Infinite Peano Derivatives, extensions, and the Baire one property

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Abstract

If $H \subset \mathbb{R}$ is a closed set and $f : H \to \mathbb{R}$ is k times Peano differentiable with infinite values allowed for f_k then we show that f_k is Baire one for $k = 1, 2$. For $k \geq 3$ we give a counterexample showing that it is possible that $f_k = +\infty$ on a countable dense subset of the nonempty perfect set H, while $f_k = -\infty$ everywhere else. In case f can be extended onto a set $H_0 \supset H$ so that the extended function (still denoted by f) is k times Peano differentiable on H_0 with infinite values allowed and each point of H is a twosided accumulation point of H_0 then we prove that f_k is Baire one for any k.

1 Introduction

Let $H \subset \mathbb{R}$ be a closed set and let $f : H \to \mathbb{R}$ be differentiable with $f'(x) \in \mathbb{R}$. (At isolated points of H, f' can take arbitrary values.) By a theorem of V. Jarník

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[7] improved by J. Mařík [10] there is a differentiable extension $F : \mathbb{R} \to \mathbb{R}$ such that $F|_H = f$, $F'(x) = f'(x)$ for $x \in H$. Of course for every $x \in H$,

$$F'(x) = \lim_{n\to\infty} \frac{F(x + \frac{1}{n}) - F(x)}{\frac{1}{n}}$$

and hence f' is Baire one. However, if f' can take infinite values, it may happen that f is not continuous and the argument used to verify the Baire one property of f' does not work. In Theorem 2 we show that f' is Baire one in this more general case as well. In case H is an interval this result was shown by Z. Zahorski, [13].

Of course, one is interested in the case of higher derivatives. For a function defined on an arbitrary closed set H the ordinary higher order derivative is not appropriate because it gives no information about the function. The appropriate notion is the kth Peano derivative.

Definition 1. Assume $H \subset \mathbb{R}$ and $f : H \to \mathbb{R}$. We say that f is k times Peano differentiable at $x \in H$ if there are $k + 1$ real numbers, $f_0(x), f_1(x)$, $f_2(x), \ldots$, $f_k(x)$ such that $f_0(x) = f(x)$

$$\lim_{\substack{y\to x\\ y\in H}} \frac{f(y) - \sum_{\ell=0}^{k} \frac{f_\ell(x)}{\ell!}(y - x)^\ell}{(y - x)^k} = 0. \tag{1}$$

At limit points of H, $f_\ell(x)$ is unique, while at isolated points of H we can choose and fix them in an arbitrary fashion. Observe that for non-isolated points of H, (1) implies

$$\lim_{\substack{y\to x\\ y\in H}} \left(f(y) - \sum_{\ell=0}^{k-1} \frac{f_\ell(x)}{\ell!}(y - x)^\ell \right) \frac{k!}{(y - x)^k} = f_k(x). \tag{2}$$

This motivates the following definition of infinite Peano derivatives

Definition 2. Assume $H \subset \mathbb{R}$ and $f : H \to \mathbb{R}$ is $k-1$ times Peano differentiable at $x \in H$. If the limit in (2) equals $+\infty$ or $-\infty$, then we say that f has an infinite kth Peano derivative at x; that is, $f_k(x) = +\infty$ or $f_k(x) = -\infty$. Of course at isolated points of H we can assign $+\infty$, $-\infty$ or any number for $f_k(x)$ in an arbitrary fashion.

If f is defined on an open interval I it was first proved by Denjoy [5] that if $f : I \to \mathbb{R}$ is k times Peano differentiable on I (with finite values for f_k) then f_k is of Baire class one. H. W. Oliver [11] had a very simple proof of this fact based on the observation

$$f_k(x) = \lim_{h\to 0} \frac{\sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{k}{\ell} f(x + \ell h)}{h^k}. \tag{3}$$

This and other similar methods do not work if we allow infinite values for f_k. One is also confronted with serious problems if f is defined only on a closed set. The idea of first extending a function onto $\mathbb{R}$ as was done in order one for finite derivatives and then using (3) will not work since in [1] a non-empty perfect set H and a twice Peano differentiable function $f : H \to \mathbb{R}$ is given so that no twice Peano differentiable $F : \mathbb{R} \to \mathbb{R}$ agrees with f on H.

