_t \tag{7}$$

on $\mathbb{R}_+$.

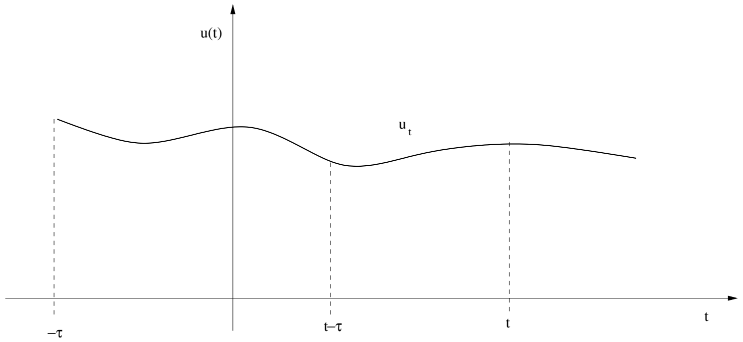

Figure 1: The history segment

A delay differential equation is an equation of the form

$$u'(t) = \frac{d}{dt}u(t) = \varphi(u(t), u_t), \tag{8}$$

where $\varphi(\cdot, \cdot)$ is an X-valued mapping. Here follows the explanation for this terminology.

In many concrete situations (see the examples later) the derivative $u'(t)$ actually depends on $u(t)$ and on $u(t-\tau)$ for some fixed $\tau > 0$ (often with τ normalized to $\tau = 1$) and one has to study differential equations of the form

$$u'(t) = \Psi(u(t), u_t(-\tau)) \tag{9}$$

for some function Ψ from $X \times X$ into X. Thus, interpreting t as time, values of u have an effect on u' with a certain delay τ. If we now define φ as

$$\varphi(u(t), u_t) := \Psi(u(t), u_t(-\tau)),$$

we arrive at Equation (8).

We will show in this section that this point of view allows us to read delay equations as vector-valued ordinary differential equations.

We discuss this, throughout this text, in the special situation where the mapping φ in (8) is the sum of a linear operator B acting on X and a linear operator Φ (the delay operator), acting on a space of functions (the history space) in which the history segments of u must lie. We are now going to make this precise by formulating our standing hypotheses, the notation, and the terminology which we will use throughout the lectures.

The standing hypotheses

(H_1) X is a Banach space,

(H_2) $B : D(B) \subseteq X \longrightarrow X$ is a closed, densely defined, linear operator,

(H_3) $1 \leq p < \infty$, $f \in L^p([-1,0],X)$ and $x \in X$,

(H_4) $\Phi : W^{1,p}([-1,0],X) \longrightarrow X$ is a bounded linear operator, called the *delay operator*.

(H_5) $\mathcal{E} := X \times L^p([-1,0],X)$.

Under these hypotheses, and for given elements $x \in X$ and $f \in L^p([-1,0],X)$, the following initial value problem will be called an *(abstract) delay equation* (with history parameter p)

$$(\mathrm{DE}_p) \qquad \begin{cases} u'(t) = Bu(t) + \Phi u_t, \quad t \geq 0, \\ u(0) = x, \\ u_0 = f. \end{cases}$$

Remark 1.1. Take $X := \mathbb{C}^n$, $B = 0$, and $\Phi := A\,\delta_{-\tau}$, where $\delta_{-\tau}$ denotes the point evaluation at $-\tau$. Then Equation (5) is of the form

$$u'(t) = Bu(t) + \Phi u_t\,.$$

Here is the natural notion of a classical solution to (DE_p).

Definition 1.2. We say that a function $u : [-1,\infty) \longrightarrow X$ is a *classical solution* of (DE_p) if

(i) $u \in C([-1,\infty),X) \cap C^1([0,\infty),X)$,

(ii) $u(t) \in D(B)$ and $u_t \in W^{1,p}([-1,0],X)$ for all $t \geq 0$,

(iii) u satisfies (DE_p) for all $t \geq 0$.

The following is a simple observation which uses a well known fact on translation semigroups on L^p-spaces. Through its (obvious) corollary we obtain the key to an interpretation of (DE_p) as an abstract Cauchy problem.

Lemma 1.3. *Let* $u : [-1,\infty) \to X$ *be a function which belongs to* $W^{1,p}_{loc}([-1,\infty), X)$. *Then the history function* $h_u : t \mapsto u_t$ *of* u *is continuously differentiable from* $\mathbb{R}_+$ *into* $L^p([-1,0], X)$ *with derivative*

$$\frac{d}{dt} h_u(t) = \frac{d}{d\sigma} u_t.$$

Proof. Let $(A, D(A))$ be the generator of the left shift group $(L(t))_{t\geq 0}$ on the space $L^p(\mathbb{R}, X)$, i.e., $D(A) = W^{1,p}(\mathbb{R}, X)$ and $A = \frac{d}{d\sigma}$ (see [30, Section II.2.10]).

Let $t \in \mathbb{R}$ and $T > 1$. We can extend $u_{|[t-T,t+T]}$ to a function $v \in W^{1,p}(\mathbb{R}, X) = D(A)$, so $\frac{d}{dt} L(t)v = AL(t)v$. Note that $(L(s)v)(\sigma) = u(s+\sigma) = u_s(\sigma) = h_u(s)(\sigma)$ for $\sigma \in [-1,0]$ and $|s-t| < T-1$. So we have

$$\begin{aligned} 0 &= \lim_{h\to 0} \left\| \frac{L(t+h)v - L(t)v}{h} - AL(t)v \right\|^p_{L^p(\mathbb{R},X)} \\ &\geq \lim_{h\to 0} \left\| \frac{h_u(t+h) - h_u(t)}{h} - \frac{d}{d\sigma} u_t \right\|^p_{L^p([-1,0],X)}, \end{aligned}$$

which implies

$$\frac{d}{dt} h_u(t) = \frac{d}{d\sigma} u_t.$$

Moreover, the map $t \mapsto \frac{d}{d\sigma} u_t = \frac{d}{dt} h_u(t)$ is continuous from $\mathbb{R}_+$ into $L^p([-1,0], X)$ since the map $t \mapsto AL(t)v = L(t)Av$ is continuous from $\mathbb{R}$ into $L^p(\mathbb{R}, X)$. $\square$

Corollary 1.4. *Let* $u : [-1,\infty) \longrightarrow X$ *be a classical solution of (DE*$_p$*). Then the function*

$$\mathfrak{U} : t \mapsto \begin{pmatrix} u(t) \\ u_t \end{pmatrix} \in \mathcal{E}_p \tag{10}$$

from $\mathbb{R}_+$ *into* $X \times L^p([-1,0], X)$ *is continuously differentiable with derivative*

$$\mathfrak{U}'(t) = \mathcal{A}\mathfrak{U}(t),$$

$$\mathcal{A} := \begin{pmatrix} B & \Phi \\ 0 & \frac{d}{d\sigma} \end{pmatrix},$$

where $\frac{d}{d\sigma}$ *denotes the distributional derivative, with domain*

$$D(\mathcal{A}) := \left\{ \begin{pmatrix} x \\ f \end{pmatrix} \in D(B) \times W^{1,p}([-1,0], X) \ : \ f(0) = x \right\}.$$

Thus every classical solution u *of (DE*$_p$*) yields a classical solution of the abstract Cauchy problem*

$$(ACP_p) \qquad \begin{cases} \mathfrak{U}'(t) = \mathcal{A}\,\mathfrak{U}(t), & t \geq 0, \\ \mathfrak{U}(0) = \begin{pmatrix} x \\ f \end{pmatrix}, \end{cases}$$

on $X \times L^p([-1,0], X)$.

We now show that (DE_p) and (ACP_p) are actually equivalent in the sense that the converse of Corollary 1.4 holds, i.e., every classical solution $t \mapsto \mathcal{U}(t)$ of (ACP_p) is of the form

$$\mathcal{U}(t) = \left(\begin{smallmatrix} u(t) \\ u_t \end{smallmatrix}\right)$$

where the function u is a classical solution of (DE_p). To that purpose, we first fix the Banach space setting for (ACP_p) by adding the following to our standing hypotheses.

(H_6) $(\mathcal{A}, D(\mathcal{A}))$ is the operator on $\mathcal{E}_p$ defined as

$$\mathcal{A} := \begin{pmatrix} B & \Phi \\ 0 & \frac{d}{d\sigma} \end{pmatrix}, \tag{11}$$

with domain

$$D(\mathcal{A}) := \left\{ \left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in D(B) \times W^{1,p}([-1,0], X) \ : \ f(0) = x \right\}. \tag{12}$$

Lemma 1.5. *Under the Hypotheses (H_1)-(H_6), the operator $(\mathcal{A}, D(\mathcal{A}))$ is closed and densely defined on $\mathcal{E}_p$.*

The proof of Lemma 1.5 is left as an exercise.

In view of Lemma 1.5, we can formulate the following corollary which is a straightforward consequence of [30, Theorem II.6.7].

Corollary 1.6. *The abstract Cauchy problem (ACP_p) associated to the operator $(\mathcal{A}, D(\mathcal{A}))$ on the space $\mathcal{E}_p$ is wellposed (in the sense of [30, Definition II.6.8]) if and only if $(\mathcal{A}, D(\mathcal{A}))$ is the generator of a strongly continuous semigroup $(\mathcal{T}(t))_{t \geq 0}$ on $\mathcal{E}_p$.*

In this case, the classical and mild solutions of (ACP_p) are given by the functions $\mathcal{U}(t) := \mathcal{T}(t)\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right)$ for $t \geq 0$.

We now introduce the following notation.

Definition 1.7. By $\pi_1 : \mathcal{E}_p \longrightarrow X$ we denote the canonical projection from $\mathcal{E}_p$ onto X.

Similarly, by $\pi_2 : \mathcal{E}_p \longrightarrow L^p([-1,0], X)$ we denote the canonical projection from $\mathcal{E}_p$ onto $L^p([-1,0], X)$.

Proposition 1.8. *For every classical solution $\mathcal{U}$ of (ACP_p) the function*

$$u(t) := \begin{cases} (\pi_1 \circ \mathcal{U})(t) & \text{if } \ t \geq 0, \\ f(t) & \text{if } \ t \in [-1, 0), \end{cases} \tag{13}$$

is a classical solution of (DE_p) and $(\pi_2 \circ \mathcal{U})(t) = u_t$ for all $t \geq 0$.

Proof. Since $\mathcal{U}$ is a classical solution of (ACP_p), it follows that $\pi_2 \circ \mathcal{U}$ is in $C^1\left(\mathbb{R}_+, L^p([-1,0],X)\right)$ and is a classical solution of the problem

$$\begin{cases} \frac{d}{dt}(\pi_2 \circ \mathcal{U})(t) = \frac{d}{d\sigma}(\pi_2 \circ \mathcal{U})(t), & t \geq 0, \\ (\pi_2 \circ \mathcal{U})(t)(0) = z(t), & t \geq 0, \\ (\pi_2 \circ \mathcal{U})(0) = f \end{cases} \tag{14}$$

in the space $L^p([-1,0],X)$.

We now observe that by definition

$$u_t(\sigma) = u(t+\sigma) = \begin{cases} (\pi_1 \circ \mathcal{U})(t+\sigma) & \text{for } t+\sigma \geq 0, \\ f(t+\sigma) & \text{for } t+\sigma < 0, \end{cases}$$

where $z \in C^1([0,\infty),X)$, $f \in W^{1,p}([-1,0],X)$, and $f(0) = x = z(0)$ by assumption. Hence, $u \in W^{1,p}_{loc}([-1,\infty),X)$. We can extend u to a function in $W^{1,p}_{loc}(\mathbb{R},X)$ and, by Lemma 1.3, we have

$$\frac{d}{dt}h_u(t) = \frac{d}{d\sigma}u_t \qquad \text{for all } t \geq 0$$

in the space $L^p([-1,0],X)$. Moreover, by definition of u_t we have

$$u_t(0) = u(t) = z(t) \qquad \text{for all } t \geq 0,$$

and

$$u_0 = f.$$

Hence, the map $t \mapsto u_t$ is also a classical solution of the problem (14) in the space $L^p([-1,0],X)$.

Now define $w(t) := u_t - (\pi_2 \circ \mathcal{U})(t)$ for $t \geq 0$. Then w is a classical solution of the problem

$$\begin{cases} \frac{d}{dt}w(t) = \frac{d}{d\sigma}w(t) & \text{for } t \geq 0, \\ w(t)(0) = 0 & \text{for } t \geq 0, \\ w(0) = 0, \end{cases} \tag{15}$$

in the space $L^p([-1,0],X)$. Since (15) is the abstract Cauchy problem associated to the generator of the (nilpotent) left shift semigroup on $L^p([-1,0],X)$ with initial value 0, we have that $w(t) = 0$ for all $t \geq 0$. Therefore, $u_t = (\pi_2 \circ \mathcal{U})(t) \in W^{1,p}([-1,0],X)$ and $\mathcal{U}(t) = \left(\begin{smallmatrix} u(t) \\ u_t \end{smallmatrix}\right)$ for all $t \geq 0$, and u is a classical solution of (DE$_p$). □

The equivalence of (DE$_p$) and (ACP$_p$) established above puts us into a position to use methods and results of semigroup theory in order to deal with (DE$_p$). This is our main idea.

At present, we transfer the notions of wellposedness and of mild solutions, known from abstract Cauchy problems and semigroups, to (DE$_p$).

Definition 1.9. (a) (DE_p) is called *wellposed* if (ACP_p) is wellposed, i.e. if $(\mathcal{A}, D(\mathcal{A}))$ generates a strongly continuous semigroup on $\mathcal{E}_p$.

(b) Suppose (DE_p) is wellposed and let $(\mathcal{T}(t))_{t\geq 0}$ be the semigroup generated by $(\mathcal{A}, D(\mathcal{A}))$ on $\mathcal{E}_p$. Then for every $x \in X$ and every $f \in L^p([-1,0],X)$ the function u defined by (13) is called a *mild solution* of (DE_p).

Proposition 1.10. *Let u be a mild solution of (DE_p). Then u satisfies $\int_0^t u(s)\,ds \in D(B)$, $\int_0^t u_s\,ds \in W^{1,p}([-1,0],X)$ and the integral equation*

$$u(t) = \begin{cases} x + B\int_0^t u(s)\,ds + \Phi\int_0^t u_s\,ds & \text{for } \ t \geq 0, \\ f(t) & \text{for a.e. } \ t \in [-1,0). \end{cases} \tag{16}$$

Proof. 1. Step: We first show that

$$u_t = \pi_2\big(\mathcal{T}(t)\big(\begin{smallmatrix} x \\ f \end{smallmatrix}\big)\big) \tag{17}$$

for every $\big(\begin{smallmatrix} x \\ f \end{smallmatrix}\big) \in \mathcal{E}_p$ and every $t \geq 0$.
For $\big(\begin{smallmatrix} x \\ f \end{smallmatrix}\big) \in D(\mathcal{A})$ the identity (17) holds by Proposition 1.8. Take now $\big(\begin{smallmatrix} x \\ f \end{smallmatrix}\big) \in \mathcal{E}_p$ and a sequence $\big(\begin{smallmatrix} x_n \\ f_n \end{smallmatrix}\big)$ in $D(\mathcal{A})$ converging to $\big(\begin{smallmatrix} x \\ f \end{smallmatrix}\big)$. Since the semigroup $(\mathcal{T}(t))_{t\geq 0}$ is strongly continuous, the sequence $\mathcal{T}(t)\big(\begin{smallmatrix} x_n \\ f_n \end{smallmatrix}\big)$ converges to $\mathcal{T}(t)\big(\begin{smallmatrix} x \\ f \end{smallmatrix}\big)$ in $\mathcal{E}_p$ uniformly for t in compact subsets of $[0,\infty)$.

Now, let

$$u_n(t) := \begin{cases} \pi_1\big(\mathcal{T}(t)\big(\begin{smallmatrix} x_n \\ f_n \end{smallmatrix}\big)\big) & \text{if } t \geq 0, \\ f_n(t) & \text{if } t \in [-1,0). \end{cases}$$

Since $\big(\begin{smallmatrix} x_n \\ f_n \end{smallmatrix}\big) \in D(\mathcal{A})$, by Proposition 1.8 we have $(u_n)_t = \pi_2\big(\mathcal{T}(t)\big(\begin{smallmatrix} x_n \\ f_n \end{smallmatrix}\big)\big)$.

For $t \geq 1$, we have that

$$(u_n)_t(\sigma) = u_n(t+\sigma) = \pi_1\big(\mathcal{T}(t+\sigma)\big(\begin{smallmatrix} x_n \\ f_n \end{smallmatrix}\big)\big) \tag{18}$$

converges to $\pi_1\big(\mathcal{T}(t+\sigma)\big(\begin{smallmatrix} x \\ f \end{smallmatrix}\big)\big) = u_t(\sigma)$ uniformly for $\sigma \in [-1,0]$. Hence, $(u_n)_t$ converges to u_t in $L^p([-1,0],X)$ and we have

$$u_t = \lim_{n\to\infty} (u_n)_t = \lim_{n\to\infty} \pi_2\big(\mathcal{T}(t)\big(\begin{smallmatrix} x_n \\ f_n \end{smallmatrix}\big)\big) = \pi_2\big(\mathcal{T}(t)\big(\begin{smallmatrix} x \\ f \end{smallmatrix}\big)\big).$$

In particular, $u_t \in L^p([-1,0],X)$.

