

A MAXIMAL INEQUALITY FOR THE TAIL OF THE BILINEAR HARDY-LITTLEWOOD FUNCTION

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ABSTRACT. Let (X, \mathcal{B}, μ, T) be an ergodic dynamical system on a non-atomic finite measure space. Consider the maximal function $R^* : (f, g) \in L^p \times L^q \rightarrow R^*(f, g)(x) = \sup_n \frac{f(T^n x)g(T^{2n} x)}{n}$. We obtain the following maximal inequality. For each $1 < p \leq \infty$ there exists a finite constant C_p such that for each $\lambda > 0$, and nonnegative functions $f \in L^p$ and $g \in L^1$

$$\mu\{x : R^*(f, g)(x) > \lambda\} \leq C_p \left(\frac{\|f\|_p \|g\|_1}{\lambda} + \left(\frac{\|f\|_p \|g\|_1}{\lambda} \right)^{1/2} \right).$$

We also show that for each $\alpha > 2$ the maximal function $R^*(f, g)$ is a.e. finite for pairs of functions $(f, g) \in (L(\log L)^{2\alpha}, L^1)$.

1. INTRODUCTION

In [1] we proved the following maximal inequality about the maximal function $R^*(f, g)(x) = \sup_n \frac{f(T^n x)g(T^{2n} x)}{n}$. For each $1 < p \leq \infty$, there exists a finite constant C'_p such that for each $\lambda > 0$, for every $f \in L^p, f > 1$ and $g \in L^1, g > 1$

$$(1) \quad \mu\{x : R^*(f, g)(x) > \lambda\} \leq C'_p \left(\frac{\|f\|_p^p \|g\|_1}{\lambda} \right)^{1/2}.$$

Furthermore the constant C'_p behaves like $\frac{1}{p-1}$ when p tends to 1. To be more precise, we will use that there exists \tilde{C}' such that for any $1 < p < 2$ we have

$$(2) \quad C'_p \leq \frac{\tilde{C}'}{p-1}.$$

The first author acknowledges support by NSF grant DMS 0456627. The second listed author was partially supported by NKTH and by the Hungarian National Foundation for Scientific Research T049727.

2000 Mathematics Subject Classification: Primary 37A05; Secondary 37A45.

Keywords: Maximal inequality, maximal function .



Inequality (1) was enough to prove the a.e. convergence to zero of the tail $\frac{f(T^n x)g(T^{2n} x)}{n}$ of the double recurrence averages $\frac{1}{n} \sum_{k=1}^n f(T^k x)g(T^{2k} x)$ for pairs of functions (f, g) in $L^p \times L^1$ (or $L^1 \times L^p$) as soon as $p > 1$. On the other hand, the tail was used to show that these averages do not converge a.e. for pairs of (L^1, L^1) functions.

During the 2007 Ergodic Theory workshop at UNC-Chapel Hill, J.P. Conze asked if this inequality could be made homogeneous with respect to f and g . In this paper first we derive from (1) the following homogeneous version.

Theorem 1. *For each $1 < p < \infty$ there exists a finite constant C_p such that for each $f, g \geq 0$ and for all $\lambda > 0$ we have*

$$(3) \quad \mu\left\{x : \sup_n \frac{f(T^n x)g(T^{2n} x)}{n} > \lambda\right\} \leq C_p \left[\left(\frac{\|f\|_p \|g\|_1}{\lambda} \right)^{1/2} + \frac{\|f\|_p \|g\|_1}{\lambda} \right],$$

and there exists \tilde{C} such that for any $1 < p < 2$ we have

$$(4) \quad C_p \leq \frac{\tilde{C}}{p-1}.$$

At the same meeting a question was raised about the a.e. finiteness of $R^*(f, g)$ for pairs of functions in $(L \log L, L^1)$. Our second result is based on an adaptation of Zygmund's extrapolation method [3] (vol. II, ch. XII, pp. 119-120) to $R^*(f, g)$. With somewhat crude estimates we prove the following theorem.