The first attempt to show that f_k defined on an interval with infinite values allowed is of Baire class one was done by P. S. Bullen and S. N. Mukhopadhyay [2], but unfortunately as it was pointed out in [6] there was an error in the proof. In [3] a correct proof was presented. Meanwhile in [8] M. Laczkovich, D. Preiss and C. E. Weil gave an argument which is simpler than [3]. It was claimed by these authors that their method works for functions defined on arbitrary closed sets E. However, as it is shown in this paper in Theorem 9 if $k \geq 3$ there are closed perfect sets E and functions $f : E \to \mathbb{R}$ which are k times Peano differentiable with infinite values allowed and f_k is not of Baire class one on E. In fact, $f_k = +\infty$ on a dense set (at the right endpoints of the contiguous intervals to E) and $f_k = -\infty$ anywhere else on E. If $k = 1$ in Theorem 2 we prove that f_1 is of Baire class one even when it can take infinite values and the function is defined only on a closed set. In Theorem 5 we investigate what can be said for higher values of k. It turns out that extension properties/possibilities play a key role. In Theorem 5 we show that if H is a closed set and $f : H \to \mathbb{R}$ can be extended to a function $f : H_0 \to \mathbb{R}$ so that it is k times Peano differentiable (with infinite values allowed) on the set $H_0 \supset H$, and each point of H is a twosided accumulation point of H_0 then f_k is of Baire class one. In Theorem 6 we prove that if $k = 2$ then there is always such an extension and hence f_2 is of Baire class one, however the result of Theorem 9 shows that for $k \geq 3$ sometimes there are no such extensions. While if H is a closed interval, or a finite union of such intervals then we always have such extensions and hence on these sets f_k is always of Baire class one. It would be interesting to see a further characterization of sets having this extension property.

2 First Order Baire one Property

We will use Cousin's lemma (Lemma 10 of [4]) which was first stated for a complex variables argument in dimension two but was used many times later in the theory of Kurzweil-Henstock integrals and in many other situations; see for example [9].

Lemma 1. *Let $\delta : [a, b] \to (0, +\infty)$. Then there is a finite partition $t_0 = a < t_1 < \cdots < t_p = b$ of $[a, b]$ and for $i = 1, 2, \ldots, p$, a point $x_i \in [t_{i-1}, t_i]$ such that*

$$[t_{i-1}, t_i] \subset B(x_i, \delta(x_i)) = (x_i - \delta(x_i), x_i + \delta(x_i)). \tag{4}$$

A tagged partition $\{([t_{i-1}, t_i], x_i)\}_{i=1}^p$ satisfying (4) is called a δ-fine partition.

Theorem 2. *Let $H \subset \mathbb{R}$ be a closed set and let $f : H \to \mathbb{R}$ be differentiable on H with infinite values allowed. Then f' is in Baire class one.*

Proof. Without loss of generality, we may assume that $H \subset [0,1]$. We wish to define a sequence $\{\phi_m\}$ converging pointwise to f' on H. To that end, let $m \in \mathbb{N}$. We will apply Cousin's Lemma. Let $x \in [0,1]$. If $x \in [0,1] \setminus H$, choose $\delta_m(x) \in (0, 1/m)$ such that $B(x, \delta_m(x)) \cap H = \emptyset$. If $x \in H$ is an isolated point of H, then choose $\delta_m(x) \in (0, 1/m)$ such that $B(x, \delta_m(x)) \cap H = \{x\}$. Now assume that $x \in H$ is a limit point of H. If $f'(x) \in \mathbb{R}$, choose $\delta_m(x) \in (0, 1/m)$ such that if $y \in B(x, \delta_m(x)) \cap H$ with $y \neq x$, then

$$\left| \frac{f(y) - f(x)}{y - x} - f'(x) \right| < \frac{1}{m}. \tag{5}$$

If $f'(x) = \pm\infty$, choose $\delta_m(x) \in (0, 1/m)$ such that if $y \in B(x, \delta_m(x)) \cap H$ with $y \neq x$, then

$$\pm\Big(\frac{f(y) - f(x)}{y - x} \Big) > m. \tag{6}$$

Now by Cousin's lemma there is a δ_m-fine partition $0 = t_{m,0} < t_{m,1} < \cdots < t_{m,p_m} = 1$, with tagged points $x_{m,i} \in [t_{m,i-1}, t_{m,i}]$ for $i = 1, 2, \ldots, p_m$. For $i = 1, 2, \ldots, p_m$, let $I_{m,i} = [t_{m,i-1}, t_{m,i}]$, let $K_m = \{i \in \{1, 2, \ldots, p_m\}; x_{m,i} \in H\}$. Then clearly $H \subset \cup_{i \in K_m} I_{m,i}$. To facilitate the definition of ϕ_m, for $i \in K_m$ we select $J_{m,i} = [u_{m,i}, v_{m,i}] \subset I_{m,i}$. Let $i \in K_m$. If $i + 1 \notin K_m$ or if $i = p_m$, then let $v_{m,i} = t_{m,i}$. So assume $i + 1 \in K_m$. Then $x_{m,i+1} - \delta_m(x_{m,i+1}) < t_{m,i}$ and we select $v_{m,i} \in (x_{m,i+1} - \delta_m(x_{m,i+1}), t_{m,i})$. The endpoint $u_{m,i}$ is defined similarly.

We now define $\phi_m : \cup_{i \in K_m} I_{m,i} \to \mathbb{R}$. Let $i \in K_m$ and first for all $x \in J_{m,i}$ let $\phi_m(x) = f'(x_{m,i})$ if $f'(x_{m,i}) \in \mathbb{R}$ and let $\phi_m(x) = \pm m$ if $f'(x_{m,i}) = \pm\infty$. To extend ϕ_m to $I_{m,i}$, first assume $i + 1 \in K_m$ for otherwise $v_{m,i} = t_{m,i}$. Then $v_{m,i} < t_{m,i} < u_{m,i+1}$ and for $x \in [v_{m,i}, u_{m,i+1}]$ let

$$\phi_m(x) = \frac{(x - v_{m,i})\phi_m(u_{m,i+1}) + (u_{m,i+1} - x)\phi_m(v_{m,i})}{u_{m,i+1} - v_{m,i}}.$$

if Clearly ϕ_m is continuous on $\cup_{i \in K_m} I_{m,i}$ and hence on H.