Let now $0 \leq t < 1$. By Corollary 1.6 and (13) we have

$$(u_n)_t(\sigma) = \begin{cases} \pi_1(\mathcal{T}(t+\sigma)\big(\begin{smallmatrix} x_n \\ f_n \end{smallmatrix}\big) & \text{for } \sigma \in [-t,0], \\ f_n(t+\sigma) & \text{for } \sigma \in [-1,-t). \end{cases}$$

This formula in conjunction with the calculation given in (18), this implies that $(u_n)_t(\sigma)$ converges to $\pi_1\big(\mathcal{T}(t+\sigma)\big(\begin{smallmatrix} x_n \\ f_n \end{smallmatrix}\big)\big)$ uniformly for $\sigma \in [-t,0]$. Moreover, by

assumption, $(u_n)_t$ converges to $f(t+\cdot)$ in $L^p([-1,-t),Z)$. Hence, $(u_n)_t$ converges to u_t in $L^p([-1,0],X)$ and, by the same argument as above, $u_t = \pi_2\big(\mathcal{T}(t)\binom{x}{f}\big)$.
2. Step: Take the first component of the identity

$$\mathcal{T}(t)\tbinom{x}{f} - \tbinom{x}{f} = \mathcal{A}\int_0^t \mathcal{T}(s)\tbinom{x}{f}\,ds, \qquad t \geq 0,$$

to obtain (16). □

Theorem 1.11. *The following assertions are equivalent.*

(a) (DE$_p$) is wellposed.

(b) For every $\binom{x}{f} \in D(\mathcal{A})$,

  (i) there is a unique (classical) solution $u(x,f,\cdot)$ of (DE$_p$) and

  (ii) the solutions depend continuously on the initial values, that is, if a sequence $\binom{x_n}{f_n}$ in $D(\mathcal{A})$ converges to $\binom{x}{f} \in D(\mathcal{A})$ in the space $\mathcal{E}_p = X \times L^p([-1,0],X)$, then $u(x_n,f_n,t)$ converges to $u(x,f,t)$ in X uniformly for t in compact intervals.

Proof. We first show $(b) \Rightarrow (a)$. Assume that for every $\binom{x}{f} \in D(\mathcal{A})$, equation (DE$_p$) has a unique solution u. Then Proposition 1.4 guarantees that for every $\binom{x}{f} \in D(\mathcal{A})$ the abstract Cauchy problem (ACP$_p$) has a classical solution which is unique by Proposition 1.8. It is easy to see that these solutions depend continuously on the initial values. Finally, by Lemma 1.5, $(\mathcal{A}, D(\mathcal{A}))$ is closed and densely defined. Therefore, it generates a strongly continuous semigroup on $\mathcal{E}_p$ by [30, Theorem II.6.7].

Conversely, if $\mathcal{A}$ is a generator, we have by Corollary 1.6 and Proposition 1.8 that for every initial value $\binom{x}{f} \in D(\mathcal{A})$ there is a unique solution u of (DE$_p$) which is given by (13). This implies that the solutions depend continuously on the initial values. □

We finish this lecture by introducing the following terminology.

Definition 1.12. In the case that (DE$_p$) is well-posed, we call the semigroup $(\mathcal{T}(t))_{t\geq 0}$ generated by $(\mathcal{A}, D(\mathcal{A}))$ on $\mathcal{E}_p$ the *delay semigroup* corresponding to (DE$_p$).

Example 1.13. Consider a diffusion process on an open, bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary. Assume that there is no dispersion at the boundary, i.e., take Neumann boundary conditions. Assume moreover that there is a delayed feedback depending on the flux. Then the equation modelling this system becomes a partial differential equation with delay given by

$$\begin{cases} \partial_t u(t,s) = \Delta u(t,s) + \sum_{i=1}^n c_i\, u(t-h_i,s), & t \geq 0,\ s \in \Omega, \\ \frac{\partial u}{\partial \nu}(t,s) = 0, & t \geq 0,\ s \in \partial\Omega, \\ u(t,s) = f(t,s), & t \in [-1,0],\ s \in \Omega, \end{cases} \tag{19}$$

for some constants $c_i \in \mathbb{R}$ and $h_i \in [0,1]$. Moreover, we assume that there exists $1 \le p < \infty$ such that $f(t,\cdot) \in L^2(\Omega)$ for a.e. $t \in [-1,0]$ and the map $t \mapsto f(t,\cdot)$ belongs to $L^p([-1,0], L^2(\Omega))$. In order to write Equation (19) as an abstract delay equation, we introduce

- the Hilbert space $X := L^2(\Omega)$ with its usual scalar product,
- the operator $B := \Delta$ be the distributional Laplacian with Neumann boundary conditions,
- the *delay operator* $\Phi : W^{1,p}([-1,0], X) \to X$ defined as

  $$\Phi f := \sum_{i=1}^{n} c_i \partial_i f(-h_i),$$

  and
- $x := f(\cdot, 0)$.

Then Hypotheses (H_1)-(H_4) are satisfied. Therefore, Equation (19) is well-posed if and only if the associated operator matrix $(\mathcal{A}, D(\mathcal{A}))$ defined in (H_5) generates a semigroup on the space $L^2(\Omega) \times L^p([-1,0], L^2(\Omega))$.

Exercises 1.14. (1) Write the following reaction-diffusion equation with delay (see Wu [81, Section 2.1]) as an abstract delay differential equation of the form (DE_p) and show that Hypotheses (H_1)-(H_4) are satisfied.

$$\begin{cases} \partial_t w(t,s) = \Delta w(t,s) + c \int_{-1}^{0} w(t+\tau, s)\, dg(\tau), \\ \quad s \in \Omega,\ t \ge 0, \\ w(t,s) = 0, \quad s \in \partial\Omega,\ t \ge 0, \\ w(t,s) = f(t,s), \quad (s,t) \in \Omega \times [-1,0], \end{cases} \tag{20}$$

where c is a constant, $\Omega \subset \mathbb{R}^n$ an open set, and $g : [-1,0] \to [0,1]$ is a function of bounded variation. The integral on the right hand side is understood as a Riemann-Stieltjes integral. Moreover, we assume that f belongs to $L^2([-1,0] \times \Omega)$.

(2) Write the following Lotka-Sharpe equation with delay as an abstract delay differential equation of the form (DE_p) and show that Hypotheses (H_1)-(H_4) are satisfied.

$$\begin{cases} \partial_t u(t,a) + \partial_a u(t,a) = -\mu(a)\, u(t,a) + \nu(a)\, u(t-1,a), \\ \quad t \ge 0,\ a \in \mathbb{R}_+, \\ u(t,0) = \int_0^{+\infty} \beta(a)\, u(t,a)\, da, \qquad t \ge 0, \\ u(t,a) = f(t,a), \qquad (t,a) \in [-1,0] \times \mathbb{R}_+, \end{cases} \tag{21}$$

where $\mu, \nu, \beta \in L^\infty(\mathbb{R}_+)$, μ, β are positive and f is in $L^1([-1,0] \times \mathbb{R}_+)$.

(3) Prove Lemma 1.5.

1.2 Well-posedness

In Lecture 1.1 we transformed the problem of solving the delay equation (DE_p) into the following problem:

When does $(\mathcal{A}, \mathbf{D}(\mathcal{A}))$ generate a strongly continuous semigroup on $\mathcal{E}_\mathbf{p}$?

We proceed to give sufficient conditions on $(B, D(B))$ and Φ such that this is true.

Let $1 \leq p < \infty$. We write

$$\mathcal{A} := \begin{pmatrix} B & \Phi \\ 0 & \frac{d}{d\sigma} \end{pmatrix}$$

with domain

$$D(\mathcal{A}) := \left\{ \left(\begin{smallmatrix} x \\ f \end{smallmatrix} \right) \in D(B) \times W^{1,p}([-1,0], X) \ : \ f(0) = x \right\}$$

as the sum $\mathcal{A}_0 + \mathcal{B}$, where

$$\mathcal{A}_0 := \begin{pmatrix} B & 0 \\ 0 & \frac{d}{d\sigma} \end{pmatrix} \tag{22}$$

with domain

$$D(\mathcal{A}_0) := D(\mathcal{A}) \tag{23}$$

and

$$\mathcal{B} := \begin{pmatrix} 0 & \Phi \\ 0 & 0 \end{pmatrix} \in \mathcal{L}(D(\mathcal{A}_0), \mathcal{E}_p). \tag{24}$$

The idea is to show first that $\mathcal{A}_0$ becomes a generator under appropriate assumptions, and then to apply perturbation results to show that the sum $\mathcal{A}_0 + \mathcal{B}$ is a generator as well. The first step is quite straightforward.

Theorem 1.15. *The following are equivalent.*

(a) The operator $(B, D(B))$ generates a strongly continuous semigroup $(S(t))_{t\geq 0}$ on X.

(b) The operator matrix $(\mathcal{A}_0, D(\mathcal{A}_0))$ generates a strongly continuous semigroup $(\mathcal{T}_0(t))_{t\geq 0}$ on the space $\mathcal{E}_p$ for all $1 \leq p < \infty$.

(c) The operator matrix $(\mathcal{A}_0, D(\mathcal{A}_0))$ generates a strongly continuous semigroup $(\mathcal{T}_0(t))_{t\geq 0}$ on the space $\mathcal{E}_p$ for one $1 \leq p < \infty$.

In this case, the semigroup $(\mathcal{T}_0(t))_{t\geq 0}$ is given by

$$\mathcal{T}_0(t) := \begin{pmatrix} S(t) & 0 \\ S_t & T_0(t) \end{pmatrix}, \tag{25}$$

where $(T_0(t))_{t\geq 0}$ *is the nilpotent left shift semigroup on* $L^p([-1,0],X)$ *and* $S_t : X \to L^p([-1,0],X)$ *is defined by*

$$(S_t\,x)(\tau) := \begin{cases} S(t+\tau)x & \text{if } -t < \tau \leq 0, \\ 0 & \text{if } -1 \leq \tau \leq -t. \end{cases}$$

Proof. For the proof of (a)⇒(b) do Exercise 1.22 (1).

The implication (b)⇒(c) is trivial.

We show (c)⇒(a). Assume that $(\mathcal{A}_0, D(\mathcal{A}_0))$ generates a semigroup on $\mathcal{E}_p$ for one $1 \leq p < \infty$. Then by Theorem 1.11, the equation (i.e. the delay equation with $\Phi = 0$)

$$\begin{cases} u'(t) = B\,u(t) \text{ for } t \geq 0, \\ u(0) = x, \\ u_0 = f \end{cases} \tag{26}$$

is well posed, that is, for all $\binom{x}{f} \in D(\mathcal{A}_0) \subseteq D(B) \times W^{1,p}([-1,0],X)$ there exists a unique classical solution $u \in C([-1,\infty),X) \cap C^1([0,\infty),X)$ of (26) and the solutions depend continuously on the initial data.

The restriction $u|_{[0,\infty)}$ is a classical solution of the abstract Cauchy problem associated to the operator $(B, D(B))$

$$\begin{cases} u'(t) = Bu(t) \text{ for } t \geq 0, \\ u(0) = x. \end{cases} \tag{27}$$

In particular, the abstract Cauchy problem (27) is well-posed. Furthermore, since B is closed and densely defined by assumption, we know that $(B, D(B))$ generates a strongly continuous semigroup on X (see [30, Theorem II.6.7]). □

We are now ready to prove a sufficient condition for wellposedness of (DE$_p$). This is an application of the Perturbation Theorem of Miyadera-Voigt (see [30, Corollary III.3.16]), wich we recall here because of its importance.

Theorem 1.16. *Let* $(G, D(G))$ *be the generator of a strongly continuous semigroup* $\mathcal{T} := (T(t))_{t\geq 0}$ *on a Banach space* X *and let* $C \in \mathcal{L}((D(G), \|\cdot\|_G), X)$*. Assume that there exist constants* $t_0 > 0$*,* $0 \leq q < 1$ *such that*

$$\int_0^{t_0} \|CT(s)x\|\,ds \leq q\|x\| \qquad \text{for all } x \in D(G) \tag{28}$$

Then $(G + C, D(G))$ *generates a strongly continuous semigroup* $(U(t))_{t\geq 0}$ *on* X *which satisfies*

$$U(t)x = T(t)x + \int_0^t T(t-s)CU(s)x\,ds$$

and

$$\int_0^{t_0} \|CU(s)x\|\, ds \le \frac{q}{1-q}\|x\| \tag{29}$$

for all $x \in D(G)$ and $t \ge 0$.

We note that the perturbed semigroup $(U(t))_{t\ge 0}$ is given by the Dyson-Phillips series

$$U(t)x = \sum_0^\infty (V^n T)(t)x \quad \text{for all} \quad x \in X,\, t \ge 0, \tag{30}$$

converging uniformly on compact subsets of $\mathbb{R}_+$. Here, $V \in \mathcal{L}(C(\mathbb{R}_+, \mathcal{L}_s(X)))$ is the abstract Volterra operator on the space of all strongly continuous functions from $\mathbb{R}_+$ to $\mathcal{L}(X)$ and is defined by

$$(VF)(t)x := \int_0^t F(t-s)CT(s)x ds \quad \text{for} \quad x \in D(G),\, t \ge 0, \tag{31}$$

and by

$$(VF)(t)x := \lim_{n\to\infty} (VF)(t)x_n \quad \text{for} \quad x \in X,\, t \ge 0, \tag{32}$$

for $F \in C(\mathbb{R}_+, \mathcal{L}_s(X))$, where $(x_n) \subset D(G)$ is a sequence such that $\lim_{n\to\infty} x_n = x$.

Let us turn our attention to the delay semigroup again.

Theorem 1.17. *Let $(B, D(B))$ be the generator of a strongly continuous semigroup $(S(t))_{t\ge 0}$on X, let $1 \le p < \infty$ and let $\Phi : W^{1,p}([-1,0], X) \to X$ be a delay operator. Assume that there exist constants $t_0 > 0$ and $0 < q < 1$ such that*

$$\int_0^{t_0} \|\Phi(S_r x + T_0(r)f)\|\, dr \le q \left\|\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right)\right\| \tag{M}$$

for all $\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in D(\mathcal{A})$. Then the operator $(\mathcal{A}, D(\mathcal{A}))$ is the generator of a strongly continuous semigroup on $\mathcal{E}_p$ and so (DE$_p$) is well-posed.

Proof. By Theorem 1.15, we know that $(\mathcal{A}_0, D(\mathcal{A}_0))$ generates the strongly continuous semigroup $(\mathcal{T}_0(t))_{t\ge 0}$ given in (25). We now apply the perturbation theorem of Miyadera-Voigt (Theorem 1.16) to the operator

$$\mathcal{B} := \begin{pmatrix} 0 & \Phi \\ 0 & 0 \end{pmatrix}.$$

We need to check that the condition of the perturbation theorem of Miyadera-Voigt (see Theorem 1.16) is satisfied, that is we show that there exist $t_0 > 0$ and $0 \le q < 1$ such that

$$\int_0^{t_0} \left\|\mathcal{B}\, \mathcal{T}_0(r)\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right)\right\|\, dr \le q \left\|\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right)\right\| \tag{33}$$

for all $\binom{x}{f} \in D(\mathcal{A}_0)$. Indeed, since

$$\begin{aligned}\int_0^{t_0} \left\| \mathcal{B}\, \mathcal{T}_0(r) \tbinom{x}{f} \right\| \, dr &= \int_0^{t_0} \left\| \begin{pmatrix} 0 & \Phi \\ 0 & 0 \end{pmatrix} \begin{pmatrix} S(r) & 0 \\ S_r & T_0(r) \end{pmatrix} \begin{pmatrix} x \\ f \end{pmatrix} \right\| \, dr \\ &= \int_0^{t_0} \| \Phi(S_r x + T_0(r) f) \| \, dr \,,\end{aligned}$$

we conclude that (33) holds if and only if there exist $t_0 > 0$ and $0 \leq q < 1$ such that (M) holds for all $\binom{x}{f} \in D(\mathcal{A}_0)$. ☐

Example 1.18. Let Φ be bounded from $L^p([-1,0], X)$ to X. A typical example is

$$\Phi\, f := \int_{-1}^{0} h(\sigma) f(\sigma) \, d\sigma,$$

where the function h is an $\mathcal{L}(X)$-valued q-integrable function for $\frac{1}{p} + \frac{1}{q} = 1$, i.e., $h \in L^q([-1,0], \mathcal{L}(X))$. Then the perturbation $\mathcal{B}$ is bounded, and $(\mathcal{A}, D(\mathcal{A}))$ is a generator on $\mathcal{E}_p$.

1.3 The main example

We assume as usual that X is a Banach space and that $(B, D(B))$ generates a strongly continuous semigroup $(S(t))_{t \geq 0}$on X.