Theorem 2. *If $\alpha > 2$ and the pair of nonnegative functions (f, g) belongs to $(L(\log L)^{2\alpha}, L^1)$ then $R^*(f, g) = \sup_n \frac{f(T^n x)g(T^{2n} x)}{n}$ is a.e. finite.*

2. PROOFS

Proof of Theorem 1. First we can notice that the original inequality (1) is homogeneous with respect to the L^1 function g . Indeed, a simple change of variables shows that the case $g > t$ can easily be obtained from the case $g > 1$ with the same constant C'_p . So by approximating g with $g_n(x) = \max\{g(x), 1/n\}$ we can see that (1) holds if the assumption $g > 1$ is replaced by $g \geq 0$. Without loss of generality we can also suppose in the sequel that $\|g\|_1 = 1$.

If $\|f\|_p = 0$ we have nothing to prove. Otherwise, if we can show that (3) holds for $\tilde{f} = f/\|f\|_p$ for all $\lambda > 0$, then this implies that it is true for f as well for all $\lambda > 0$. Thus, we just need to prove (3) for $f \in L^p$ with $\|f\|_p = 1$. We split the function f into the sum $f_1 + f_2$ where $f_1 = f$ if $f \leq 1$.

We have then

$$\mu\left\{x : \sup_n \frac{f(T^n x)g(T^{2n} x)}{n} > \lambda\right\} \leq$$

$$\mu\left\{x : \sup_n \frac{f_1(T^n x)g(T^{2n} x)}{n} > \lambda/2\right\} + \mu\left\{x : \sup_n \frac{f_2(T^n x)g(T^{2n} x)}{n} > \lambda/2\right\} = M_1 + M_2.$$

First we estimate M_1 . Since

$$\sup_n \frac{f_1(T^n x)g(T^{2n} x)}{n} \leq \sup_n \frac{g(T^{2n} x)}{n}$$

by the standard maximal inequality for the ergodic averages (see [2], for instance) we obtain

$$M_1 \leq \frac{2\|g\|_1}{\lambda} = \frac{2}{\lambda} = \frac{2\|f\|_p\|g\|_1}{\lambda}.$$

To estimate M_2 , we dominate f_2 with the function $h = \max\{f_2, 1\}$. By our remark about the assumption $g \geq 0$ the maximal inequality (1) is applicable and we obtain that $M_2 \leq C'_p \left(\frac{\|h\|_p^2}{\lambda/2}\right)^{1/2}$, and (2) also holds for $1 < p < 2$. As $\|h\|_p \leq \|\mathbf{1}\|_p + \|f_2\|_p \leq \|\mathbf{1}\|_p + \|f\|_p = 2$ we have the estimate

$$M_2 \leq 2^p \sqrt{2} C'_p \left(\frac{1}{\lambda}\right)^{1/2} = 2^p \sqrt{2} C'_p \left(\frac{\|f\|_p\|g\|_1}{\lambda}\right)^{1/2},$$

with C'_p satisfying (2) for $1 < p < 2$. By combining the upper bounds for M_1 and M_2 we obtain

$$\begin{aligned} \mu\left\{x : \sup_n \frac{f(T^n x)g(T^{2n} x)}{n} > \lambda\right\} &\leq \frac{2\|f\|_p\|g\|_1}{\lambda} + 2^p \sqrt{2} C'_p \left(\frac{\|f\|_p\|g\|_1}{\lambda}\right)^{1/2} \leq \\ &C_p \left(\left(\frac{\|f\|_p\|g\|_1}{\lambda}\right)^{1/2} + \frac{\|f\|_p\|g\|_1}{\lambda} \right) \end{aligned}$$

with $C_p = \max\{2^p \sqrt{2} C'_p, 2\}$ and from (2) it follows that there exist \tilde{C} such that (4) holds for $1 < p < 2$. \square

Proof of Theorem 2. The starting point is (3) and (4).

There exists a finite constant \tilde{C} such that for every $1 < p < 2$, for each $f, g \geq 0$ and for all $\lambda > 0$ we have

$$(5) \quad \mu\left\{x : \sup_n \frac{f(T^n x)g(T^{2n} x)}{n} > \lambda\right\} \leq \frac{\tilde{C}}{p-1} \left[\left(\frac{\|f\|_p\|g\|_1}{\lambda}\right)^{1/2} + \frac{\|f\|_p\|g\|_1}{\lambda} \right].$$

We can again assume without loss of generality that $\|g\|_1 = 1$. We fix the function g and denote by $R^*(f)(x)$ the maximal function $\sup_n \frac{f(T^n x)g(T^{2n} x)}{n}$. Now we can rewrite (5) as

$$(6) \quad \mu\left\{x : R^*(f)(x) > \lambda\right\} \leq \frac{\tilde{C}}{p-1} \left[\left(\frac{\|f\|_p}{\lambda}\right)^{1/2} + \frac{\|f\|_p}{\lambda} \right].$$

The important element for the extrapolation is the factor $\frac{1}{p-1}$ in the above inequality.