To prove $\{\phi_m\}$ converges pointwise on H to f', let $z \in H$. If z is an isolated point, then there exists $r > 0$ such that $(z - r, z + r) \cap H = \{z\}$. Let $m > \frac{1}{r}$. Then there is $i \in K_m$ such that $z \in I_{m,i}$ and by the definition of δ_m it follows that $z = x_{m,i}$ and hence $\phi_m(z) = f'(z)$ if $f'(z) \in \mathbb{R}$ and $\phi_m(z) = \pm m$ if $f'(z) = \pm\infty$. Thus $\{\phi_m(z)\}$ converges to $f'(z)$.

Now assume z is not an isolated point of H and let $\epsilon > 0$. If $f'(z) \in \mathbb{R}$, choose $\eta > 0$ such that for $y \in B(z, \eta) \cap H$ with $y \neq z$,

$$\left| \frac{f(y) - f(z)}{y - z} - f'(z) \right| < \frac{\epsilon}{2}. \tag{7}$$

If $f'(z) = \pm\infty$, choose $\eta > 0$ such that for $y \in B(z, \eta) \cap H$ with $y \neq z$,

$$\pm\Big(\frac{f(y) - f(z)}{y - z} \Big) > 1 + \frac{1}{\epsilon}. \tag{8}$$

Let $m > \max\left\{\frac{2}{\epsilon}, \frac{1}{\eta}\right\}$ and if $f'(z) \in \mathbb{R}$, in addition let $m > |f'(z)| + \frac{\epsilon}{2}$. For ease of notation, in all subscripts, we drop m. For example K_m will be denoted simply by K. Let $i \in K$ with $z \in B(x_i, \delta(x_i))$ and $z \neq x_i$. First assume $f'(z) \in \mathbb{R}$. Because $\delta(x_i) < \frac{1}{m} < \eta$, by (7), $\left|\frac{f(x_i) - f(z)}{x_i - z} - f'(z)\right| < \frac{\epsilon}{2}$. Since $m > |f'(z)| + \frac{\epsilon}{2}$, it follows that (6) cannot hold. Thus $f'(x_i) \in \mathbb{R}$. Moreover even if $z = x_i$, we have

$$|f'(z) - f'(x_i)| < \frac{\epsilon}{2} + \frac{1}{m} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \tag{9}$$

Now assume $f'(z) = +\infty$. Because $\delta(x_i) < \frac{1}{m} < \eta$, by (8), $\frac{f(x_i) - f(z)}{x_i - z} > 1 + \frac{1}{\epsilon}$ if $z \neq x_i$. Since $z \in B(x_i, \delta(x_i))$, it follows by (6) that $f'(x_i) \neq -\infty$ and so either $f'(x_i) = +\infty$, or by (5)

$$f'(x_i) = f'(x_i) - \frac{f(x_i) - f(z)}{x_i - z} + \frac{f(x_i) - f(z)}{x_i - z} > 1 + \frac{1}{\epsilon} - \frac{1}{m} > \frac{1}{\epsilon}. \tag{10}$$

Note that if $x_i = z$, then the conclusion of (10) still holds since $f'(z) = +\infty$.

Now choose an $i \in K$ for which $z \in I_i$. First assume $f'(z) \in \mathbb{R}$. If $z \in J_i$, then $\phi_m(z) = f'(x_i)$ and hence by (9), $|f'(z) - \phi_m(z)| < \epsilon$. If $z \in I_i \setminus J_i$, say $z \in (v_i, t_i]$, then it follows that $z \in B(x_{i+1}, \delta(x_{i+1}))$ and by (9), $f'(x_{i+1}) \in \mathbb{R}$ and hence $\phi_m(u_{i+1}) = f'(x_{i+1})$ and also by (9), $|f'(z) - \phi_m(u_{i+1})| < \epsilon$. From the definition of ϕ_m, by a typical convex combination argument it follows that $|f'(z) - \phi_m(z)| < \epsilon$. A similar argument yields the same conclusion if $z \in [t_{i-1}, u_i)$.

Next assume that $f'(z) = +\infty$. If $z \in J_i$, then by (10), $\phi_m(z) = f'(x_i) > \frac{1}{\epsilon}$. If $z \in I_i \setminus J_i$, say $z \in (v_i, t_i]$, then we still have $f'(x_i) > \frac{1}{\epsilon}$ and it follows that $z \in B(x_{i+1}, \delta(x_{i+1}))$ and also by (10), $f'(x_{i+1}) > \frac{1}{\epsilon}$. Again from the definition of ϕ_m, by a typical convex combination argument it follows that $\phi_m(z) > \frac{1}{\epsilon}$. A similar argument yields the same conclusion if $z \in [t_{i-1}, u_i)$. Finally if $f'(z) = -\infty$, an analogous argument shows that $\phi_m(z) < -\frac{1}{\epsilon}$. □

3 Higher Order Baire one Property

This section begins with two lemmas. In the finite case only the first is needed as will be pointed out in the proof of the theorem of this section.