This section is devoted to a large class of delay operators Φ such that the matrix $(\mathcal{A}, D(\mathcal{A}))$ becomes a generator for every semigroup generator $(B, D(B))$.

In the case that $B \in \mathcal{L}(X)$, then $(\mathcal{A}, D(\mathcal{A}))$ is a generator for all $\Phi \in \mathcal{L}(W^{1,p}([-1,0], X), X)$ (see Exercise 1.22 (2)).

Whether the analogous statement is true in the case that B is unbounded is still an open question.

However, we can presently show that $(\mathcal{A}, D(\mathcal{A}))$ is a generator for a very large class of delay operators.

Let $\eta : [-1,0] \to \mathcal{L}(X)$ be of bounded variation and let $\Phi : C([-1,0], X) \to X$ be the bounded linear operator given by the Riemann-Stieltjes integral

$$\Phi(f) := \int_{-1}^{0} d\eta\, f \,. \tag{34}$$

Since $W^{1,p}([-1,0], X)$ is continuously embedded in $C([-1,0], X)$, we may note that Φ defines a bounded operator from $W^{1,p}([-1,0], X)$ to X.

Example 1.19. An important special case of (34) are the so called "discrete" delay operators, that is operators Φ of the form

$$\Phi(f) := \sum_{k=0}^{n} B_k f(-h_k), \quad f \in W^{1,p}([-1,0], X), \tag{35}$$

where $B_k \in \mathcal{L}(X)$ and $h_k \in [0,1]$ for each $k = 0, \dots, n$. In fact, we have

$$\Phi f = \int_{-1}^{0} d\eta(\sigma)\, f(\sigma)$$

for $\eta := \sum_{k=0}^{n} B_k \mathbb{1}_{[-h_k,0]}$.

We now show that all delay operators as in (34) satisfy (M) for each $1 \leq p < \infty$.

Theorem 1.20. *Let $1 \leq p < \infty$ and let Φ be as in (34). Then the operator matrix $(\mathcal{A}, D(\mathcal{A}))$ defined by*

$$\mathcal{A} := \begin{pmatrix} B & \Phi \\ 0 & \frac{d}{d\sigma} \end{pmatrix},$$

$$D(\mathcal{A}) := \left\{ \left(\begin{smallmatrix} x \\ f \end{smallmatrix} \right) \in D(B) \times W^{1,p}([-1,0], X) \ : \ f(0) = x \right\},$$

generates a strongly continuous semigroup, hence the corresponding delay equation (DE$_p$) is well-posed.

Proof. We first consider the case $1 < p < \infty$. Then for $0 < t < 1$ we obtain that

$$\begin{aligned}
&\int_0^t \|\Phi(S_r\, x + T_0(r) f)\|\, dr \\
&\quad = \int_0^t \left\| \int_{-1}^{-r} d\eta(\sigma)\, f(r+\sigma) + \int_{-r}^{0} d\eta(\sigma)\, S(r+\sigma)x \right\| dr \\
&\quad \leq \int_0^t \int_{-1}^{-r} \|f(r+\sigma)\|\, d|\eta|(\sigma)\, dr + \int_0^t \int_{-r}^{0} \|S(r+\sigma)x\|\, d|\eta|(\sigma)\, dr \\
&\quad \leq \int_{-t}^{0} \int_{\sigma}^{0} \|f(s)\|\, ds\, d|\eta|(\sigma) + \int_{-1}^{-t} \int_{\sigma}^{t+\sigma} \|f(s)\|\, ds\, d|\eta|(\sigma) \\
&\qquad + \int_0^t M\|x\| |\eta|([-1,0])\, dr \\
&\quad \leq \int_{-t}^{0} (-\sigma)^{1/p'} \|f\|_p\, d|\eta|(\sigma) + \int_{-1}^{-t} t^{1/p'} \|f\|_p\, d|\eta|(\sigma) \\
&\qquad + tM\|x\| |\eta|([-1,0]) \\
&\quad \leq \int_{-1}^{0} t^{1/p'} \|f\|_p\, d|\eta|(\sigma) + tM\|x\| |\eta|([-1,0]) \\
&\quad = (t^{1/p'} \|f\|_p + tM\|x\|)\, |\eta|([-1,0]),
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$, $M := \sup_{r \in [0,1]} \|S(r)\|$ and $|\eta|$ is the positive Borel measure on $[-1,0]$ defined by the total variation of η.

Finally, we conclude that

$$\int_0^t \|\Phi(S_r x + T_0(r)f)\|\, dr \le t^{1/p'} M\, |\eta|([-1,0])(\|f\|_p + \|x\|) \tag{36}$$

for all $0 < t < 1$. Choose now t_0 small enough such that

$$t_0^{1/p'} M\, |\eta|([-1,0]) < 1.$$

Then condition (M) is satisfied with $q := t_0^{1/p'} M\, |\eta|([-1,0])$.

We now consider the case $p = 1$. The proof of this case is due to L. Maniar and J. Voigt [56].

Without loss of generality, we may assume that η has no mass at zero, that is we assume

$$\eta(0) = \lim_{\tau \to 0-} \eta(\tau) = 0. \tag{37}$$

If this is not the case, then define $\eta_0 := \eta(0) - \lim_{\tau\to 0-} \eta(\tau) \in \mathcal{L}(X)$. Then it follows that

$$\mathcal{A} = \begin{pmatrix} B & \Phi \\ 0 & \frac{d}{d\sigma} \end{pmatrix} = \begin{pmatrix} B + \eta_0 & \Phi - \eta_0\delta_0 \\ 0 & \frac{d}{d\sigma} \end{pmatrix} = \begin{pmatrix} B & \Phi - \eta_0\delta_0 \\ 0 & \frac{d}{d\sigma} \end{pmatrix} + \begin{pmatrix} \eta_0 & 0 \\ 0 & 0 \end{pmatrix},$$

where $\begin{pmatrix} \eta_0 & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{L}(\mathcal{E}_1)$. Thus by replacing B by $B + \eta_0$ and Φ by $\Phi - \eta_0\delta_0$ we obtain (37).

Moreover, we may assume that η is continuous from the left. Indeed, the set of points where a function of bounded variation is not continuous is always countable and at the points of discountinuity the left and right hand limits exist. Hence, η can be ‘made’ left continuous by changing its value on a countable set. This procedure does not affect the definition of Φ (see e.g. Diekmann et al. [29, Section I.1]).

The main idea of the proof is to add $\mathcal{B}$ to $\mathcal{A}_0$ in two steps. For $0 < \alpha \le 1$ we define

$$\eta_\alpha(\tau) := \begin{cases} \eta(\tau) & \text{for } -1 \le \tau < -\alpha, \\ \eta(-\alpha) & \text{for } -\alpha \le \tau \le 0, \end{cases}$$

and $\tilde{\eta}_\alpha := \eta - \eta_\alpha$. Then

$$|\eta_\alpha|([-1,0]) = |\eta|([-1,-\alpha]) \le |\eta|([-1,0])$$

for $\alpha \in (0,1]$. Furthermore, we see from (37) that $|\tilde{\eta}_\alpha|([-1,0]) = |\eta|([-\alpha,0]) \to 0$ as $\alpha \to 0$.

We now define the operators Φ_α and $\tilde{\Phi}_\alpha$ analogously to (34) and the operators $\mathcal{B}_\alpha := \begin{pmatrix} 0 & \Phi_\alpha \\ 0 & 0 \end{pmatrix}$, $\tilde{\mathcal{B}}_\alpha := \begin{pmatrix} 0 & \tilde{\Phi}_\alpha \\ 0 & 0 \end{pmatrix}$ analogously to (24). Note that

$$\Phi_\alpha f = \int_{-1}^0 d\eta_\alpha\, f = \int_{-1}^{-\alpha} d\eta\, f$$

and

$$\tilde{\Phi}_\alpha f = \int_{-1}^{0} d\tilde{\eta}_\alpha \, f = \int_{-\alpha}^{0} d\eta \, f.$$

Let us first show that $\mathcal{A}_\alpha := \mathcal{A}_0 + \mathcal{B}_\alpha$ generates a strongly continuous semigroup. Taking $\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in D(\mathcal{A}_0)$, we obtain that

$$\begin{aligned}
\int_0^\alpha \left\| \mathcal{B}_\alpha \mathcal{T}_0(r) \left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \right\| dr &= \int_0^\alpha \| \Phi_\alpha (S_r \, x + T_0(r) f) \| \, dr \\
&= \int_0^\alpha \| \Phi_\alpha T_0(r) f \| \, dr \\
&= \int_0^\alpha \left\| \int_{-1}^{-\alpha} d\eta(\tau) f(r+\tau) \right\| dr \\
&\leq \int_{-1}^{-\alpha} \int_0^\alpha \| f(r+\tau) \| \, dr d|\eta|(\tau) \\
&\leq |\eta|([-1,-\alpha]) \| f \|_1 \\
&\leq |\eta|([-1,-\alpha]) \left\| \left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \right\| .
\end{aligned}$$

Choose now $n \in \mathbb{N}$ such that $|\eta|([-1,-\alpha]) < n$. Then the above estimate just shows that $\frac{1}{n}\mathcal{B}_\alpha$ is a Miyadera perturbation of $\mathcal{A}_0$, and therefore by Theorem 1.16 we have that $\mathcal{A}_0 + \frac{1}{n}\mathcal{B}_\alpha$ generates a strongly continuous semigroup $(\mathcal{T}_{\alpha,1}(t))_{t \geq 0}$.

Hence we know that $(\mathcal{T}_{\alpha,1}(t))_{t \geq 0}$ solves the abstract Cauchy problem associated to its generator $((\mathcal{A}_0 + \frac{1}{n}\mathcal{B}_\alpha), D(\mathcal{A}_0))$:

$$\begin{cases} \mathcal{U}'(t) = (\mathcal{A}_0 + \frac{1}{n}\mathcal{B}_\alpha) \, \mathcal{U}(t), & t \geq 0, \\ \mathcal{U}(0) = \left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right). \end{cases}$$

Note also that we take $0 < \alpha \leq 1$. Utilizing these observations together with Formulae (13) and (17) of Section 1.1 we obtain

$$(\pi_2 \mathcal{T}_{\alpha,1}(t) \left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right))(\sigma) = f(t+\sigma) = (\pi_2 \mathcal{T}_0(t) \left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right))(\sigma)$$

for all $0 \leq t \leq \alpha$ and all $\sigma \in [-1,-\alpha]$. There π_2 denotes the projection onto the second coordinate (see Definition 1.7).

Since η_α has no mass in $[-\alpha, 0]$, it follows that

$$\mathcal{B}_\alpha \mathcal{T}_{\alpha,1}(t) \left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) = \mathcal{B}_\alpha \mathcal{T}_0(t) \left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \qquad \text{for } 0 \leq t \leq 0.$$

Therefore, we obtain

$$\int_0^\alpha \left\| \mathcal{B}_\alpha \mathcal{T}_{\alpha,1}(r) \left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \right\| dr \leq |\eta|([-1,-\alpha]) \left\| \left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \right\|$$

for all $\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in D(\mathcal{A}_0) = D(\mathcal{A}_0 + \frac{1}{n}\mathcal{B}_\alpha)$. Thus, $(\mathcal{A}_0 + \frac{1}{n}\mathcal{B}_\alpha) + \frac{1}{n}\mathcal{B}_\alpha = \mathcal{A}_0 + \frac{2}{n}\mathcal{B}_\alpha$ generates a strongly continuous semigroup. Repeating this argument n times we obtain that $\mathcal{A}+\mathcal{B}_\alpha$ is the generator of a strongly continuous semigroup $(\mathcal{T}_\alpha(t))_{t\geq 0}$ on the space $\mathcal{E}_1$.

Let

$$c := \sup\{\|\pi_2\mathcal{T}_\alpha(t)\|_1 \,:\, 0 < t \leq \alpha \leq 0\} < \infty. \tag{38}$$

Since η_α has no mass in $[-\alpha, 0]$, the delay equation

$$u'(t) = Bu(t) + \Phi_\alpha u_t$$

with initial values

$$x \in D(B) \text{ and } f \in W^{1,1}([-1,0],X) \text{ such that } f(0) = x,$$

has a solution $u(t)$ given by

$$u(t) = S(t)x + \int_0^t S(t-s)\int_{-1}^{-\alpha} d\eta(\tau)f(s+\tau)ds$$

for $0 \leq t \leq \alpha$. Using this notation we have

$$\mathcal{T}_\alpha(t)\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) = \left(\begin{smallmatrix} u(t) \\ u_t \end{smallmatrix}\right).$$

In the last step of this proof we show that $\tilde{\mathcal{B}}_\alpha$ satisfies a Miyadera-type estimate. This follows from

$$\begin{aligned}
&\int_0^\alpha \left\|\tilde{\mathcal{B}}_\alpha\mathcal{T}_\alpha(t)\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right)\right\| dt = \int_0^\alpha \left\|\int_{-\alpha}^0 d\eta(\tau)\, u_t(\tau)\right\| dt \\
&\leq \int_0^\alpha \int_{-\alpha}^0 \|u(t+\tau)\|\, d|\eta|(\tau)\, dt = \int_{-\alpha}^0 \int_0^\alpha \|u(t+\tau)\|\, dt\, d|\eta|(\tau) \\
&\leq \int_{-\alpha}^0 \|u_{\tau+\alpha}\|_1\, d|\eta|(\tau) \leq c\,|\eta|([-\alpha,0])\,\left\|\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right)\right\|,
\end{aligned}$$

where c is the constant defined in (38).

We finish the proof by choosing α small enough such that $c\,|\eta|([-\alpha,0]) < 1$. Then, from the previous estimate, it follows that $\tilde{\mathcal{B}}_\alpha = \begin{pmatrix} 0 & \tilde{\Phi}_\alpha \\ 0 & 0 \end{pmatrix}$ is a Miyadera perturbation of $\mathcal{A}_0+\mathcal{B}_\alpha$, which, by the perturbation Theorem of Miyadera-Voigt, implies that $\mathcal{A}_0 + \mathcal{B}_\alpha + \tilde{\mathcal{B}}_\alpha = \mathcal{A}_0 + \mathcal{B} = \mathcal{A}$ generates a strongly continuous semigroup. □

Hence the reaction-diffusion equation with delay (20) in Exercise 1.14 (1) and the age-structured population equation with delay (21) in Exercise 1.14 (2) are well-posed.

Notes and references 1.21. The systematic treatment of delay differential equation via semigroups was started by J. Hale [40] and later developed by many authors. We only quote here the monographs by O. Diekmann et al. [29], J. Hale and S. Verduyn-Lunel [42] and J. Wu [81]. For a general approach to integrodifferential equations (and in particular delay differential equations) see J. Prüß [66].

The study of delay equations on the state space $X \times L^p([-1,0],X)$ with $X = \mathbb{C}^n$ was started by Coleman and Mizel [15] and then developed further by many authors. In particular, for applications to control theory see Delfour [25, 26], Delfour and Mitter [27, 28], Nakagiri [62, 63] and the informative survey by Curtain and Zwart [16, Section 2.4]. However, all these authors first solve the delay equation by using the variation of parameter formula and a fixed point argument and then show that the solutions form a semigroup on the product space and that its generator is the operator $(\mathcal{A}, D(\mathcal{A}))$ defined in (11) and (12). The converse approach is due to Webb. In [80] he considers the operator $(\mathcal{A}, D(\mathcal{A}))$ (even for nonlinear B and Φ but for X a Hilbert space) and shows that for Φ given as in (34) this generates a (nonlinear) semigroup on the Hilbert space $X \times L^p([-1,0], \mu dx, X)$ for a suitable weight function μ. The suitable weight is necessary in his approach to obtain a norm, where the operator $(\mathcal{A}, D(\mathcal{A}))$ becomes dissipative. He then proves that the semigroup actually gives the solutions of the delay equation. For a review of this treatment see also the article by Kappel [47].

The equivalence of functional differential equations and certain abstract Cauchy problems on product spaces was studied by Burns, Herdman and Stech [12]. A version of the equivalence stated in Theorem 1.11 for neutral differential equations and $X = \mathbb{C}^n$ was proved by Kappel and Zhang [48].

The results of Section 1.1 are all taken from [6, 7].

Equation (43) is called *characteristic equation.* A systematic abstract approach to operator valued characteristic equations on Banach spaces is due to Nagel [60].

The generator property of $\mathcal{A}$ has been proved in special cases by Kunisch and Schappacher (see [53, Proposition 4.2]).

For different approaches to the well-posedness of delay equations see also Di Blasio, Kunisch, Sinestrari [10], Diekmann et al. [29], Hale, Verduyn Lunel [42], Kunisch, Mastinšek [52], Maniar, Rhandi [55], Rhandi [68], Ruess [69, 70], Tanabe [73], Travis, Webb [74], Webb [79].

For delay equations with infinite delay see Hale, Kato [41] and Hino, Murakami, Naito [46] or Fragnelli and Nickel [33, 34] for nonautonomous past.

Exercises 1.22. (1) Prove the implication $(a) \Rightarrow (b)$ in Theorem 1.15.

(To this purpose, there are two strategies. The first one is to prove that the family $(\mathcal{T}_0(t))_{t\geq 0}$ forms indeed a strongly continuous semigroup. Then, using the formula for the resolvent in Exercise 1.14 (5), show that its generator is given by $(\mathcal{A}_0, D(\mathcal{A}_0))$.