Our goal is to prove that for $\alpha > 2$ there is C_α such that for any $f \in L(\log L)^{2\alpha}$ we have for each $\lambda > 0$

$$(7) \quad \mu\{x : R^*(f)(x) > \lambda\} \leq C_\alpha \left(\frac{[1 + (f|f|(\log^+ |f|)^{2\alpha})^{1/2}]}{\lambda^{1/2}} + \frac{[1 + \int |f|(\log^+ |f|)^\alpha]}{\lambda} \right).$$

Let γ_j be a positive sequence of numbers such that $\sum_{j=0}^{\infty} \gamma_j = 1$.

The function f being in $L(\log L)^{2\alpha}$ we have $\sum_{j=1}^{\infty} j^{2\alpha} 2^j \mu\{x : 2^j \leq f < 2^{j+1}\} < \infty$.

We denote by t_j the quantity $\mu\{2^j \leq f < 2^{j+1}\}$, by f_j the function $2^j \mathbf{1}_{\{x: 2^j \leq f < 2^{j+1}\}}$ and by p_j the number $1 + \frac{1}{j}$. We set $f_0(x) = f(x)$ if $0 \leq f(x) < 2$, otherwise we put $f_0(x) = 0$. Then

$$(8) \quad f \leq 2 \sum_{j=0}^{\infty} f_j.$$

We also have

$$(9) \quad \mu\{x : R^*(f_0)(x) > \frac{\lambda\gamma_0}{2}\} \leq \mu\{x : R^*(2 \cdot \mathbf{1}_X)(x) > \frac{\lambda\gamma_0}{2}\} \leq \frac{4\|g\|_1}{\lambda\gamma_0} = \frac{4}{\lambda\gamma_0}$$

by the standard maximal inequality for the ergodic averages.

For $j \geq 1$ by (6) used with $p_j = 1 + \frac{1}{j}$ we obtain

$$(10) \quad \mu\{x : R^*(f_j)(x) > \frac{\lambda\gamma_j}{2}\} \leq \tilde{C} \frac{1}{(1 + (1/j)) - 1} \left(\frac{2^{j/2} [t_j]^{1/2p_j}}{(\lambda\gamma_j/2)^{1/2}} + \frac{2^j [t_j]^{1/p_j}}{\lambda\gamma_j/2} \right) \leq 2\tilde{C} \left(\frac{j2^{j/2} [t_j]^{1/2p_j}}{(\lambda\gamma_j)^{1/2}} + \frac{j2^j [t_j]^{1/p_j}}{\lambda\gamma_j} \right).$$

We choose $\gamma_0 = 1/2$ and $\gamma_j = \frac{C_\gamma}{j(\log(j+1))^\beta}$ with $\beta > 1$ and C_γ such that $\sum_{j=0}^{\infty} \gamma_j = 1$.

Set $\hat{C} = 2\tilde{C} \max\{\frac{1}{C_\gamma^{1/2}}, \frac{1}{C_\gamma}\}$.

Using (8) and adding (9) and (10) for all j we obtain

$$\begin{aligned}
(11) \quad \mu\{x : R^*(f)(x) > \lambda\} &\leq \sum_{j=0}^{\infty} \mu\{R^*(f_j) > \frac{\lambda\gamma_j}{2}\} \leq \\
&\frac{8}{\lambda} + 2\tilde{C} \left(\sum_{j=1}^{\infty} \frac{j2^{j/2} [t_j]^{1/2p_j}}{(\lambda\gamma_j)^{1/2}} + \sum_{j=1}^{\infty} \frac{j2^j [t_j]^{1/p_j}}{(\lambda\gamma_j)} \right) \leq \\
&\frac{8}{\lambda} + \widehat{C} \left(\frac{\sum_{j=1}^{\infty} j^{3/2} [\log(j+1)]^{\beta/2} 2^{j/2} [t_j]^{1/2p_j}}{\lambda^{1/2}} + \frac{\sum_{j=1}^{\infty} j^2 [\log(j+1)]^{\beta} 2^j [t_j]^{1/p_j}}{\lambda} \right) = \\
&\frac{8}{\lambda} + \widehat{C} \left(\frac{A_1}{\lambda^{1/2}} + \frac{A_2}{\lambda} \right).
\end{aligned}$$