Lemma 3. *Let $\epsilon > 0$ and for $j = 0, 1, \ldots, k$ let $x_j \in [-1, 1]$ with*

$$|x_j| < \frac{1}{2}|x_{j+1}| \text{ for } j = 0, 1, \ldots, k. \tag{11}$$

Let $P(x) = \sum_{\ell=0}^{k} a_\ell x^\ell$. *Assume*

$$|P(x_j)| < \epsilon |x_j|^k \text{ for } j = 0, 1, \ldots, k. \tag{12}$$

Then

$$|a_\ell| < \epsilon(k+1)4^k \text{ for } \ell = 0, 1, \ldots, k. \tag{13}$$

Proof. Denote the associated Lagrange interpolating polynomials by

$$L_n(x) = \frac{\prod_{\substack{j=0 \\ j \neq n}}^{k} (x - x_j)}{\prod_{\substack{j=0 \\ j \neq n}}^{k} (x_n - x_j)} = \sum_{\ell=0}^{k} b_{\ell,n} x^\ell.$$

We have $P(x) = \sum_{n=0}^{k} P(x_n) L_n(x)$ by the Lagrange interpolation formula. By (11)

$$\frac{1}{\left| \prod_{\substack{j=0 \\ j \neq n}}^{k} (x_n - x_j) \right|} \leq \frac{2^k}{|x_n|^k}$$

and if we set $\prod_{\substack{j=0 \\ j \neq n}}^{k} (x - x_j) = \sum_{\ell=0}^{k} c_{\ell,n} x^\ell$, then by using (11) again we obtain

$$|c_{\ell,n}| = \left| \sum_{\substack{0 \leq j_1 < j_2 < \cdots < j_{k-\ell} \leq k \\ j_\ell \neq n,\ \ell = 1, \ldots, k-\ell}} (-1)^{k-\ell} x_{j_1} \ldots x_{j_{k-\ell}} \right| \leq \binom{k}{k-\ell} \prod_{t=0}^{k-\ell+1} 2^{-t} < 2^k.$$

Hence $|b_{\ell,n}| \leq 4^k / |x_n^k|$ and using (12)

$$\begin{aligned} |a_\ell| = &\left| \sum_{n=0}^{k} P(x_n) b_{\ell,n} \right| \leq \epsilon \sum_{n=0}^{k} |x_n|^k |b_{\ell,n}| \\ \leq &\epsilon \sum_{n=0}^{k} |x_n|^k \frac{4^k}{|x_n|^k} \leq \epsilon(k+1)4^k. \end{aligned}$$

□

Lemma 4. *Let* $\epsilon > 0$, *let* $x_0, \ldots, x_k \in [-1, 1] \setminus \{0\}$ *and for* $\ell = 0, \ldots, k$, *let* $|a_\ell| \leq M$. *Let* $P(x) = \sum_{\ell=0}^{k} a_\ell x^\ell$. *Assume satisfies*

$$|x_\ell| < \frac{|x_{\ell+1}|^k}{k} \text{ for } \ell = 0, \ldots, k-1 \tag{14}$$

and

$$|P(x_\ell)| < \epsilon |x_\ell|^k \text{ for } \ell = 0, \dots, k. \tag{15}$$

Then

$$|a_\ell| < |x_\ell| k(\epsilon + M) \text{ for } \ell = 0, \dots, k-1, \tag{16}$$

and

$$|a_k| < \epsilon + |x_k| k(\epsilon + M). \tag{17}$$

Proof. Because $a_0 = P(x_0) - \left(\sum_{\ell=1}^k a_\ell x_0^\ell\right)$, by (15)

$$|a_0| < \epsilon |x_0|^k + |x_0| \cdot |a_1 + a_2 x_0 + \cdots + a_k x_0^{k-1}| \le |x_0|(\epsilon + kM).$$

Now let $0 < \ell \le k$ and assume that we have verified $|a_j| < |x_j|((j+1)\epsilon + kM)$ for $j = 0, \dots, \ell - 1$. Then $|P(x_\ell)| < \epsilon |x_\ell|^k$ implies

$$\begin{aligned} |a_\ell| <& \epsilon |x_\ell|^{k-\ell} + |x_\ell| \cdot |a_{\ell+1} + \cdots + a_k x_\ell^{k-\ell-1}| + \frac{|a_0|}{|x_\ell|^\ell} + \cdots + \frac{|a_{\ell-1}|}{|x_\ell|} \\ <& \epsilon |x_\ell|^{k-\ell} + |x_\ell|(k-\ell)M + (\ell\epsilon + kM)\left(\frac{|x_0|}{|x_\ell|^\ell} + \cdots + \frac{|x_{\ell-1}|}{|x_\ell|}\right). \end{aligned}$$