The second is to show first the following decomposition

$$\mathcal{A}_0 = \begin{pmatrix} B & 0 \\ 0 & A_0 \end{pmatrix} \begin{pmatrix} \mathrm{Id} & 0 \\ -(\mathbb{1} \otimes \mathrm{Id}) & \mathrm{Id} \end{pmatrix},$$

where A_0 denotes the generator of the nilpotent left shift semigroup on $L^p([-1,0], X)$. Then apply the perturbation theorem of Desch-Schappacher (see [30, Corollary III.3.16] and [30, Corollary III.3.4]) to show the generator property. Finally, use the Dyson-Phillips series (see [30, Corollary III.3.2]) to compute the perturbed semigroup.)

(2) Show that for $B \in \mathcal{L}(X)$ the operator $(\mathcal{A}, D(\mathcal{A}))$ defined in (11) and (12) generates a strongly continuous semigroup for all delay operators $\Phi \in \mathcal{L}(W^{1,p}([-1,0], X), X)$. (Hint: Use the following decomposition

$$\lambda - \mathcal{A} = \begin{pmatrix} \mathrm{Id} & -\Phi R(\lambda, A_0) \\ 0 & \mathrm{Id} \end{pmatrix} \begin{pmatrix} \lambda - B - \Phi_\lambda & 0 \\ 0 & \lambda - A_0 \end{pmatrix} \begin{pmatrix} \mathrm{Id} & 0 \\ -\epsilon_\lambda \otimes \mathrm{Id} & \mathrm{Id} \end{pmatrix}$$

from Lemma 1.24 and then apply the perturbation theorem of Desch-Schappacher (see [30, Corollary III.3.16] and [30, Corollary III.3.4]).

1.4 Spectral theory for delay equations

We will see in Chapter 3 that many qualitative properties of the solutions of (DE_p) can be characterized by the spectrum of the operator $(\mathcal{A}, D(\mathcal{A}))$. As a first step towards this goal, we calculate the resolvent $R(\lambda, \mathcal{A})$ and the resolvent set $\rho(\mathcal{A})$ of the operator $\mathcal{A}$.

Let $1 \leq p < \infty$ be fixed for the rest of this section. We first introduce some notation. Here and in the following $(A_0, D(A_0))$ is the generator of the nilpotent left shift semigroup $(T_0(t))_{t\geq 0}$ on $L^p([-1,0], X)$. For $\lambda \in \mathbb{C}$, ϵ_λ denotes the function

$$\epsilon_\lambda(s) := e^{\lambda s} \text{ for } s \in [-1,0].$$

Let now $\lambda \in \mathbb{C}$ and $\left(\begin{smallmatrix} y \\ g \end{smallmatrix}\right) \in \mathcal{E}_p$, we have to find $\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in D(\mathcal{A})$ such that

$$(\lambda - \mathcal{A})\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) = \begin{pmatrix} (\lambda - B)x - \Phi f \\ \lambda f - f' \end{pmatrix} = \begin{pmatrix} y \\ g \end{pmatrix}. \tag{39}$$

By the variation of parameter formula, we can integrate the second equation of (39) and obtain

$$f = \epsilon_\lambda\, f(0) + R(\lambda, A_0)\, g, \tag{40}$$

where, $R(\lambda, A_0)$ is the resolvent of $(A_0, D(A_0))$, which exists since its spectrum $\sigma(A_0) = \emptyset$. Since $f(0) = x$, equation (40) becomes

$$f = \epsilon_\lambda\, x + R(\lambda, A_0)\, g. \tag{41}$$

Hence, x has to satisfy the equation

$$(\lambda - B - \Phi_\lambda)x = \Phi R(\lambda, A_0)g + y, \tag{42}$$

where $\Phi_\lambda \in \mathcal{L}(Z, X)$ is defined by $\Phi_\lambda x := \Phi(e^{\lambda \cdot}\, x)$ for $x \in Z$.

This leads to the following proposition.

Proposition 1.23. *For $\lambda \in \mathbb{C}$ and for all $1 \leq p < \infty$ we have*

$$\lambda \in \rho(\mathcal{A}) \textit{ if and only if } \lambda \in \rho(B + \Phi_\lambda). \tag{43}$$

Moreover, for $\lambda \in \rho(\mathcal{A})$ the resolvent $R(\lambda, \mathcal{A})$ is given by

$$\begin{pmatrix} R(\lambda, B + \Phi_\lambda) & R(\lambda, B + \Phi_\lambda)\Phi R(\lambda, A_0) \\ \epsilon_\lambda\, R(\lambda, B + \Phi_\lambda) & [\epsilon_\lambda\, R(\lambda, B + \Phi_\lambda)\Phi + Id]R(\lambda, A_0) \end{pmatrix}. \tag{44}$$

Proof. Let $\lambda \in \rho(B+\Phi_\lambda)$. Then the matrix defined by (44) is a bounded operator from $\mathcal{E}_p$ to $D(\mathcal{A})$ defining the inverse of $(\lambda - \mathcal{A})$.

To see this, we first show that the range of the matrix defined by (44) is contained in $D(\mathcal{A})$. Take $\left(\begin{smallmatrix} y \\ g \end{smallmatrix}\right) \in \mathcal{E}_p$ and consider the vector

$$\begin{pmatrix} x \\ f \end{pmatrix} := \begin{pmatrix} R(\lambda, B + \Phi_\lambda)y + R(\lambda, B + \Phi_\lambda)\Phi R(\lambda, A_0)g \\ \epsilon_\lambda\, R(\lambda, B + \Phi_\lambda)y + \epsilon_\lambda\, R(\lambda, B + \Phi_\lambda)\Phi R(\lambda, A_0)g + R(\lambda, A_0)g \end{pmatrix}.$$

Since $R(\lambda, B+\Phi_\lambda)$ maps into $D(B)$, we know that $x \in D(B)$. For the second component, note that the function ϵ_λ is smooth, and the range of $R(\lambda, A_0)$ is contained in $W_0^{1,p}([-1,0], Z)$. Hence, $f \in W^{1,p}([-1,0], X)$. Since $R(\lambda, A_0)g(0) = 0$ and $(\epsilon_\lambda\, x)(0) = x$, it is now easy to see that $f(0) = x$. So we have $\binom{x}{f} \in D(\mathcal{A})$.

Write $R_\lambda := R(\lambda, B+\Phi_\lambda)$ and compute the matrix product

$$\begin{aligned}
&(\lambda - \mathcal{A}) \begin{pmatrix} R(\lambda, B+\Phi_\lambda) & R(\lambda, B+\Phi_\lambda)\Phi R(\lambda, A_0) \\ \epsilon_\lambda\, R(\lambda, B+\Phi_\lambda) & [\epsilon_\lambda\, R(\lambda, B+\Phi_\lambda)\Phi + Id]R(\lambda, A_0) \end{pmatrix} \\
&= \begin{pmatrix} (\lambda - B)R_\lambda - \Phi\epsilon_\lambda R_\lambda & (\lambda - B)R_\lambda \Phi R(\lambda, A_0) - \Phi[\epsilon_\lambda R_\lambda \Phi + Id]R(\lambda, A_0) \\ \left(\lambda - \frac{d}{d\sigma}\right)\epsilon_\lambda R_\lambda & \left(\lambda - \frac{d}{d\sigma}\right)[\epsilon_\lambda R_\lambda \Phi + Id]R(\lambda, A_0) \end{pmatrix}.
\end{aligned}$$

The identities

$$(\lambda - B)R_\lambda - \Phi(\epsilon_\lambda\, R_\lambda) = (\lambda - B - \Phi_\lambda)R_\lambda = Id$$

and

$$\begin{aligned}
&(\lambda - B)R_\lambda \Phi R(\lambda, A_0) - \Phi[\epsilon_\lambda\, R_\lambda \Phi + \mathrm{Id}]R(\lambda, A_0) = \\
&\qquad\qquad (\lambda - B - \Phi_\lambda)R_\lambda \Phi R(\lambda, A_0) - \Phi R(\lambda, A_0) = 0
\end{aligned}$$

hold. Moreover, we have $\left(\lambda - \frac{d}{d\sigma}\right)(\epsilon_\lambda\, R_\lambda) = 0$. Using this identity we also obtain

$$\left(\lambda - \frac{d}{d\sigma}\right)[\epsilon_\lambda\, R_\lambda \Phi + \mathrm{Id}]R(\lambda, A_0) = \left(\lambda - \frac{d}{d\sigma}\right)R(\lambda, A_0) = Id.$$

So the operator (44) is a right inverse of $(\lambda - \mathcal{A})$. In an analogous way, we can prove that (44) is also a left inverse of $(\lambda - \mathcal{A})$ and is hence an inverse of $(\lambda - \mathcal{A})$.

Conversely, if $\lambda \in \rho(\mathcal{A})$, then for every $\binom{y}{g} \in \mathcal{E}_p$ there exists a unique $\binom{x}{f} \in D(\mathcal{A})$ such that (41) and (42) hold. In particular, for $g = 0$ and for every $y \in X$, there exists a unique $x \in D(B)$ such that

$$(\lambda - B - \Phi_\lambda)x = y.$$

This means that $(\lambda - B - \Phi_\lambda)$ is invertible, i.e., $\lambda \in \rho(B + \Phi_\lambda)$. □

The complementary statement to Equation (43) in Proposition 1.23 yields

$$\lambda \in \sigma(\mathcal{A}) \text{ if and only if } \lambda \in \sigma(B + \Phi_\lambda). \tag{45}$$

We call Equation (45) the *characteristic equation* of the delay equation. An important consequence of the characteristic equation is that the spectrum of the operator $\mathcal{A}$ in $\mathcal{E}_p$ can be determined in the smaller space X. In particular, if the dimension of X is finite, we have

$$\lambda \in \sigma(\mathcal{A}) \text{ if and only if } \det(\lambda - B - \Phi_\lambda) = 0. \tag{46}$$

An analogous characteristic equation also holds for subsets of the spectrum such as the point spectrum $P\sigma$, approximative point spectrum $A\sigma$, the residual spectrum $R\sigma$ and the essential spectrum σ_{ess}.

Lemma 1.24. *For $\lambda \in \mathbb{C}$ the following hold*

$$\begin{aligned} \lambda \in P\sigma(\mathcal{A}) &\text{ if and only if } \lambda \in P\sigma(B+\Phi_\lambda), \\ \lambda \in A\sigma(\mathcal{A}) &\text{ if and only if } \lambda \in A\sigma(B+\Phi_\lambda), \\ \lambda \in R\sigma(\mathcal{A}) &\text{ if and only if } \lambda \in R\sigma(B+\Phi_\lambda), \\ \lambda \in \sigma_{ess}(\mathcal{A}) &\text{ if and only if } \lambda \in \sigma_{ess}(B+\Phi_\lambda) \end{aligned}$$

for all $1 \le p < \infty$.

Proof. Let $\lambda \in \mathbb{C}$. We first show that the following decomposition

$$\lambda - \mathcal{A} = \begin{pmatrix} \mathrm{Id} & -\Phi R(\lambda, A_0) \\ 0 & \mathrm{Id} \end{pmatrix} \begin{pmatrix} \lambda - B - \Phi_\lambda & 0 \\ 0 & \lambda - A_0 \end{pmatrix} \begin{pmatrix} \mathrm{Id} & 0 \\ -\epsilon_\lambda \otimes \mathrm{Id} & \mathrm{Id} \end{pmatrix}$$

holds.

Let

$$\mathcal{A}_\lambda := \begin{pmatrix} \lambda - B - \Phi_\lambda & 0 \\ 0 & \lambda - A_0 \end{pmatrix}$$

with domain $D(\mathcal{A}_\lambda) := D(B) \times D(A_0)$. Moreover, let

$$T := \begin{pmatrix} \mathrm{Id} & -\Phi R(\lambda, A_0) \\ 0 & \mathrm{Id} \end{pmatrix} \text{ and } S := \begin{pmatrix} \mathrm{Id} & 0 \\ -\epsilon_\lambda \otimes \mathrm{Id} & \mathrm{Id} \end{pmatrix},$$

where Id denotes the identity on X.

Taking $\binom{x}{f} \in D(\mathcal{A})$, then $S\binom{x}{f} \in D(\mathcal{A}_\lambda)$ and we obtain

$$\begin{aligned} &\begin{pmatrix} \mathrm{Id} & -\Phi R(\lambda, A_0) \\ 0 & \mathrm{Id} \end{pmatrix} \begin{pmatrix} \lambda - B - \Phi_\lambda & 0 \\ 0 & \lambda - A_0 \end{pmatrix} \begin{pmatrix} \mathrm{Id} & 0 \\ -\epsilon_\lambda \otimes \mathrm{Id} & \mathrm{Id} \end{pmatrix} \begin{pmatrix} x \\ f \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{Id} & -\Phi R(\lambda, A_0) \\ 0 & \mathrm{Id} \end{pmatrix} \begin{pmatrix} \lambda - B - \Phi_\lambda & 0 \\ 0 & \lambda - A_0 \end{pmatrix} \begin{pmatrix} x \\ f - \epsilon_\lambda x \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{Id} & -\Phi R(\lambda, A_0) \\ 0 & \mathrm{Id} \end{pmatrix} \begin{pmatrix} \lambda x - Bx - \Phi_\lambda x \\ (\lambda - A_0)\,(f - \epsilon_\lambda x) \end{pmatrix} \\ &= \begin{pmatrix} \lambda x - Bx - \Phi_\lambda x - \Phi R(\lambda, A_0)(\lambda - A_0)(f - \epsilon_\lambda x) \\ (\lambda - A_0)\,(f - \epsilon_\lambda x) \end{pmatrix} \\ &= \begin{pmatrix} \lambda x - Bx - \Phi f \\ \lambda f - f' \end{pmatrix} = (\lambda - \mathcal{A})\binom{x}{f}. \end{aligned}$$

Conversely, let $\binom{x}{f} \in \mathcal{E}_p$ such that $S\binom{x}{f} \in D(\mathcal{A}_\lambda)$. Then necessarily $x \in D(B)$, $f \in W^{1,p}([-1,0], X)$ and $f(0) = x$. Hence, $\binom{x}{f} \in D(\mathcal{A})$.

Notice that the operators T and S are invertible on the space $\mathcal{E}_p$ with inverses given by

$$T^{-1} = \begin{pmatrix} \mathrm{Id} & \Phi R(\lambda, A_0) \\ 0 & \mathrm{Id} \end{pmatrix} \text{ and } S^{-1} = \begin{pmatrix} \mathrm{Id} & 0 \\ \epsilon_\lambda \otimes \mathrm{Id} & \mathrm{Id} \end{pmatrix}$$

respectively. Hence the operator

$$\mathcal{A}_\lambda := \begin{pmatrix} \lambda - B - \Phi_\lambda & 0 \\ 0 & \lambda - A_0 \end{pmatrix}$$

with $D(\mathcal{A}_\lambda) := D(B) \times D(A_0)$ has the same spectral properties as $(\lambda - \mathcal{A}, D(\mathcal{A}))$. That is, the two operators $\mathcal{A}$ and $\mathcal{A}_\lambda$ are simultaneously injective, bounded from below, have closed range or are Fredholm. Regarding the operator $\mathcal{A}$, these properties are equivalent to $\lambda \notin P\sigma(\mathcal{A})$, $\lambda \notin A\sigma(\mathcal{A})$, $\lambda \notin R\sigma(\mathcal{A})$, and $\lambda \notin \sigma_{ess}(\mathcal{A})$, respectively. These statements in turn are equivalent to $0 \notin P\sigma(\mathcal{A}_\lambda)$, $0 \notin A\sigma(\mathcal{A}_\lambda)$, $0 \notin R\sigma(\mathcal{A}_\lambda)$, and $0 \notin \sigma_{ess}(\mathcal{A}_\lambda)$, respectively. For example, this means for the essential spectrum that $\lambda \in \sigma_{ess}(\mathcal{A})$ if and only if $0 \in \sigma_{ess}(\mathcal{A}_\lambda)$. But using the fact that $\mathcal{A}_\lambda$ is a diagonal operator and that $\sigma(A_0) = \emptyset$, we see that $0 \in \sigma_{ess}(\mathcal{A}_\lambda)$ if and only if $\lambda \in \sigma_{ess}(B + \Phi_\lambda)$. ☐

Example 1.25. We consider here a simple example. Let $X := \mathbb{C}$, $B = 0$ and $\Phi := c\,\delta_{-1}$, where $c \in \mathbb{C}$ is a constant and δ_{-1} is the point evaluation at $\sigma = -1$. The delay equation becomes

$$\begin{cases} u'(t) = c\,u(t-1) \text{ for } t \geq 0, \\ u(0) = x, \\ u_0 = f \end{cases}$$

and the associated operator matrix on the space $\mathbb{C} \times L^p[-1,0]$ is

$$\mathcal{A} = \begin{pmatrix} 0 & c\,\delta_{-1} \\ 0 & \frac{d}{d\sigma} \end{pmatrix}$$

with domain

$$D(\mathcal{A}) = \{ \left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in \mathbb{C} \times W^{1,p}[-1,0] \ : \ f(0) = x \}.$$

By Proposition 1.23 and the characteristic equation (46), the spectrum of $\mathcal{A}$ is given by

$$\sigma(\mathcal{A}) = \{ \lambda \in \mathbb{C} \ : \ \lambda = c\,e^{-\lambda} \}.$$

Remark 1.26. Note that all the above results hold for all $1 \leq p < \infty$; hence $\mathcal{A}$ has the same spectral properties in each $\mathcal{E}_p$.