First we estimate A_1 . Denote by J_1 the set of those j for which $t_j^{1/2p_j} \leq 3^{-j}$. Then

$$(12) \quad \sum_{j \in J_1} j^{3/2} [\log(j+1)]^{\beta/2} 2^{j/2} [t_j]^{1/2p_j} \leq \sum_{j=1}^{\infty} j^{3/2} [\log(j+1)]^{\beta/2} 2^{j/2} 3^{-j} \stackrel{\text{def}}{=} C_s.$$

If $j \notin J_1$ then $t_j^{1/2p_j} > 3^{-j}$, that is,

$$3 > t_j^{\frac{-\frac{1}{2}}{2p_j}} = t_j^{\frac{1-(1+\frac{1}{j})}{2p_j}} = t_j^{\frac{1}{2p_j} - \frac{1}{2}},$$

which implies $t_j^{1/2p_j} < 3t_j^{1/2}$. Hence

$$(13) \quad \sum_{j \notin J_1} j^{3/2} [\log(j+1)]^{\beta/2} 2^{j/2} [t_j]^{1/2p_j} \leq 3 \sum_{j=1}^{\infty} j^{3/2} [\log(j+1)]^{\beta/2} 2^{j/2} [t_j]^{1/2} \stackrel{\text{def}}{=} B_1.$$

Suppose that $\alpha > \delta > 2$. By rewriting and applying the Cauchy–Schwartz inequality we obtain with a suitable constant C_δ that

$$\begin{aligned}
B_1 &= 3 \sum_{j=1}^{\infty} [j^{3/2} j^{-\delta}] j^\delta [\log(j+1)]^{\beta/2} 2^{j/2} [t_j]^{1/2} \leq \\
&3 \left[\sum_{j=1}^{\infty} j^{3-2\delta} \right]^{1/2} \left[\sum_{j=1}^{\infty} j^{2\delta} [\log(j+1)]^\beta 2^j t_j \right]^{1/2} = \\
&C_\delta \left[\sum_{j=1}^{\infty} j^{2\delta} [\log(j+1)]^\beta 2^j t_j \right]^{1/2} \stackrel{\text{def}}{=} B_2.
\end{aligned}$$

There exists $C_{\delta,\alpha,\beta}$ such that for all $j = 1, 2, \dots$

$$[\log(j+1)]^\beta \leq C_{\delta,\alpha,\beta} j^{2(\alpha-\delta)}.$$

Hence,

$$(14) \quad B_1 \leq B_2 \leq C_\delta C_{\delta, \alpha, \beta} \left(\int |f| (\log^+ |f|)^{2\alpha} d\mu \right)^{1/2}.$$

To estimate A_2 a similar argument can be used. We denote by J'_1 the set of those j for which $t_j^{1/p_j} \leq 3^{-j}$. Then

$$(15) \quad \sum_{j \in J'_1} j^2 [\log(j+1)]^\beta 2^j [t_j]^{1/p_j} \leq \sum_{j=1}^{\infty} j^2 [\log(j+1)]^\beta 2^j 3^{-j} \stackrel{\text{def}}{=} C'_s.$$

If $j \notin J'_1$ then $t_j^{1/p_j} > 3^{-j}$, that is,

$$3 > t_j^{\frac{-1}{p_j}} = t_j^{\frac{1-(1+\frac{1}{j})}{p_j}} = t_j^{\frac{1}{p_j} - 1},$$

which implies $t_j^{1/p_j} < 3t_j$. Hence,

$$(16) \quad \sum_{j \notin J'_1} j^2 [\log(j+1)]^\beta 2^j [t_j]^{1/p_j} \leq 3 \sum_{j=1}^{\infty} j^2 [\log(j+1)]^\beta 2^j t_j \stackrel{\text{def}}{=} B'_1.$$