Using (14) this implies $|a_\ell| < \epsilon |x_\ell|^{k-\ell} + (\ell\epsilon + kM)|x_\ell|$ which yields (16) if $l \le k-1$ and (17) if $\ell = k$. □

Theorem 5. *Let $H_0 \subset \mathbb{R}$ be an arbitrary set, $H \subset H_0$ be a closed set and let $f : H_0 \to \mathbb{R}$ be k-times Peano differentiable on H with infinite values allowed for f_k. Suppose each limit point of H is a bilateral limit point of H_0. Then f_k is in Baire class one on H.*

Proof. Without loss of generality we may assume $H \subset [0,1]$. The proof is patterned after that of Theorem 2. We assume $k \ge 2$. We let $m \in \mathbb{N}$ and, as was done in the proof of Theorem 2, we define a positive function δ_m on $[0,1]$. At points belonging to $[0,1] \setminus H$ and at isolated points of H, we define δ_m as in the proof of Theorem 2. The process of defining δ_m at limit points of H is far more involved in the present setting. Set

$$c_k = \frac{1}{2k!(k-1)^2(1 + k4^{k-1})}. \tag{18}$$

Because f is k times Peano differentiable at x, there is a $\sigma \in (0, \frac{1}{m})$ such that for $y \in H_0 \cap B(x, \sigma)$, with $y \ne x$,

$$\left| f(y) - \sum_{\ell=0}^{k-1} \frac{f_\ell(x)}{\ell!}(y-x)^\ell \right| < c_k |y - x|^{k-1} \tag{19}$$

and if $f_k(x) \in \mathbb{R}$, then

$$\left| f(y) - \sum_{\ell=0}^{k} \frac{f_\ell(x)}{\ell!}(y-x)^\ell \right| < \frac{1}{m}|y-x|^k, \tag{20}$$

or, if $f_k(x) = \pm\infty$, then

$$\pm\Big(f(y) - \sum_{\ell=0}^{k-1} \frac{f_\ell(x)}{\ell!}(y-x)^\ell \Big) \frac{k!}{(y-x)^k} > m. \tag{21}$$

Since x is a bilateral limit point of H_0, for $j = 0, 1, \ldots, k$, there are $w_j^\pm \in H_0$ such that

$$x - \sigma < w_k^- < w_{k-1}^- < \cdots < w_0^- < x < w_0^+ < \cdots < w_{k-1}^+ < w_k^+ < x + \sigma \tag{22}$$

and

$$|w_j^\pm - x| < \min\left\{ \frac{|w_{j+1}^\pm - x|}{2}, \frac{|w_{j+1}^\pm - x|^k}{k-1} \right\} \text{ for } j = 0, 1, \ldots, k-1. \tag{23}$$

By continuity, there is a $0 < \delta_m(x) < |w_0^\pm - x|$ such that, for $z \in H_0 \cap B(x, \delta_m(x))$

$$|f(w_j^\pm) - \sum_{\ell=0}^{k-1} \frac{f_\ell(x)}{\ell!}(w_j^\pm - z)^\ell| < c_k |w_j^\pm - z|^{k-1} \text{ for } j = 0, 1, \ldots, k \tag{24}$$

and if $f_k(x) \in \mathbb{R}$, then

$$|f(w_j^\pm) - \sum_{\ell=0}^{k} \frac{f_\ell(x)}{\ell!}(w_j^\pm - z)^\ell| < \frac{1}{m}|w_j^\pm - z|^k \text{ for } j = 0, 1, \ldots, k \tag{25}$$

or if $f_k(x) = (\pm)\infty$, then with $z \neq x$,

$$(\pm)\Big(f(w_j^\pm) - \sum_{\ell=0}^{k-1} \frac{f_\ell(x)}{\ell!}(w_j^\pm - z)^\ell \Big) \frac{k!}{(w_j^\pm - z)^k} > m \text{ for } j = 0, 1, \ldots, k, \tag{26}$$

(in the above formula $(\pm)$ and $\pm$ can take positive and negative values independently). Moreover, for $z \in H_0 \cap B(x, \delta_m(x))$,

$$|w_j^\pm - z| < \min\left\{ \frac{|w_{j+1}^\pm - z|}{2}, \frac{|w_{j+1}^\pm - z|^k}{k-1} \right\} \text{ for } j = 0, 1, \ldots k-1. \tag{27}$$

Now that δ_m has been defined, we proceed exactly as in the proof of Theorem 2 applying Cousin's Lemma, defining K_m, $I_{m,i}$ and $J_{m,i}$. The function ϕ_m is defined as it was in the proof of Theorem 2 except that f_k is used in place of f'. The proof that $\{\phi_m(z)\}$ converges to $f_k(z)$ if z is an isolated point of H is identical to that in the proof of Theorem 2 and is omitted. Let z be a limit point of H and let $0 < \epsilon < 1$. Then there is an $\eta > 0$ such that for $y \in H_0 \cap B(z, \eta)$ with $y \neq z$,

$$\Big|f(y) - \sum_{\ell=0}^{k-1} \frac{f_\ell(z)}{\ell!}(y-z)^\ell\Big| < c_k|y-z|^{k-1} \tag{28}$$

and if $f_k(z) \in \mathbb{R}$, then

$$\Big|f(y) - \sum_{\ell=0}^{k} \frac{f_\ell(z)}{\ell!}(y-z)^\ell\Big| < \frac{\epsilon}{2(k+1)4^k k!}|y-z|^k, \tag{29}$$

or if $f_k(z) = \pm\infty$, then

$$\pm\Big(f(y) - \sum_{\ell=0}^{k-1} \frac{f_\ell(z)}{\ell!}(y-z)^\ell\Big)\frac{k!}{(y-z)^k} > \frac{1}{\epsilon} + 2. \tag{30}$$