Notes and references 1.27. Versions of Proposition 1.23 has been proved also in Nakagiri [62, 63]. An abstract version of Lemma 1.24 was shown in [3].

For further references on the characteristic equation (Equation (45)) see Diekmann et al. [29, Section I.3], Hale, Verduyn Lunel [42, Section 7.3] and Wu [81, Section 3.1] for functional differential equations, or Nagel [59, 60] for abstract Cauchy problems.

2 Regularity properties of the delay semigroup

In this lecture we investigate the regularity of the delay semigroup. Such regularity will be an important tool for the study of the asymptotic behaviour of the delay semigroup and therefore of the solution of the delay equation (DE). Our main interest is whether the delay semigroup is eventually norm continuous[1] since for such semigroups the Spectral Mapping Theorem holds (see [30, Section IV.2]).

In Section 2.1 we prove abstract regularity results for perturbed semigroups.

In Section 2.2 we show that, if the operator $(B, D(B))$ generates an immediately norm continuous semigroup and the delay operator Φ is of the form (34) in Section 1.3, then the delay semigroup becomes eventually norm continuous.

2.1 Norm continuity and Miyadera perturbation

We first show under which conditions a perturbed semigroup is immediately or eventually norm continuous. We will do this in the case of perturbations of Miyadera-Voigt type (see [30, Corollary III.3.16]).

Let $(G, D(G))$ be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space X and let $C \in \mathcal{L}(X_1^G, X)$, i.e., C is relatively bounded by G. The following hypothesis will be used, strengthening slightly the assumptions in the perturbation theorem of Miyadera-Voigt.

Hypothesis 2.1. Assume that there exists a function $q : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim\limits_{t\searrow 0} q(t) = 0$ and

$$\int_0^t \|C\,T(s)x\|\,ds \leq q(t)\,\|x\| \tag{47}$$

for every $x \in D(G)$ and $t \geq 0$.

Estimates of this type were introduced in this abstract setting by J. Voigt [77]. By the Miyadera-Voigt perturbation theorem we know that $(G + C, D(G))$ generates a strongly continuous semigroup $(U(t))_{t\geq 0}$ on X given by the Dyson-Phillips series, i.e.,

$$U(t)x = \sum_0^\infty (V^n T)(t)x \quad \text{for all} \quad x \in X,\, t \geq 0,$$

where V is the abstract Volterra operator defined in (31) and (32).

Lemma 2.2. *Let* $\varepsilon > 0$. *If there exists a function* $\tilde{q} : (0, \varepsilon) \to \mathbb{R}_+$ *such that* $\lim\limits_{t\searrow 0} \tilde{q}(t) = 0$ *and*

$$\int_0^t \|C\,T(s)x\|\,ds \leq \tilde{q}(t)\,\|x\|$$

---

[1] For the terminology see [30, Section II.4] .

for every $x \in D(G)$ and $t \in [0, \varepsilon)$, then Hypothesis 2.1 is satisfied.

For the proof of Lemma 2.2 do Exercise 2.13 (1).

We can now prove results on the permanence of eventual norm continuity under perturbations satisfying condition (2.1).

Proposition 2.3. *If $(T(t))_{t\geq 0}$ is immediately norm continuous and C satisfies Hypothesis (2.1), then also the perturbed semigroup $(U(t))_{t\geq 0}$ is immediately norm continuous.*

For the proof of Proposition 2.3 do Exercise 2.13 (2).

For the permanence of eventual norm continuity, we have to make some additional assumptions.

Lemma 2.4. *If $(T(t))_{t\geq 0}$ is norm continuous for $t > \alpha > 0$, then for every $n \in \mathbb{N}$ V^nT is norm continuous for $t > (n+1)\,\alpha$.*

Proof. The proof is by induction on $n \in \mathbb{N}$. The assertion holds for $n = 0$. Assume now that it is true for some n, i.e., V^nT is norm continuous for $t > (n+1)\,\alpha$. We show that it also holds for $n+1$.

Take $t > (n+2)\,\alpha$, $x \in D(G)$ and $0 < h < 1$. We have

$$\begin{aligned}
&\|(V^{n+1}T)(t+h)x - (V^{n+1}T)(t)x\| \\
&\quad\leq \int_t^{t+h} \|(V^nT)(t+h-s)\|\,\|CT(s)x\|\,ds \\
&\qquad+ \left\| \int_0^t ((V^nT)(t+h-s) - (V^nT)(t-s))CT(s)x\,ds \right\| \\
&\quad\leq K \int_0^h \|CT(s)T(t)x\|\,ds \\
&\qquad+ \left\| \int_0^t ((V^nT)(t+h-s) - (V^nT)(t-s))CT(s)x\,ds \right\|,
\end{aligned}$$

where $K := \sup_{s\in[0,t+1]} \|(V^nT)(s)\|$. By assumption (2.1), the first integral converges to 0 as $h \searrow 0$ uniformly for $\|x\| \leq 1$. So it remains to estimate the second integral. As in the first step, we have

$$\begin{aligned}
&\left\| \int_0^t ((V^nT)(t+h-s) - (V^nT)(t-s))CT(s)x\,ds \right\| \\
&\quad\leq \int_0^{t-(n+1)\alpha} \|(V^nT)(t+h-s) - (V^nT)(t-s)\|\,\|CT(s)x\|\,ds \\
&\qquad+ \left\| \int_{t-(n+1)\alpha}^t ((V^nT)(t+h-s) - (V^nT)(t-s))CT(s)x\,ds \right\| \\
&\quad\leq q(t-(n+1)\alpha) \sup_{s\in[0,t-(n+1)\alpha]} \|(V^nT)(t+h-s) - (V^nT)(t-s)\|\,\|x\| \\
&\qquad+ \left\| \int_{t-(n+1)\alpha}^t ((V^nT)(t+h-s) - (V^nT)(t-s))CT(s)x\,ds \right\|.
\end{aligned}$$

The first term of the above inequality converges to 0 as $h \searrow 0$ uniformly for $\|x\| \leq 1$ by the induction assumption and by Hypothesis 2.1. We now estimate the second term as

$$\begin{aligned}
&\left\| \int_{t-(n+1)\alpha}^{t} ((V^nT)(t+h-s) - (V^nT)(t-s))CT(s)x\,ds \right\| \\
&= \left\| \int_{t-(n+1)\alpha-h}^{t-h} (V^nT)(t-s)CT(s+h)x\,ds - \int_{t-(n+1)\alpha}^{t} (V^nT)(t-s)CT(s)x\,ds \right\| \\
&\quad \leq \int_{t-(n+1)\alpha-h}^{t-(n+1)\alpha} \|(V^nT)(t-s)CT(s+h)x\|\,ds \\
&\qquad + \int_{t-(n+1)\alpha}^{t-h} \|(V^nT)(t-s)\,C(T(s+h)-T(s))x\|\,ds \\
&\qquad + \int_{t-h}^{t} \|(V^nT)(t-s)CT(s)x\|\,ds \\
&\quad \leq K \int_0^h \|CT(s)\,T(t-(n+1)\alpha)x\|\,ds \\
&\qquad + \int_{t-(n+1)\alpha}^{t-h} \|(V^nT)(t-s)\,C(T(s+h)-T(s))x\|\,ds \\
&\qquad + K \int_0^h \|CT(s)\,T(t-h)x\|\,ds \\
&\quad \leq q(h)\,K\,\|T(t-(n+1)\alpha)\|\,\|x\| \\
&\qquad + \int_{t-(n+1)\alpha}^{t-h} \|(V^nT)(t-s)\,C(T(s+h)-T(s))x\|\,ds \\
&\qquad + q(h)\,K\,\|T(t-h)\|\,\|x\|\,.
\end{aligned}$$

By assumption (2.1), the first and the last term converges to 0 uniformly for $\|x\| \leq 1$ as $h \searrow 0$. For the second term, we have

$$\begin{aligned}
&\int_{t-(n+1)\alpha}^{t-h} \|(V^nT)(t-s)\,C(T(s+h)-T(s))x\|\,ds \\
&\quad \leq K \int_{t-(n+1)\alpha}^{t} \|C(T(s+h)-T(s))x\|\,ds \\
&\quad = M \int_0^{(n+1)\alpha} \|CT(s)(T(t-(n+1)\,\alpha+h) - T(t-n\,\alpha))x\|\,ds \\
&\quad \leq M\,q((n+1)\alpha)\,\|(T(t-(n+1)\alpha+h) - T(t-(n+1)\alpha))\,x\|\,.
\end{aligned}$$

This converges to 0 uniformly for $\|x\| \leq 1$ as $h \searrow 0$ by the norm continuity of the semigroup $(T(t))_{t\geq 0}$ for $t > \alpha$. In an analogous way one can show the left continuity for $t > 0$. Since $D(G)$ is dense in X, the assertion follows. □

We can now prove the main theorem of this section.

Theorem 2.5. *If* $(T(t))_{t\geq 0}$ *is norm continuous for* $t > \alpha$*,* C *satisfies condition (2.1) and there exists* $n \in \mathbb{N}$ *such that* $V^n T$ *is norm continuous for* $t > 0$*, then the perturbed semigroup* $(U(t))_{t\geq 0}$ *is norm continuous for* $t > n\alpha$*.*

Proof. In the same way as in the proof of Proposition 2.3, we can prove that $V^m T$ is norm continuous for $t > 0$ for every $m \geq n$.
From Proposition 2.4 we have that $V^{(n-1)}T$ is norm continuous for $t > n\alpha$. Since the Dyson-Phillips series (30) converges uniformly on compact intervals, we have that $(U(t))_{t\geq 0}$ is norm continuous for $t > n\alpha$. ☐

2.2 Regularity of the delay semigroup

In this section we apply the abstract theorem above and give sufficient conditions such that the delay semigroup becomes eventually norm continuous (see Theorem 2.8) or/and has compact resolvent (see Proposition 2.10).

We will assume that the delay operator Φ satisfies the following condition being slightly stronger than condition (M) in Theorem 1.17 on page 12 and corresponding to Condition (47) in the case of the delay semigroup.

There exists $q : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ with $\lim_{t\to 0^+} q(t) = 0$ and

$$\int_0^t \|\Phi(S_s x + T_0(s)f)\|\, ds \leq q(t) \left\|\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right)\right\| \tag{K}$$

for all $\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in D(\mathcal{A}_0)$ and $t > 0$.

We then have that $(\mathcal{A}, D(\mathcal{A}))$ is a generator on each $\mathcal{E}_p$ by Theorem 1.17.

Remark 2.6. The main example of Theorem 1.20, in which the delay operator is given by

$$\Phi\, f := \int_{-1}^{0} d\eta\, f$$

for a function $\eta : [-1, 0] \to \mathcal{L}(X)$ of bounded variation, satisfies condition (K) for all $1 < p < \infty$ (see the proof of Theorem 1.20). Unfortunately, in case $p = 1$ the result is not known at this moment.

Since we will use perturbation, we recall that $(\mathcal{A}_0, D(\mathcal{A}_0))$ is the operator defined in Lecture 2, (22) and (23) as

$$\mathcal{A}_0 := \begin{pmatrix} B & 0 \\ 0 & \frac{d}{d\sigma} \end{pmatrix}$$

with domain

$$D(\mathcal{A}_0) := D(\mathcal{A}) = \{\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in X \times L^p([-1, 0], X) \ :\ f(0) = x\}.$$

It generates the strongly continuous semigroup $(\mathcal{T}_0(t))_{t\geq 0}$ given by

$$\mathcal{T}_0(t) := \begin{pmatrix} S(t) & 0 \\ S_t & T_0(t) \end{pmatrix}, \tag{25}$$

where $(T_0(t))_{t\geq 0}$ is the nilpotent left shift semigroup on $L^p([-1,0],X)$.

Proposition 2.7. *If $(S(t))_{t\geq 0}$ is norm continuous for $t > t_0 \geq 0$, then $(\mathcal{T}_0(t))_{t\geq 0}$ is norm continuous for $t > t_0 + 1$ for all $1 \leq p < \infty$.*

For the proof of Proposition 2.7 do Exercise 2.13 (3).

We now apply a perturbation argument (see Proposition 2.4) to show eventual norm continuity of the delay semigroup. It is clear from Formula (25) that we cannot expect immediate norm continuity since even for $B = 0$ and $\Phi = 0$ the semigroup is norm continuous only for $t > 1$.

Theorem 2.8. *If the semigroup $(S(t))_{t\geq 0}$ generated by $(B, D(B))$ is immediately norm continuous and Φ satisfies condition* (K)*, then the delay semigroup $(\mathcal{T}(t))_{t\geq 0}$ is norm continuous for $t > 1$ for all $1 \leq p < \infty$.*

Proof. By Proposition 2.7, $(\mathcal{T}_0(t))_{t\geq 0}$ is norm continuous for $t > 1$. We now show that $V\mathcal{T}_0$, where V is the abstract Volterra operator defined in (31) and (32), is norm continuous for $t \geq 0$. In fact, for $t \geq 0$ and $\binom{x}{f} \in D(\mathcal{A}_0)$ we have

$$\begin{aligned}
V\mathcal{T}_0(t)\tbinom{x}{f} &= \int_0^t \mathcal{T}_0(t-s)\mathcal{B}\mathcal{T}_0(s)\tbinom{x}{f}\,ds \\
&= \int_0^t \mathcal{T}_0(t-s)\begin{pmatrix} 0 & \Phi \\ 0 & 0 \end{pmatrix}\begin{pmatrix} S(s)x \\ S_s x + T_0(s)f \end{pmatrix} ds \\
&= \int_0^t \begin{pmatrix} S(t-s) & 0 \\ S_{t-s} & T_0(t-s) \end{pmatrix}\begin{pmatrix} \Phi(S_s x + T_0(s)f) \\ 0 \end{pmatrix} ds \\
&= \int_0^t \begin{pmatrix} S(t-s)\Phi(S_s x + T_0(s)f) \\ S_{t-s}\Phi(S_s x + T_0(s)f) \end{pmatrix} ds.
\end{aligned}$$

We prove norm continuity of both components separately.

1. Let $t \geq 0$ and $0 < h < 1$. Then we have

$$\begin{aligned}
&\left\| \int_0^{t+h} S(t+h-s)\Phi(S_s x + T_0(s)f)\, ds - \int_0^t S(t-s)\Phi(S_s x + T_0(s)f)\, ds \right\| \\
&\qquad \leq \left\| \int_t^{t+h} S(t+h-s)\Phi(S_s x + T_0(s)f)\, ds \right\| \\
&\qquad\quad + \left\| \int_0^t (S(t+h-s) - S(t-s))\Phi(S_s x + T_0(s)f)\, ds \right\| \\
&\qquad \leq \int_0^h \|S(h-s)\|\, \|\Phi(S_{s+t}x + T_0(s+t)f)\|\, ds \\
&\qquad\quad + \int_0^t \|S(t+h-s) - S(t-s)\|\, \|\Phi(S_s x + T_0(s)f)\|\, ds. \\
&\qquad \leq \sup_{0\leq r\leq 1} q(h)\ \|S(r)\|\ \left\|\mathcal{T}_0(t)\left(\begin{smallmatrix} x\\ f\end{smallmatrix}\right)\right\| \\
&\qquad\quad + \int_0^t \|S(t+h-s) - S(t-s)\|\, \|\Phi(S_s x + T_0(s)f)\|\, ds.
\end{aligned}$$

By condition (K), the Lebesgue dominated convergence theorem, and by the immediate norm continuity of $(S(t))_{t\geq 0}$, we obtain that

$$\sup_{0\leq r\leq 1} \|S(r)\| q(h) \left\|\mathcal{T}_0(t)\left(\begin{smallmatrix} x\\ f\end{smallmatrix}\right)\right\| + \int_0^t \|S(t+h-s) - S(t-s)\|\, \|\Phi(S_s x + T_0(s)f)\|\, ds$$

tends to 0 as $h \to 0^+$ uniformly for $\left(\begin{smallmatrix} x\\ f\end{smallmatrix}\right) \in D(\mathcal{A}_0)$ with $\left\|\left(\begin{smallmatrix} x\\ f\end{smallmatrix}\right)\right\| \leq 1$. The proof for $h \to 0^-$ is analogous.
Since $D(\mathcal{A})$ is dense in $\mathcal{E}_p$, the first component of $V\mathcal{T}_0$ is immediately norm continuous.

2. To prove immediate norm continuity of the second component of $V\mathcal{T}_0$ one proceeds in a similar way. We only have to use the norm continuity of the map $t \mapsto S_t$ which was proved in Proposition 2.7.