Since $\alpha > 2$ there exists $C'_{\alpha, \beta}$ such that for all $j = 1, 2, \dots$

$$[\log(j+1)]^\beta \leq C'_{\alpha, \beta} j^{(\alpha-2)}.$$

Hence,

$$(17) \quad B'_1 \leq 3 \cdot C'_{\alpha, \beta} \sum_{j=1}^{\infty} j^\alpha 2^j t_j \leq 3 \cdot C'_{\alpha, \beta} \int |f| (\log^+ |f|)^\alpha d\mu.$$

By (11-17) we have

$$\mu\{x : R^*(f)(x) > \lambda\} \leq \widehat{C} \left(\frac{C_s + C_\delta C_{\delta, \alpha, \beta} \left(\int |f| (\log^+ |f|)^{2\alpha} d\mu \right)^{1/2}}{\lambda^{1/2}} + \frac{(8/\widehat{C}) + C'_s + 3C'_{\alpha, \beta} \int |f| (\log^+ |f|)^\alpha d\mu}{\lambda} \right)$$

this implies (7) with a suitable C_α . □

Remark 1. Using the fact that we are dealing with finite measure spaces one can assume that $\mu(X) = 1$. With this assumption one can obtain a more compact form of (3):

$$(18) \quad \mu\left\{x : \sup_n \frac{f(T^n x)g(T^{2n} x)}{n} > \lambda\right\} \leq 2C_p \left[\left(\frac{\|f\|_p \|g\|_1}{\lambda} \right)^{1/2} \right].$$

Indeed, we can distinguish two cases. If $\frac{\|f\|_p\|g\|_1}{\lambda} \geq 1$ then (18) is always true. If $\frac{\|f\|_p\|g\|_1}{\lambda} < 1$, then $\left(\frac{\|f\|_p\|g\|_1}{\lambda}\right)^{1/2} > \left(\frac{\|f\|_p\|g\|_1}{\lambda}\right)$ and (18) is also true.

Remark 2. Equation (7) implies also that for the pair of nonnegative functions (f, g) in $(L(\log L)^{2\alpha}, L^1)$ we have

$$(19) \quad \lim_n \frac{f(T^n x)g(T^{2n} x)}{n} = 0.$$

Indeed, consider a sequence of bounded functions $0 \leq f_M \leq f$ converging monotone increasingly to $f \in L(\log L)^{2\alpha}$. Then we have

$$(20) \quad \lim_n \frac{f_M(T^n x)g(T^{2n} x)}{n} = 0.$$

Given $\varepsilon \in (0, 1)$ choose M so large that

$$(21) \quad I(M, \varepsilon, 1/2) \stackrel{\text{def}}{=} \left(\int \frac{2}{\varepsilon^2} |f - f_M| (\log^+ \frac{2}{\varepsilon^2} |f - f_M|)^{2\alpha} d\mu \right)^{1/2} < 1, \text{ and}$$

$$I(M, \varepsilon, 1) \stackrel{\text{def}}{=} \int \frac{2}{\varepsilon^2} |f - f_M| (\log^+ \frac{2}{\varepsilon^2} |f - f_M|)^\alpha d\mu < 1.$$

Then

$$\mu\{x : \limsup_{n \rightarrow \infty} \frac{f(T^n x)g(T^{2n} x)}{n} > \varepsilon\} \leq \mu\{x : \limsup_{n \rightarrow \infty} \frac{(f - f_M)(T^n x)g(T^{2n} x)}{n} > \frac{\varepsilon}{2}\} + \mu\{x : \limsup_{n \rightarrow \infty} \frac{f_M(T^n x)g(T^{2n} x)}{n} > \frac{\varepsilon}{2}\} \leq$$

(by using (20))

$$\mu\{x : R^*((f - f_M), g)(x) > \frac{\varepsilon}{2}\} = \mu\{x : R^*(\frac{2}{\varepsilon^2}(f - f_M), g)(x) > \frac{1}{\varepsilon}\} \leq$$

(by using (7) and (21))

$$C_\alpha \sqrt{\varepsilon} (2 + I(M, \varepsilon, 1/2) + I(M, \varepsilon, 1)) \leq 4C_\alpha \sqrt{\varepsilon}.$$

Since this holds for any $\varepsilon \in (0, 1)$ we obtained (19).

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