Let $m > \max\Big\{\frac{2(k+1)4^k k!}{\epsilon}, \frac{1}{\eta}\Big\}$ and if $f_k(z) \in \mathbb{R}$, we also assume that $m > |f_k(z)| + 2$. As in the proof of Theorem 2, we drop the m in the subscripts. Let $i \in K$ with $z \in B(x_i, \delta(x_i))$. Next we employ Lemmas 3 and 4. First assume that $f_k(x_i)$ and $f_k(z) \in \mathbb{R}$. In this case set

$$P_k(y) = \sum_{\ell=0}^{k} \frac{f_\ell(x_i) - f_\ell(z)}{\ell!} y^\ell = \sum_{\ell=0}^{k} a_\ell y^\ell.$$

Then by (29) and (25) for $j = 0, 1, \dots, k$,

$$\begin{aligned} |P_k(w_j^\pm - z)| &\le |f(w_j^\pm) - \sum_{\ell=0}^{k} \frac{f_\ell(z)}{\ell!}(w_j^\pm - z)^\ell| + |f(w_j^\pm) - \sum_{\ell=0}^{k} \frac{f_\ell(x_i)}{\ell!}(w_j^\pm - z)^\ell| \\ &< \frac{\epsilon}{2(k+1)4^k k!}|w_j^\pm - z|^k + \frac{1}{m}|w_j^\pm - z|^k \\ &< \frac{\epsilon}{(k+1)4^k k!}|w_j^\pm - z|^k. \end{aligned}$$

By Lemma 3 for $\ell = 0, 1, \dots, k$ we have $|a_\ell| = \Big|\frac{f_\ell(x_i) - f_\ell(z)}{\ell!}\Big| < \frac{\epsilon}{k!}$. In particular, $|f_k(x_i) - f_k(z)| < \epsilon$.

Remark. From the above conclusion it can be deduced that if $f_k(x) \in \mathbb{R}$ for all $x \in H$, then f_k is a function of Baire class one without the existence of the set H_0.

Next we drop the assumption that $f_k(x_i)$ and $f_k(z) \in \mathbb{R}$ and investigate

$$P_{k-1}(y) = \sum_{\ell=0}^{k-1} \frac{f_\ell(x_i) - f_\ell(z)}{\ell!} y^\ell = \sum_{\ell=0}^{k-1} a_\ell y^\ell.$$

In this case arguing as above but using (28) and (24),

$$|P_{k-1}(w_j^\pm - z)| < 2c_k|w_j^\pm - z|^{k-1}.$$

By Lemma 3, $|a_\ell| < 2c_k k4^{k-1}$ and then by Lemma 4 for $\ell = 0, 1, \ldots, k-2$, we have $|a_\ell| < (k-1)2c_k(1 + k4^{k-1})|w_\ell^\pm - z|$. Thus by the definition of c_k and because $|w_\ell^\pm - z| < |w_k^\pm - z|^k$ by (27),

$$\begin{aligned}\Big|\sum_{\ell=0}^{k-2} a_\ell (w_j^\pm - z)^\ell\Big| &< \frac{1}{k!(k-1)}\sum_{\ell=0}^{k-2}|w_\ell^\pm - z||w_k^\pm - z|^\ell \\ &< \frac{|w_k^\pm - z|^k}{k!(k-1)}\sum_{\ell=0}^{k-2}|w_k^\pm - z|^0 < \frac{|w_k^\pm - z|^k}{k!}.\end{aligned} \tag{31}$$

We now show how (31) is used to prove that $f_k(z) \in \mathbb{R}$ and $f_k(x_i) = +\infty$ is impossible. For if $f_k(z) \in \mathbb{R}$ and $f_k(x_i) = +\infty$, then by (29)

$$\Big(f(w_k^\pm) - \sum_{\ell=0}^{k-1}\frac{f_\ell(z)}{\ell!}(w_k^\pm - z)^\ell\Big)\frac{k!}{(w_k^\pm - z)^k} < |f_k(z)| + \frac{\epsilon}{2(k+1)4^k k!}$$

and by (26)

$$\Big(f(w_k^\pm) - \sum_{\ell=0}^{k-1}\frac{f_\ell(x_i)}{\ell!}(w_k^\pm - z)^\ell\Big)\frac{k!}{(w_k^\pm - z)^k} > m.$$

Combining these two and applying (31) yields

$$\begin{aligned}\frac{f_{k-1}(x_i) - f_{k-1}(z)}{(k-1)!}\frac{k!}{w_k^\pm - z} &< -\sum_{\ell=0}^{k-2} a_\ell (w_k^\pm - z)^\ell \frac{k!}{(w_k^\pm - z)^k} - m + |f_k(z)| + 1 \\ &< -m + |f_k(z)| + 2 < 0.\end{aligned}$$

But because $w_k^+ - z > 0$ and $w_k^- - z < 0$, the left hand side cannot be negative in both cases. In a similar fashion $f_k(z) \in \mathbb{R}$ and $f_k(x_i) = -\infty$ is impossible. This means that $f_k(z) \in \mathbb{R}$ implies $f_k(x_i) \in \mathbb{R}$ and we have already treated this case.