Hence, the map $t \mapsto V\mathcal{T}_0(t)$ is norm continuous on $\mathbb{R}_+$ and, by Theorem 2.5, we conclude that $(\mathcal{T}(t))_{t\geq 0}$ is norm continuous for $t > 1$. □

We remark that the immediate norm continuity of $(B, D(B))$ is essential here and cannot be weakened to eventual norm continuity. To show this, we consider the following simple example due to R. Shvidkoy.

Example 2.9. Consider the space $X = L^1[0,2]$, the operators

$$Bx = -x', \quad D(B) := \left\{x \in W^{1,1}[0,2] : x(0) = 0\right\},$$

which is the generator of the nilpotent right-shift operator, and

$$Cx(s) := \begin{cases} x(s+1), & 0 \leq s+1 \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Notice that the functions

$$x_n(s) := \begin{cases} s \cdot e^{-2\pi i n s}, & 0 \leq s+1 \leq 1, \\ e^{-2\pi n s}, & \text{otherwise}, \end{cases}$$

satisfy

$$(B+C)x_n = 2\pi i n x_n \quad \text{for each } n \in \mathbb{N}.$$

Taking $\Phi := C\delta_{-1}$ and

$$F_n := \begin{pmatrix} x_n \\ e^{2\pi i n \cdot} x_n \end{pmatrix},$$

we obtain that $F_n \in D(\mathcal{A})$ and

$$\mathcal{A}F_n = 2\pi i n F_n, \text{ i.e. }, 2\pi i n \in P\sigma(\mathcal{A}).$$

Hence, by [30, Theorem II.4.18], the semigroup generated by $(\mathcal{A}, D(\mathcal{A}))$ is not eventually norm continuous.

We now turn our attention to another regularity property: compactness of the resolvent. This is also an important property for the study of the asymptotic behaviour of the solutions and will be used in the next lectures.

Proposition 2.10. *If the semigroup $(S(t))_{t\geq 0}$ generated by $(B, D(B))$ is immediately compact, then $R(\lambda, B+\Phi_\lambda)$ is compact for all $\lambda \in \rho(\mathcal{A})$.*

Proof. By [30, Theorem II.4.29] we have that the resolvent $R(\mu, B)$ is compact for all $\mu \in \rho(B)$. Take $\lambda \in \rho(\mathcal{A})$. Then $R(\mu, B+\Phi_\lambda)$ exists and is compact for all $\mu \in \rho(B+\Phi_\lambda)$. This holds in particular for $\mu = \lambda$, since by (43) Exercise 1.14 (5) $\lambda \in \rho(B+\Phi_\lambda)$. □

Example 2.11. We recall from Exercise 1.14 (1) the following reaction-diffusion equation with delay

$$\begin{cases} \partial_t w(s,t) = \Delta w(s,t) + c\int_{-1}^{0} w(s,t+\tau)\, dg(\tau), \\ \quad s \in \Omega,\ t \geq 0, \\ w(s,t) = 0, \quad s \in \partial\Omega,\ t \geq 0, \\ w(s,t) = f(s,t), \quad (s,t) \in \Omega \times [-1,0], \end{cases} \tag{20}$$

where c is a constant, $\Omega \subset \mathbb{R}^n$ an open set with smooth boundary, and $g : [-1,0] \to [0,1]$ is a function of bounded variation.

In this case the generator $(B, D(B))$ is the Laplacian with Dirichlet boundary conditions, hence it generates an analytic and in particular immediately norm continuous semigroup $(T(t))_{t\geq 0}$ on $L^2(\Omega)$. Condition (K) is satisfied by Remark 2.6.

Hence, the delay semigroup solving (20) is norm continuous for $t > 1$. Moreover, if Ω is bounded, then the resolvent of $(B, D(B))$ is compact and by [30, Theorem II.4.29] we have that $(T(t))_{t\geq 0}$ is immediately compact. Therefore, by Lemma 2.10 the resolvent $R(\lambda, B+\Phi_\lambda)$ is compact for all $\lambda \in \rho(B+\Phi_\lambda)$.

Notes and references 2.12. The results of Section 2.1 are taken from [65]. The case of bounded perturbations has been treated in [61].

The results of Section 2.2 are essentially taken from [6, 7]. A result analogous to Theorem 2.8 was already proved by a different technique, and for the special case $\Phi := \sum_{k=0}^{n} B_k \delta_{h_k}$, by Fischer and van Neerven [32, Proposition 3.5].

Example 2.9 is a modification of Brendle, Nagel, Poland [11, Example 5.1] and has been observed by R. Shvidkoy.

Exercises 2.13. (1) Prove Lemma 2.2.

(2) Prove Proposition 2.3 (Hint: look at the proof of Lemma 2.4).

(3) Prove Proposition 2.7.

3 Asymptotic behaviour of the delay semigroup

In the previous two lectures we have seen how to associate to a delay differential equation

$$(\mathrm{DE}_p) \qquad \begin{cases} u'(t) = Bu(t) + \Phi u_t, \quad t \geq 0, \\ u(0) = x, \\ u_0 = f. \end{cases}$$

a one-sided coupled operator matrix

$$\mathcal{A} := \begin{pmatrix} B & \Phi \\ 0 & \frac{d}{d\sigma} \end{pmatrix}, D(\mathcal{A}) := \left\{ \left(\begin{smallmatrix} x \\ f \end{smallmatrix} \right) \in D(B) \times W^{1,p}([-1,0],X) \ : \ f(0) = x \right\}.$$

We used semigroup theory and the theory of one-sided coupled operator matrices to show well-posedness and, under additional assumptions on $(B, D(B))$, the eventual norm continuity of the so called delay semigroup generated by $(\mathcal{A}, D(\mathcal{A}))$.

However, investigating a time dependent differential equation coming from some "real" problem a main interest is in the long time behaviour of the solutions. The aim of the following lectures will be to give a brief introduction to some questions of this kind. Due to our personal taste and interest, uniform exponential stability and connected asymptotic properties will be investigated.

Before you go on, dear reader, please make sure that you are familiar with the key notions of semigroup theory that will be used later on. For your reference, as mentioned in the introduction, suggest the corresponding sections of the monograph Engel-Nagel [30]. We do not ask you to learn all this material right now, just have a feeling for the methods we will apply later on.

In Section 3.1 we consider the situation where the spectral mapping theorem holds, see e.g., [30, Section IV.3], and therefore the growth bound of the delay semigroup is equal to the spectral bound of its generator. Thus, the uniform exponential stability is characterized by the negativity of the real part of the spectrum. Main examples for this situation are finite dimensional delay equations, see e.g., Hale, Verduyn Lunel [42], or parabolic equations with delay, see e.g., Milota [58], Wu [81].

In Section 3.2 we try to develop some techniques for more general semigroups where the spectral mapping theorem cannot be expected to hold. Finally, in Section 3.3 we address, as an application, a special problem, the so called problem of small delays.

3.1 Asymptotic behaviour via the spectral mapping theorem

We start by setting the stage, i.e., formulating the main hypotheses for this lecture, and by formulating some immediate consequences. As we will see, the actual hard work has already been done in the previous lectures, we can now enjoy their fruits.

Theorem 3.1. *Assume that* $(B, D(B))$ *generates an immediately norm continuous semigroup and that the delay operator* Φ *satisfies condition* (K). *Then*

$$s(\mathcal{A}) = \omega_0(\mathcal{A}).$$

In particular, the delay semigroup is uniformly exponentially stable if and only if the implication

$$\lambda \in \sigma(B + \Phi_\lambda) \Rightarrow \Re\lambda < 0$$

holds.

Proof. It follows from Proposition 2.8 that the delay semigroup is norm continuous for $t > 1$. By [30, Theorem IV.3.10] the spectral mapping theorem, i.e.,

$$\sigma(\mathcal{T}(t)) \setminus \{0\} = e^{t\sigma(\mathcal{A})} \qquad \text{holds for all } t \geq 0. \tag{SMT}$$

holds and, as a consequence, the equality $s(\mathcal{A}) = \omega_0(\mathcal{A})$ follows, see [30, Corollary IV.3.11]. Using the spectral characterization of (43), we obtain that $\lambda \in \sigma(\mathcal{A})$ if and only if $\lambda \in \sigma(B + \Phi_\lambda)$. Using the fact that the spectrum of the generator of an eventually norm continuous semigroup is bounded along imaginary lines (see [30, Theorem II.4.18]), it cannot happen that the points in the spectrum approach the imaginary axis. Hence, we obtain the desired condition for the negativity of the spectral bound. ☐

Before going on with the investigation of the stability, we have to clarify a technical detail. If we are given a delay equation with a smooth initial function, does it really matter which $\mathcal{E}_p$ space we choose? Looking at Theorem 3.1 we immediately see that, at least concerning uniform exponential stability, it does not.

Corollary 3.2. *Under the assumptions of Theorem 3.1, if the delay operator* Φ *is given by a function of bounded variation* η *as in Theorem 1.20, then the growth bound of the delay semigroup is independent of* $1 < p < \infty$.

Though we have a complete characterization of uniform exponential stability via the spectrum of the generator, it is sometimes rather complicated to use it effectively. However, the situation is much easier if we have compactness.

Corollary 3.3. *Assume that* $(B, D(B))$ *generates an immediately compact semigroup and that the delay operator* Φ *satisfies condition* (K). *Then the delay semigroup is uniformly exponentially stable if and only if the implication*

$$\lambda x - Bx - \Phi_\lambda x = 0 \;\; \textit{for some } 0 \neq x \in D(B) \implies \Re\lambda < 0$$

holds.

Proof. By Lemma 2.10 we have that $\sigma(\mathcal{A}) = \sigma_p(\mathcal{A})$. The result follows immediately from Proposition 3.1. $\square$

Example 3.4. We illustrate the above corollary by the following equation modelling heat conduction in a rod where the two endpoints of the rod are kept at a constant temperature.

$$\begin{cases} \partial_t u(x,t) = \partial_x^2 u(x,t) - au(x,t) - bu(x,t-r),\ x \in [0,l],\ t \geq 0, \\ u(0,t) = u(l,t) = 0,\ t \geq 0, \\ u(x,t) = f(x,t),\ x \in [0,l],\ t \in [-r,0], \end{cases}$$

where $l, r > 0$, $a, b \in \mathbb{R}$. Recalling the well-posedness treated in Exercises 1.14 (1) in Chapter 1, we define the following spaces and operators:

- the Hilbert space $X := L^2(0,l)$,
- the operator $B := (\Delta - a)$ with Dirichlet boundary conditions, i.e., $D(B) := H_0^1(0,l) \cap H^2(0,l)$,
- the functions $\mathbb{R}_+ \ni t \mapsto u(t) = u(\cdot,t) \in L^2(0,l)$ and $u_t : [-1,0] \to L^2(0,l)$, $u_t(s) := u(t+s)$,
- $X := L^2(0,l)$,
- the operator $\Phi : W^{1,p}([-1,0],X) \to X$ defined as $\Phi f := -b\delta_{-r} f = -bf(-r)$.

In order to apply Corollary 3.3, we have to find all $\lambda \in \mathbb{C}$ such that the equation

$$(\lambda + be^{-\lambda r})x - Bx = 0$$

has a solution $0 \neq x \in D(B)$. We recall that $\sigma(B) = \left\{ -\frac{n^2\pi^2}{l^2} - a : n \in \mathbb{N} \right\}$. Hence, the solution semigroup is uniformly exponentially stable if and only if all solutions $\lambda \in \mathbb{C}$ of the equation

$$\lambda + be^{-\lambda r} = -\frac{n^2\pi^2}{l^2} - a, \quad n \in \mathbb{N},$$

have negative real part.

The literature on the investigation of the roots of such transcendental equations is vast. We recall here a useful result from Hale and Verduyn Lunel [42, Theorem A.5], which should help you to solve the exercises.

Theorem 3.5. *All roots of the equation* $(\lambda+\alpha)e^{\lambda}+\beta=0$*, where* α *and* β *are real numbers, have negative real parts if and only if*

$$\begin{aligned} \alpha >& -1 \\ \alpha+\beta >&0 \\ \beta <&\rho\sin\rho-\alpha\cos\rho, \end{aligned}$$

where $\rho=\frac{\pi}{2}$ *if* $\alpha=0$*, or* ρ *is the root of* $\rho=-\alpha\tan\rho$ *in* $(0,\pi)$ *if* $\alpha\neq 0$*.*

Though we said in the introduction to this section that we will concentrate on uniform exponential stability, we cannot resist the temptation to show a so called decomposition result. It says that, if we assume the same compactness on the on the semigroup generated by $(B, D(B))$ as before, then the space $\mathcal{E}_p$ naturally decomposes into three subspaces which are invariant under the delay semigroup.

Theorem 3.6. *Assume that* $(B, D(B))$ *generates an immediately compact semigroup and that* Φ *satisfies condition (K). Then there exist subspaces* $\mathcal{E}_S$*,* $\mathcal{E}_U$ *and* $\mathcal{E}_C$ *which are invariant under the delay semigroup such that* $\mathcal{E}=\mathcal{E}_S\oplus\mathcal{E}_C\oplus\mathcal{E}_U$*,* $\dim\mathcal{E}_C<\infty$*,* $\dim\mathcal{E}_U<\infty$*, and*

- *the semigroup* $T_S(t)=T(t)|_{\mathcal{E}_S}$ *is uniformly exponentially stable,*
- *the semigroup* $T_U(t)=T(t)|_{\mathcal{E}_U}$ *is invertible and the semigroup* $T_U^{-1}(t)$ *is uniformly exponentially stable,*
- *the semigroup* $T_C(t)=T(t)|_{\mathcal{E}_C}$ *is a group being polynomially bounded in both time directions and hence having growth bound* 0 *in both directions.*

Proof. We first show that under the above conditions the operator $(\mathcal{A}, D(\mathcal{A}))$ has pure point spectrum with finite dimensional spectral subspaces. To this end, recall that $\lambda\in\sigma_{ess}(\mathcal{A})$ if and only if $\lambda\in\sigma_{ess}(B+\Phi_\lambda)=\emptyset$ since $B+\Phi_\lambda$ has compact resolvent by Lemma 2.10. Hence, by the characterization of the essential and point spectrum in Lemma 1.24, this is equivalent to the equality $\sigma(\mathcal{A})=\sigma_p(\mathcal{A})$.

Further, since

$$\lambda x=Bx+\Phi_\lambda x \quad\Leftrightarrow\quad \lambda\begin{pmatrix} x\\ \epsilon_\lambda x\end{pmatrix}=\mathcal{A}\begin{pmatrix} x\\ \epsilon_\lambda x\end{pmatrix},$$

the corresponding eigenspaces have the same finite dimension.

Consider now the following decomposition of $\sigma(\mathcal{A})$:

$$\begin{aligned} \Sigma_U &:= \sigma(\mathcal{A})\cap\{\lambda\in\mathbb{C}:\,\Re\lambda>0\}, \\ \Sigma_C &:= \sigma(\mathcal{A})\cap i\mathbb{R}, \\ \Sigma_S &:= \sigma(\mathcal{A})\cap\{\lambda\in\mathbb{C}:\,\Re\lambda<0\}. \end{aligned}$$

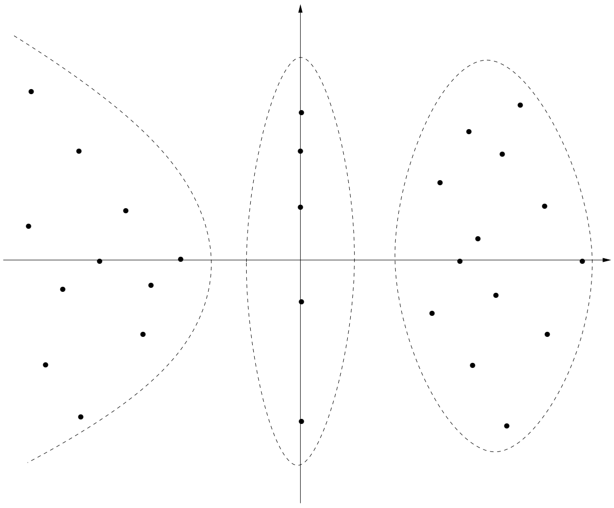

Figure 2: $\sigma(\mathcal{A}_S)$, $\sigma(\mathcal{A}_C)$ and $\sigma(\mathcal{A}_U)$

By our previous considerations the sets Σ_U and Σ_C are finite and the corresponding spectral projections P_U and P_C are finite dimensional (see e.g., [30, Section IV.1]). Let us denote the corresponding spectral subspaces by $\mathcal{E}_U := P_U\mathcal{E}$, $\mathcal{E}_C := P_C\mathcal{E}$ and $\mathcal{E}_S := P_S\mathcal{E}$ where P_S is the spectral projection corresponding to Σ_S. Then, by the construction,

$$\mathcal{E} = \mathcal{E}_S \oplus \mathcal{E}_C \oplus \mathcal{E}_U.$$

All these subspaces are invariant under the delay semigroup, and the parts of the generator $\mathcal{A}$ in the corresponding subspaces have spectrum Σ_S, Σ_C and Σ_U. Since the delay semigroup is eventually norm continuous, the spectral mapping theorem holds and the asymptotic behaviour is determined by the spectrum. The assertion follows since $\mathcal{A}_C$ has purely imaginary spectrum in the finite dimensional space $\mathcal{E}_C$, $\mathcal{A}_U$ has spectral values with strictly positive real part in the finite dimensional space $\mathcal{E}_U$, and $\mathcal{A}_S$ has strictly negative spectrum and generates an eventually norm continuous semigroup in the space $\mathcal{E}_S$. $\square$

The subspaces $\mathcal{E}_S$, $\mathcal{E}_U$ and $\mathcal{E}_C$ are usually referred to as the corresponding *stable*, *unstable* and *center* manifolds.