So assume $f_k(z) = +\infty$. In this case (31) will be employed to show that $f_k(x_i) > \frac{1}{\epsilon}$. Suppose to the contrary that $f_k(x_i) \leq \frac{1}{\epsilon}$. If $f_k(x_i) = -\infty$, then by (26)

$$\Big(f(w_k^\pm) - \sum_{\ell=0}^{k-1}\frac{f_\ell(x_i)}{\ell!}(w_k^\pm - z)^\ell\Big)\frac{k!}{(w_k^\pm - z)^k} < -m \tag{32}$$

while if $f_k(x_i) \in \mathbb{R}$, then by (25) and by the assumption $f_k(x_i) \leq \frac{1}{\epsilon}$

$$\Big(f(w_k^\pm) - \sum_{\ell=0}^{k-1}\frac{f_\ell(x_i)}{\ell!}(w_k^\pm - z)^\ell\Big)\frac{k!}{(w_k^\pm - z)^k} < \frac{k!}{m} + \frac{1}{\epsilon}. \tag{33}$$

Because (32) implies (33), we need only assume that (33) holds. Since $f_k(z) = +\infty$, by (30)

$$\Big(f(w_k^\pm) - \sum_{\ell=0}^{k-1} \frac{f_\ell(z)}{\ell!}(w_k^\pm - z)^\ell\Big)\frac{k!}{(w_k^\pm - z)^k} > \frac{1}{\epsilon} + 2. \tag{34}$$

Combining (33) and (34) and using (31) we get

$$\begin{aligned}\frac{1}{\epsilon} + 2 - \frac{k!}{m} - \frac{1}{\epsilon} <& \Big(\sum_{\ell=0}^{k-2} a_\ell (w_k^\pm - z)^\ell \frac{k!}{(w_k^\pm - z)^k}\Big) + \frac{f_{k-1}(x_i) - f_{k-1}(z)}{(k-1)!}\frac{k!}{w_k^\pm - z}\\ <& 1 + \frac{(f_{k-1}(x_i) - f_{k-1}(z))k}{(w_k^\pm - z)}.\end{aligned}$$

Thus $\dfrac{(f_{k-1}(x_i) - f_{k-1}(z))k}{(w_k^\pm - z)} > 0$, which leads to the same contradiction as before. In a similar fashion it can be shown that if $f_k(z) = -\infty$, then $f_k(x_i) < -\frac{1}{\epsilon}$. Now the remainder of the proof proceeds as in the proof of Theorem 2. □

Theorem 6. *Let $H \subset \mathbb{R}$ be a closed set and let $f : H \to \mathbb{R}$ be twice Peano differentiable on H with infinite values allowed for f_2. Then there exists $H_0 \supset H$ and an $F : H_0 \to \mathbb{R}$ such that F is twice Peano differentiable on H_0, $F|_{H_0} = f$ and each point of H is a twosided accumulation point of H_0.*

Proof. Suppose H has only finitely many contiguous intervals. For an interval (a, b) contiguous to H one can choose $\delta > 0$ such that $a + \delta < b - \delta$ and define F on $(a, a + \delta] \cup [b - \delta, b)$ so that it is twice Peano differentiable on $H \cup (a, a+\delta] \cup [b-\delta, b)$ and $F|_H = f$. It is clear that by repeating this process on all contiguous intervals we obtain the required extension.

So assume H has infinitely many contiguous intervals (a_n, b_n) and $b_{n+1} - a_{n+1} \le b_n - a_n$ for $n = 1, 2, \dots$. We denote by $\mathcal{E}$ the set consisting of the (finite) endpoints of these contiguous intervals. For $n = 1$, or 2 we allow $a_n = -\infty$ or $b_n = +\infty$ to take care of contiguous intervals of possibly infinite length. For each n we will choose $\delta_n > 0$ such that $a_n + \delta_n < b_n - \delta_n$. If $f_2(e) \in \mathbb{R}$ we will set

$$F(x) = f(e) + f_1(e)(x - e) + \frac{f_2(e)}{2!}(x - e)^2 \tag{35}$$

on $[e - \delta_n, e + \delta_n] \cap (a_n, b_n)$. If $f_2(e) = \pm\infty$, then we set

$$F(x) = f(e) + f_1(e)(x - e) \pm |x - e|^{3/2} \tag{36}$$

on $[e - \delta_n, e + \delta_n] \cap (a_n, b_n)$.