Notes and references 3.7. The material presented in this section can be found with some variations in almost every textbook on delay differential equations, though their approach is usually different to ours.

For stability criteria based on the spectral mapping theorem and the characteristic equations there exists an enormous literature, mainly using compactness. In particular, we mention the monographs by Diekmann et al. [29], Hale, Verduyn Lunel [42] and Wu [81]. For a deep spectral analysis using compactness see Verduyn Lunel [75, 76]. For a generalization of the decomposition result to delay equations with infinite delay see Milota [58].

Quite recently, after the preparation of this manuscript, T. Mátrai (see [57]) showed that under the assumtions of Theorem 3.6 the delay semigroup is eventually compact, which makes the proof of the decomposition theorem considerably easier.

Exercises 3.8. (1) Consider the scalar delay differential equation

$$u'(t) = -au(t) - bu(t-r)$$

with $a, b \in \mathbb{R}$ and analyze the stability of the solutions using Theorem 3.5 if

- (a) $a = 0$
- (b) $a = 2$, $b = 1$
- (c) $a = 1$, $b = 1$.

(2) Consider the heat equation treated in Example 3.4 and analyze its stability behaviour for some concrete values of a, b and r.

(3) Consider the following equation modelling heat conduction in a rod where the two endpoints of the rod are isolated.

$$\begin{cases} \partial_t u(x,t) = \partial_x^2 u(x,t) - bu(x,t-r),\ x \in [0,1],\ t \geq 0, \\ \partial_x u(0,t) = \partial_x u(1,t) = 0,\ t \geq 0, \\ u(x,t) = f(x,t),\ x \in [0,1],\ t \in [-r,0], \end{cases}$$

where $r > 0$, $b \in \mathbb{R}$.

- (a) Show that if $b > 0$ and $br < \frac{\pi}{2}$, then the solutions are uniformly exponentially stable.
- (b) Show that if $b > 0$ and $br = \frac{\pi}{2}$, then the unstable manifold is trivial and the center manifold is two dimensional. Identify it as a subspace of $\mathcal{E}_p$.

3.2 Stability via perturbation

In this lecture we continue to investigate the asymptotic behaviour of the solutions of delay differential equations. Last week we characterized uniform exponential stability in case of the generation of a compact semigroup, using so called "characteristic equations" as in Corollary 3.3. However, the exact analysis of the roots of this characteristic equation turns out to be rather complicated even for a simple scalar equation with two point delay or even with distributed delay. One needs a powerful and deep machinery even to get some partial results. However, it is not the main stream of these lectures but to use more functional analytic methods.

In many cases, even when the spectral mapping theorem holds, see e.g., formula (SMT) in Theorem 3.1, resolvent estimates are much easier to obtain rather than the computation of the spectral bound via the characteristic equation. Such resolvent estimates and their consequences for the stability of the delay semigroup are the topic of this section.

Our main tool are estimates for the abscissa of uniform boundedness of the resolvent

$$s_0(\mathcal{A}) := \inf \left\{ \omega \in \mathbb{R} \ : \ \{\Re\lambda > \omega\} \subset \rho(\mathcal{A}) \text{ and } \sup_{\Re\lambda > \omega} \|R(\lambda, \mathcal{A})\| < \infty \right\}$$

. In Hilbert spaces and for $p = 2$, this gives a stability result according to the theorem of Gearhart (see [30, Theorem V.1.11]). At the end of this section we extend these results to hyperbolicity.

To this purpose we restrict our attention to a class of delay operators Φ for which (DE) is wellposed for each generator B and a boundedness condition is satisfied. We start with a small but important technical detail which is emphasized here because only the presented properties of Φ will be used later on.

Lemma 3.9. *Assume the delay operator* $\Phi \in \mathcal{L}(W^{1,p}([-1,0],X),X)$ *is given by the Riemann-Stieltjes integral* (34)*. Then*

- *(a) the operator matrix* $(\mathcal{A}, D(\mathcal{A}))$ *generates a strongly continuous semigroup for all generators* $(B, D(B))$ *and*
- *(b) the function* $\lambda \mapsto \Phi R(\lambda, A_0)$ *is an analytic function which is bounded on each halfplane* $\{\lambda \in \mathbb{C} : \Re\lambda > \omega\}$ *for all* $\omega \in \mathbb{R}$*.*

It seems to be an **open question** whether there are examples of delay operators Φ satisfying condition (a) but not condition (b).

Proof. Assume that Φ is of the form (34), i.e.,

$$\Phi\, f := \int_{-1}^{0} d\eta\, f, \qquad f \in W^{1,p}([-1,0],X),$$

where $\eta : [-1,0] \to \mathcal{L}(X)$ is of bounded variation.

We have already shown in Theorem 1.20 that, for such delay operators Φ, $(\mathcal{A}, D(\mathcal{A}))$ generates a strongly continuous semigroup for all generators $(B, D(B))$. Hence we only have to verify the boundedness condition (b). This follows from the estimate

$$\begin{aligned}\|\Phi R(\lambda, A_0) f\| &= \left\| \int_{-1}^0 d\eta(\sigma) \int_\sigma^0 e^{\lambda(\sigma-\tau)} f(\tau) d\tau \right\| \le \int_{-1}^0 \int_\sigma^0 \|e^{\lambda(\sigma-\tau)} f(\tau)\| d\tau d|\eta|(\sigma) \\ &= \int_{-1}^0 \int_\sigma^0 e^{\Re\lambda(\sigma-\tau)} \|f(\tau)\| d\tau \, d|\eta|(\sigma) \le \int_{-1}^0 \int_\sigma^0 e^{-\omega} \|f(\tau)\| d\tau \, d|\eta|(\sigma) \\ &\le \int_{-1}^0 \int_{-1}^0 e^{-\omega} \|f(\tau)\| d\tau \, d|\eta|(\sigma) \le e^{-\omega} \, |\eta|([-1,0]) \|f\|_p\end{aligned}$$

for every λ with $\Re\lambda > \omega$ and every $f \in L^p([-1,0], X)$. ☐

We now obtain an estimate on the abscissa of uniform boundedness of the generator $\mathcal{A}$.

Theorem 3.10. *Assume that* Φ *is of the form* (34), $s_0(B) < 0$ *and let* $\alpha \in (s_0(B), 0]$ *such that*

$$a_{\alpha,n} := \sup_{\omega \in \mathbb{R}} \|(\Phi_{\alpha+i\omega} R(\alpha + i\omega, B))^n\| < \infty. \tag{48}$$

If

$$a_\alpha := \sum_{n=0}^\infty a_{\alpha,n} < \infty, \tag{49}$$

then $s_0(\mathcal{A}) < \alpha \le 0$.

Proof. We have to show the boundedness on the halfplane $\{\Re\lambda > \alpha\}$ of the resolvent operator given in (44).

Under our assumptions, this is equivalent to the existence and boundedness of $R(\lambda, B + \Phi_\lambda)$ on the halfplane $\{\Re\lambda > \alpha\}$.

Since $s_0(B) < \alpha$, using the appropriate version of the maximum principle we obtain that $a_{\beta,n} \le a_{\alpha,n}$ for $\beta > \alpha$.

Defining $M_\alpha := \sup_{\Re\lambda > \alpha} \|R(\lambda, B)\|$, we obtain for all $\lambda \in \mathbb{C}$ with $\Re\lambda > \alpha$ that

$$R(\lambda, B) \sum_{n=0}^\infty (\Phi_\lambda R(\lambda, B))^n \in \mathcal{L}(X)$$

and

$$\begin{aligned}\left\| R(\lambda, B) \sum_{n=0}^\infty (\Phi_\lambda R(\lambda, B))^n \right\| &\le M_\alpha \sum_{n=0}^\infty \| (\Phi_\lambda R(\lambda, B))^n \| \\ &\le M_\alpha a_{\Re\lambda} \le M_\alpha a_\alpha.\end{aligned}$$

This operator is the inverse of $(\lambda - B - \Phi_\lambda)$ and is bounded on the halfplane $\{\Re\lambda > \alpha\}$. Hence, using the fact that the infimum $s_0(\mathcal{A})$ is never attained, we have $s_0(\mathcal{A}) < \alpha$. ☐

We now consider a direct application of Theorem 3.10 and treat the special case where $\Phi := C\delta_{-1}$ for an operator $C \in \mathcal{L}(X)$ commuting with $(B, D(B))$.

Corollary 3.11. *Assume that* $\Phi = C\delta_{-1}$ *for some* $C \in \mathcal{L}(X)$ *commuting with* $(B, D(B))$, $s_0(B) < 0$ *and,*

$$r(C) < \frac{1}{\sup_{\omega\in\mathbb{R}} \|R(i\omega, B)\|}.$$

Then $s_0(\mathcal{A}) < 0$.

Proof. Take $M := \sup_{\omega\in\mathbb{R}} \|R(i\omega, B)\|$ and assume that

$$r(C) \cdot M < q < 1.$$

We then obtain that there exists $n_0 \in \mathbb{N}$ such that

$$\|C^n\|^{\frac{1}{n}} \cdot M < q < 1 \text{ for all } n \geq n_0.$$

This implies

$$\begin{aligned} a_{0,n} &= \sup_{\omega\in\mathbb{R}} \|(\Phi_{i\omega} R(i\omega, B))^n\| \leq \sup_{\omega\in\mathbb{R}} \|R(i\omega, B)^n\| \sup_{\omega\in\mathbb{R}} \|\Phi_{i\omega}^n\| \\ &\leq \|C^n\| \cdot M^n < q^n \end{aligned}$$

for all $n \geq n_0$.

We conclude that (49) holds for $\alpha = 0$ and the assertion follows by Theorem 3.10. ☐

We now formulate an important special case of Theorem 3.10 which will be applied to estimate the growth bound of the delay semigroup if X is a Hilbert space and $p = 2$.

Corollary 3.12. *Assume that* Φ *is of the form* (34), $s_0(B) < 0$, *and let* $\alpha \in (s_0(B), 0]$. *If*

$$\sup_{\omega\in\mathbb{R}} \|\Phi_{\alpha+i\omega} R(\alpha + i\omega, B)\| < 1,$$

or in particular if

$$\sup_{\omega\in\mathbb{R}} \|\Phi_{\alpha+i\omega}\| < \frac{1}{\sup_{\omega\in\mathbb{R}} \|R(\alpha + i\omega, B)\|}, \tag{50}$$

then $s_0(\mathcal{A}) < \alpha \leq 0$.

Proof. Defining $q_\alpha := \sup_{\omega\in\mathbb{R}} \|\Phi_{\alpha+i\omega}R(\alpha+i\omega,B)\| < 1$, we obtain that $a_{\alpha,n} \le q_\alpha^n$, hence the series $\sum a_{\alpha,n}$ is convergent. ☐

For Hilbert spaces, these results imply stability estimates by the Theorem of Gearhart, see [30, Theorem V.1.11]. We state this only in the case (50), the other results being analogous.

Corollary 3.13. *Assume that X is a Hilbert space and that $p = 2$. If Φ is of the form* (34)*, $\alpha \in (\omega_0(B), 0]$ and*

$$\sup_{\omega\in\mathbb{R}} \|\Phi_{\alpha+i\omega}\| < \frac{1}{\sup_{\omega\in\mathbb{R}} \|R(\alpha+i\omega,B)\|}, \tag{51}$$

then $\omega_0(\mathcal{A}) < \alpha \le 0$.

Moreover, if $(B, D(B))$ is a normal operator, we can give a very useful condition implying (50).

Corollary 3.14. *Assume that X is a Hilbert space, $p = 2$, $(B, D(B))$ is a normal operator generating a strongly continuous semigroup and Φ is of the form (34), i.e., there is a function $\eta \in BV([-1,0],\mathcal{L}(X))$ such that $\Phi(f) := \int_{-1}^0 d\eta\, f$. If $\omega_0(B) < 0$ and*

$$|\eta|([-1,0]) < |s(B)|,$$

then $\omega_0(\mathcal{A}) < 0$.

Proof. We show that the inequalities

$$\sup_{\omega\in\mathbb{R}} \|\Phi_{i\omega}\| \le |\eta|([-1,0]) < |s(B)| = \frac{1}{\sup_{\omega\in\mathbb{R}} \|R(i\omega,B)\|}$$

hold. Then the assertion follows by Corollary 3.13. For $\omega \in \mathbb{R}$ we have that

$$\|\Phi_{i\omega}\| = \left\| \int_{-1}^0 e^{i\omega s} d\eta(s) y \right\| \le \|y\| \cdot \left| \int_{-1}^0 e^{i\omega s} d|\eta|(s) \right| \le \|y\| \cdot |\eta|([-1,0]). \tag{52}$$

The right hand side, since $(B, D(B))$ is a normal operator on a Hilbert space, can be computed as (see Kato [49, Section V.3.8])

$$\sup_{\omega\in\mathbb{R}} \|R(i\omega,B)\| = \sup_{\omega\in\mathbb{R}} \frac{1}{d\,(i\omega,\sigma(B))} == \frac{1}{d\,(\mathbb{R},\sigma(B))} = \frac{1}{|s(B)|}. \tag{53}$$

☐

In the following we test our results in two examples.

Example 3.15. We consider the *reaction–diffusion equation with delay*

$$\begin{cases} \partial_t w(x,t) = \Delta w(x,t) + c\int_{-1}^{0} w(x,t+\tau)dg(\tau), \\ \quad x \in \Omega,\ t \geq 0, \\ w(x,t) = 0, \quad x \in \partial\Omega,\ t \geq 0, \\ w(x,t) = f(x,t), \quad (x,t) \in \Omega \times [-1,0], \end{cases} \tag{RDD}$$

where c is a constant, $\Omega \subset \mathbb{R}^n$ a bounded open set, $f(\cdot,t) \in L^2(\Omega)$ for all $t \geq 0$, $f(\cdot,0) \in D := \{u \in H_0^1(\Omega) : \Delta u \in L^2(\Omega)\}$ and the map $[-1,0] \ni t \mapsto f(\cdot,t) \in L^2(\Omega)$ belongs to $W^{1,2}\left([-1,0], L^2(\Omega)\right)$. The function $g : [-1,0] \to [0,1]$ is supposed to be the Cantor function (see Gelbaum, Olmsted [35, Example I.8.15]), which is singular and has total variation 1.

In order to apply our results, we take

- $X := L^2(\Omega)$,
- $B := \Delta_D$ the Dirichlet-Laplacian with usual domain $D(\Delta_D) = D$ and
- $\eta := c \cdot g \cdot Id$.

Before going on, we mention that, though all conditions of Corollary 3.3 are satisfied, it is practically not possible to yield exact solutions of the corresponding characteristic equations.

We will verify the stability estimate (51). First, the expression with Φ satisfies

$$\|\Phi_\lambda y\| = |c|\|y\| \left\| \int_{-1}^{0} e^{\lambda\tau} dg(\tau) \right\| \leq |c|\|y\|.$$

The other expression, using that $B = \Delta_D$ is a normal operator on a Hilbert space, was calculated in (53). We can summarize as follows.

Corollary 3.16. *The above reaction–diffusion equation with delay* (RDD) *is well-posed. Moreover, the solutions decay exponentially if*

$$|c| < |\lambda_1|.$$

We refer, e.g., to Davies [24, Chapter 6] for estimates on λ_1 and for further references. The same result holds for general elliptic operators as considered in [24, Section 6.3].

Our next example is of importance because there neither has the generator compact resolvent nor can we expect the Spectral Mapping Theorem to hold, and hence there is no theory to justify the use of characteristic equations.

Example 3.17. Consider the following first order differential equation with delay

$$\begin{cases} \partial_t u(x,t) = -\partial_x u(x,t) - \mu u(x,t) + \int_{-1}^{0} u(x,t+s)dg(s), \\ x \in \mathbb{R},\ t \geq 0, \\ u(0,t) = 0, \quad t \geq 0, \\ u(x,s) = f(x,s), \quad x \in \mathbb{R},\ t \in [-1,0], \end{cases}$$

where $\mu > 0$ and g is a function of bounded variation. We treat this equation in Hilbert spaces by considering

- $X := L^2(\mathbb{R})$,
- $B := -\partial_x - \mu$ with domain $D(B) = H^1(\mathbb{R})$,
- $\eta := c \cdot g \cdot Id$.

The well-posedness follows again from Theorem 1.20. Next, we verify the stability estimate (51). First, the expression with Φ satisfies

$$\|\Phi_\lambda y\| = |c|\|y\| \left\| \int_{-1}^{0} e^{\lambda\tau} dg(\tau) \right\| \leq |c| \mathrm{Var}(g)_{-1}^{0} \|y\|.$$

The other expression, using that $B = \partial_x - \mu$ is a normal operator on a Hilbert space, can be computed as

$$\sup_{\omega \in \mathbb{R}} \|R(i\omega, B)\| = \frac{1}{|\mu|}.$$

We can summarize as follows.

Corollary 3.18. *The above first order partial differential equation with delay is well-posed. Moreover, the solutions decay exponentially if*

$$\mathrm{Var}(g)_{[-1,0]} < |\mu|.$$

We now formulate the analogous results on hyperbolicity in Hilbert spaces using the corresponding version of the Theorem of Gearhart, see [30, Theorem V.1.17].

Theorem 3.19. *Let X be a Hilbert space and consider the equation* (DE). *Assume that Φ is of the form* (34), *the semigroup $(B, D(B))$ generates a hyperbolic semigroup, and consider*

$$a_n := \sup_{\omega \in \mathbb{R}} \|(\Phi_{i\omega} R(i\omega, B))^n\| < \infty, \quad n \in \mathbb{N}. \tag{54}$$

If

$$a := \sum_{n=0}^{\infty} a_n < \infty, \tag{55}$$

then $(\mathcal{A}, D(\mathcal{A}))$ generates a hyperbolic semigroup.

Proof. As a consequence of the above mentioned theorem of Gearhart, the numbers a_n are defined for all $n \in \mathbb{N}$, and we only have to show the boundedness of the resolvent operator given in (44) on the line $i\mathbb{R}$. Under our assumptions, this is equivalent to the existence and boundedness of $R(\lambda, B + \Phi_\lambda)$ for $\lambda \in i\mathbb{R}$.

Let $\lambda \in i\mathbb{R}$ and $M := \sup_{\lambda \in i\mathbb{R}} \|R(\lambda, B)\|$. We then obtain

$$R(\lambda, B) \sum_{n=0}^{\infty} (\Phi_\lambda R(\lambda, B))^n \in \mathcal{L}(X)$$

and

$$\begin{aligned} \left\| R(\lambda, B) \sum_{n=0}^{\infty} (\Phi_\lambda R(\lambda, B))^n \right\| &\leq M \sum_{n=0}^{\infty} \| (\Phi_\lambda R(\lambda, B))^n \| \\ &\leq M \sum_{n=0}^{\infty} a_n = M \cdot a. \end{aligned}$$

This operator defines the inverse of $(\lambda - B - \Phi_\lambda)$ and remains bounded on $i\mathbb{R}$. □

Using the same arguments, we can formulate the analogous version of Corollary 3.12.

Corollary 3.20. *Let X be a Hilbert space and consider the equation* (DE). *Assume that Φ is of the form* (34) *and that the semigroup $(B, D(B))$ generates a hyperbolic semigroup. If*

$$\sup_{\omega \in \mathbb{R}} \|\Phi_{i\omega} R(i\omega, B)\| < 1, \tag{56}$$

or in particular if

$$\sup_{\omega \in \mathbb{R}} \|\Phi_{i\omega}\| < \frac{1}{\sup_{\omega \in \mathbb{R}} \|R(i\omega, B)\|}, \tag{57}$$

then $(\mathcal{A}, D(\mathcal{A}))$ generates a hyperbolic semigroup.

Notes and references 3.21. In Section 3.2 the results are mainly taken from [7] and [2]. Stability criteria without using characteristic equations can be found, e.g., in Fischer, van Neerven [32, Corollary 3.6] and Curtain, Zwart [16, Theorem 5.1.7]. To our knowledge the hyperbolicity and the existence of stable, center and unstable manifolds in infinite dimensions has not been investigated until now in this context.

See also the review article Schnaubelt [71] for related wellposedness and hyperbolicity results in the non-autonomous case.

Exercises 3.22. (1) Consider the reaction-diffusion equation with delay

$$\begin{cases} \partial_t w(x,t) = \Delta w(x,t) + c \int_{-1}^{0} w(x,t+\tau) dh(\tau), \\ \quad x \in \Omega, \ t \geq 0, \\ w(x,t) = 0, \quad x \in \partial\Omega, \ t \geq 0, \\ w(x,t) = f(x,t), \quad (x,t) \in \Omega \times [-1,0], \end{cases} \tag{RDU}$$

where c is a constant, h is of bounded variation and $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ a domain (not necessarily bounded) such that the Laplace operator generates an exponentially stable semigroup in $L^2(\Omega)$. We refer, e.g., to Dautray-Lions [22, Vol. 2, Proposition IV.7.1.1] for the following sufficient condition:

$$\Omega \subseteq \{x \in \mathbb{R}^n : a \leq \xi \cdot x \leq b\} \quad \text{for some } \xi \in \mathbb{R}^n \text{ and } a < b. \tag{58}$$

In this case the Dirichlet–Laplacian in $L^2(\Omega)$ is invertible, but its resolvent is not necessarily compact. Denoting with $s(\Delta)$ the spectral bound of the Dirichlet–Laplacian, show that if

$$|c| \mathrm{Var}(h)_{[-1,0]} < |s(\Delta)|,$$

then the solutions of (RDU) are (uniformly) exponentially stable.

(2) consider the reaction–diffusion equation from Example 3.15, but now in the state space $\mathcal{E} := L^r(\Omega) \times L^p([-1,0], L^r(\Omega))$ for $1 \leq r < \infty$, $1 < p < \infty$. Show that the solutions decay exponentially if

$$|c| < |\lambda_1|,$$

thereby extending our result from the Hilbert space case. (Hint: See [7, Example 5.1].)

3.3 The effect of small delays

It has been observed by Datko [20] (see also [17, 19]) that small delays may destroy stability in a partial differential equation. This phenomenon has important consequences for the applications and will be studied systematically in this section. More precisely, we start from an abstract Cauchy problem

$$(\mathrm{DE})_0 \qquad \begin{cases} u'(t) = (B+C)u(t), \quad t \geq 0, \\ u(0) = x, \end{cases}$$

where $(B, D(B))$ generates a strongly continuous semigroup $(S(t))_{t\geq 0}$and $C \in \mathcal{L}(X)$.For this problem we assume that the solutions are uniformly exponentially stable (or hyperbolic).

The question is, whether stability or hyperbolicity prevails for the solutions of the delayed equation

$$(\mathrm{DE})_\tau \qquad \begin{cases} u'(t) = Bu(t) + Cu(t-\tau), \quad t \geq 0, \\ u(0) = x, \\ u_0 = f, \end{cases}$$

where $\tau > 0$ is “small”.

Before considering the general problem, we demonstrate by some simple examples that indeed the stability can be destroyed by small delays.

Example 3.23. Let $(B, D(B))$ be the (unbounded) generator of a unitary group on an infinite dimensional Hilbert space H and let $C := d \cdot Id$ for $d < 0$. Then $(B + C, D(B))$ generates an exponentially stable semigroup. We show that there exists a sequence (τ_k), $\tau_k \searrow 0$, such that the solution semigroup of each of the corresponding equations $(\mathrm{DE})_{\tau_k}$ does not decay exponentially.

To construct this sequence, take $(\mu_k) \subset \mathbb{R}$ such that $i\mu_k \in \sigma(B)$, $|\mu_k| \to \infty$ and $\mu_k \neq -d$. If we define

$$\tau_k := \begin{cases} \frac{3\pi}{2(\mu_k+d)}, & \mu_k + d > 0, \\ \frac{-\pi}{2(\mu_k+d)}, & \mu_k + d < 0, \end{cases}$$

we obtain $\lambda_k \in \sigma(B + \Phi_{\lambda_k})$ for the numbers $\lambda_k := (\mu_k + d)i \in i\mathbb{R}$. By the spectral characterization in Exercise 1.14 (5) in Chapter 1 it follows that the associated operator $(\mathcal{A}, D(\mathcal{A}))$ cannot generate a uniformly exponentially stable semigroup.

In order to find conditions under which stability is not sensitive to small delays, we use an idea similar to Hale, Verduyn Lunel [42, Section 5.4 (4.9)] and transform the equation $(\mathrm{DE})_\tau$ into

$$u'(t) = (B + C)u(t) + C\left(u(t-\tau) - u(t)\right)$$

for $t \geq \tau$.

The main task is now to transform $\Phi u_t = C\left(u(t-\tau) - u(t)\right)$ into a form such that it is possible to use the estimates from Corollary 3.12.

Integration on $\tau \leq t_1 \leq t_2$ the original form of $(\mathrm{DE})_\tau$ yields the variation of constants formula

$$u(t_2) - u(t_1) = \left[S(t_2 - t_1) - Id\right] u(t_1) + \int_{t_1}^{t_2} S(t_2 - s) C u(s - \tau) ds, \tag{59}$$

see also Exercise 1. Here the semigroup $(S(t))_{t\geq 0}$denotes the semigroup generated by $(B, D(B))$. By taking $t_1 = t - \tau$ and $t_2 = t$, we obtain that

$$u(t) - u(t-\tau) = \left[S(\tau) - Id\right] u(t-\tau) + \int_{-\tau}^{0} S(-s) C u(t+s-\tau) ds.$$

Thus, $(\mathrm{DE})_\tau$ can be written in the form

$$\begin{aligned} u'(t) &= (B+C)u(t) \\ &\quad - C\left(\left[S(\tau) - Id\right] u(t-\tau) + \int_{-\tau}^{0} S(-s) C u(t+s-\tau) ds\right). \end{aligned} \tag{60}$$

Define now

$$\Phi f := -C\left[S(\tau) - Id\right] \delta_{-\tau} f - \int_{-\tau}^{0} C S(-s) C \delta_{s-\tau} f ds, \tag{61}$$

where $\delta_r \in \mathfrak{L}\left(W^{1,p}([-1,0], X), X\right)$ is given by $\delta_r(f) := f(r)$ for $r \in [-1, 0]$. Then $(\mathrm{DE})_\tau$ has the form

$$u'(t) = (B + C)\, u(t) + \Phi u_t, \tag{62}$$

where $(B + C, D(B))$ generates a uniformly exponentially stable or hyperbolic semigroup and Φ is of the form (34).

In order to apply the stability results from Corollary 3.12, or the hyperbolicity results from Corollary 3.20, we have to calculate for $x \in X$

$$\begin{aligned} \Phi_\lambda R(\lambda, B+C)x = &- C\left[S(\tau) - Id\right] e^{-\lambda\tau} R(\lambda, B+C)x \\ &- \int_{-\tau}^{0} C S(-s) C e^{-\lambda(s-\tau)} R(\lambda, B+C)x ds. \end{aligned} \tag{63}$$

As we saw in Example 3.23, the unboundedness of the spectrum of $(B, D(B))$ along imaginary axes may cause trouble. Therefore, we make an additional assumption on the semigroup generated by B implying that $\sigma(B)$ is bounded along the imaginary lines, see [30, Theorem II.4.18].

Theorem 3.24. *Assume that* $(B, D(B))$ *generates an immediately norm continuous semigroup* $(S(t))_{t\geq 0}$ *and that the semigroup generated by* $(B+C, D(B))$ *is exponentially stable (hyperbolic, resp.) on the Hilbert space* X*. Then there exists* $\kappa > 0$ *such that the solution semigroup of* $(\mathrm{DE})_\tau$ *is exponentially stable (hyperbolic, resp.) for all* $\tau \in (0, \kappa)$*. Thus, in this situation stability and hyperbolicity are not sensitive to small delays.*

Proof. Define

$$I_1^\omega(\tau) := C\left[S(\tau) - Id\right] e^{-i\omega\tau} R(i\omega, B+C) \tag{64}$$

and

$$I_2^\omega(\tau)x := \int_{-\tau}^{0} CS(-s)Ce^{-i\omega(s-\tau)} R(i\omega, B+C)xds \tag{65}$$

for $x \in X$.

We show that there exists $\kappa > 0$ such that $\sup_{\omega\in\mathbb{R}} \|I_i^\omega(\tau)\| < \frac{1}{2}$ for $i = 1, 2$ and all $\tau \in (0, \kappa)$. Then, using Corollary 3.12 or Corollary 3.20 and (63), the assertion follows since $\sup_{\omega\in\mathbb{R}} \|\Phi_{i\omega} R(i\omega, B+C)\| \leq \sup_{\omega\in\mathbb{R}} \|I_1^\omega(\tau)\| + \sup_{\omega\in\mathbb{R}} \|I_2^\omega(\tau)\| < 1$.

The estimate on I_2^ω is

$$\|I_2^\omega(\tau)\| \leq \tau \|C\|^2 K \|R(i\omega, B+C)\|, \tag{66}$$

where $K := \sup_{0\leq t\leq 1} \|S(t)\|$. Since $\|R(i\omega, B+C)\|$ is uniformly bounded for all $\omega \in \mathbb{R}$, there exists $\kappa_2 > 0$ such that for all $\tau \in (0, \kappa_2)$ the estimate $\sup_{\omega\in\mathbb{R}} \|I_2^\omega(\tau)\| < \frac{1}{2}$ holds.

The estimate on I_1^ω is

$$\begin{aligned} \|I_1^\omega(\tau)\| &\leq \|C\| \cdot \|(S(\tau) - Id)R(i\omega, B+C)\| \\ &\leq \|C\| \cdot \|(S(\tau) - Id)R(\lambda, B)\| \cdot \|(\lambda - B)R(i\omega, B+C)\|, \end{aligned} \tag{67}$$

where $\lambda > \max\{\omega_0(B), 0\}$ is fixed.

Since $\|(\lambda - B)R(i\omega, B+C)\|$ is independent of τ, we only have to consider $(S(\tau) - Id)R(\lambda, B)$.

But then it follows from

$$\begin{aligned} \|(S(\tau) - Id)R(\lambda, B)\| &\leq \|S(\tau)\|(1 - e^{-\lambda\tau})\|R(\lambda, B)\| + \|\int_0^\tau e^{-\lambda s} S(s)ds\| \\ &\leq \tau K\left(\|R(\lambda, B)\||\lambda| + 1\right) \end{aligned}$$

that

$$\lim_{\tau\to 0} \|C\left[S(\tau) - Id\right] e^{i\omega\tau} R(i\omega, B+C)\| = 0 \tag{68}$$

for every $\omega \in \mathbb{R}$.

To finish the proof we have to show that this convergence is uniform in ω.

To this end we use the immediate norm continuity of the semigroup generated by $(B+C, D(B))$, see [30, Theorem III.1.16(i)]. An important property of there semigroups is that $\lim_{|\omega|\to\infty} \|R(i\omega, B+C)\| = 0$, see [30, Corollary II.4.19]. Thus, there exists $L > 0$ such that

$$\|R(i\omega, B+C)\| < \frac{1}{2\|C\|(K+1)} \quad \text{for } |\omega| > L,$$

where $K := \sup_{0\leq t\leq 1} \|S(t)\|$.

For $\omega \in [-L, L]$, we recall that the function

$$(\omega, \tau) \mapsto \|C\,[S(\tau) - Id]\, e^{i\omega\tau} R(i\omega, B+C)\|$$

is uniformly continuous on $[-L, L] \times [0, 1]$. Thus, there exists $\kappa_1 > 0$ such that for all $\tau \in (0, \kappa_1)$ and for all $\omega \in [-L, L]$

$$\|C\,[S(\tau) - Id]\, e^{i\omega\tau} R(i\omega, B+C)\| < \frac{1}{2}.$$

Combining these estimates we obtain the desired statement.

The proof can be finished by choosing $\kappa := \min\{\kappa_1, \kappa_2\}$. □

In the previous theorem we gave a condition on the generator $(B, D(B))$ without any restriction on the stabilizing operator C. In Exercise 2 we provide a condition also involving C.

Notes and references 3.25. The material presented here is mainly based on the articles [2, 6, 4]. The first who examined the effect of small delays was R. Datko [17, 18, 19, 20, 21]. It is known that for finite dimensional equations the stability cannot be destroyed by small delays. In addition, there exists an extensive literature on delay dependent stability conditions, see, e.g., the papers by Győri and coauthors [38, 39]. For similar questions in the non-autonomous parabolic case we refer to Gühring, Räbiger and Schnaubelt [37, 72].

There is a recent exposition of this problem by J. Hale and S. Verduyn Lunel [44, 43], where many examples of functional differential and difference equations are considered. A control theoretic investigation using transfer functions was made for compact feedback in Rebarber, Townly [67].

It is important to mention that under an additional positivity assumption the stability is delay independent, see Kerscher, Nagel [50], and hence cannot be destroyed.

Exercises 3.26. (1) Prove formula (59).

(2) Let $(B, D(B))$ be a generator of a strongly continuous semigroup $(S(t))_{t\geq 0}$on the Hilbert space X, let $C \in \mathcal{L}(X)$ be a compact operator commuting with $S(t)$, $t \geq 0$. Assume that the semigroup generated by $(B + C, D(B))$ is exponentially stable (hyperbolic, resp.). Show that there exists $\kappa > 0$ such that the solution semigroup of $(\mathrm{DE})_\tau$ is exponentially stable (hyperbolic, resp.) for all $\tau \in (0, \kappa)$. Thus, in this case stability and hyperbolicity are not sensitive to small delays.

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