For each $x \in H$ choose $\delta(x) > 0$ such that if $f_2(x) \in \mathbb{R}$, we have

$$|f(y) - f(x) - f_1(x)(y - x)| < |y - x| \text{ and} \tag{37}$$

$$\left|f(y) - f(x) - f_1(x)(y - x) - \frac{f_2(x)}{2!}(y - x)^2\right| < (y - x)^2 \tag{38}$$

for all $y \in (x-\delta(x), x+\delta(x)) \cap H$.

If $f_2(x) = \pm\infty$, then we choose $\delta(x)$ so that we have (37) and

$$\pm\left(\frac{f(y)-f(x)-f_1(x)(y-x)}{(y-x)^2}\right) > 1 \tag{39}$$

for all $y \in (x-\delta(x), x+\delta(x)) \cap H$.

If $x \in H$ is an isolated point of H, we choose $\delta(x) > 0$ so that $(x-\delta(x), x+\delta(x)) \cap H = \{x\}$ and $\delta(x) < 1$.

Assume (a, b) is an interval contiguous to H. The endpoint $e \in \{a, b\}$ is called type I if there exists a sequence $x_n \in H$, $x_n \neq e$, $x_n \to e$ such that $\delta(x_n) > 2(b-a)$, otherwise e is called type II. In this latter case we choose $\eta(e) > 0$ such that for $x \in (e-\eta(e), e+\eta(e)) \cap H$, $x \neq e$ we have $\delta(x) \leq 2(b-a)$. We can also suppose that $\eta(e) < (b-a)/3$. Note that if e is an isolated point of H, then e is of thpe II.

For $x \in H$ set $s(x) = 0$ if $f_2(x) \in \mathbb{R}$, $s(x) = -1$ if $f_2(x) = -\infty$, and $s(x) = 1$ if $f_2(x) = +\infty$.

If e is of type I and if $s(e) = 0$, we choose $\eta(e) > 0$ such that $\eta(e) < (b-a)/3$.

Now we interrupt the proof of Theorem 6 by the statement of a lemma which will be proved later. It defines $\eta(e)$ in case e is of type I and $|s(E)| = \infty$.

Lemma 7. *Assume that $M' > 20$ and e, the endpoint of a contiguous interval (a, b), is of type I and $|s(e)| = 1$. Then there exists an $\eta(e) > 0$ such that for all $x \in (e-\eta(e), e+\eta(e)) \cap H \setminus \{e\}$ from $\delta(x) > 2(b-a)$ it follows that $f_2(x) > M'$ when $s(e) = 1$, and $f_2(x) < -M'$ when $s(e) = -1$. We can also suppose that $\eta(e) < (b-a)/3$.*

We return to the proof of Theorem 6. Assume (a_n, b_n) is an interval contiguous to H and $e \in \{a_n, b_n\}$ is one of its endpoints. If e is of type II put $\eta'(e) = \min\{\eta(e), 1\}$.

If e is of type I and $|s(e)| = 1$, then by using Lemma 7 with $M' = \max\{20, n\}$ choose $\eta(e)$. Next we choose a positive $\eta'(e) \leq \eta(e)$ such that

$$\begin{gathered} |f(x)-f(e)-f_1(e)(x-e)| < \frac{1}{n}|x-e|, \text{ and} \\ \pm\left(\frac{f(x)-f(e)-f_1(e)(x-e)}{(x-e)^2}\right) > n \text{ when } f_2(e) = \pm\infty \end{gathered} \tag{40}$$

hold for all $x \in (e-\eta'(e), e+\eta'(e)) \cap H$. If e is of type I and $s(e) = 0$, choose a positive $\eta'(e) \leq \eta(e)$ such that the first inequality in (40) holds together with

$$\begin{aligned} &|f(x)-f(e)-f_1(e)(x-e)-\frac{f_2(e)}{2!}(x-e)^2| < \frac{1}{n}(x-e)^2, \text{ and} \\ &\frac{|f_2(e)|}{2!}|x-e| < \frac{1}{n} \text{ for all } x \in (e-\eta'(e), e+\eta'(e)) \cap H; \end{aligned} \tag{41}$$

furthermore, we also assume $\eta'(e)$ is so small that there exist $x_1, x_2, x_3 \in H \setminus (e - n\eta'(e), e + n\eta'(e))$ such that

$$\begin{gathered} \frac{|x_{j+1} - e|}{|x_j - e|} < \frac{1}{2} \text{ for } j = 1, 2 \\ |x_1 - e| < \eta(e) < (b_n - a_n)/3 \text{ and} \\ (41) \text{ holds with } x \text{ replaced by } x_j \ (j = 1, 2, 3). \end{gathered} \tag{42}$$

We can assume that $\delta_n > 0$ is chosen so that if e is either endpoint of $(a_n, b_n($, then

$$2n\delta_n < (\eta'(e))^2 < \eta'(e) \tag{43}$$

and using $F(x)$ defined in (35), or in (36) we have

$$|F(x) - F(e)| < \frac{1}{n}(\eta'(e))^2 \text{ for all } x \in [e - \delta_n, e + \delta_n]. \tag{44}$$

We define $H_0 \supset H$ so that $H_0 \cap (a_n, b_n) = (a_n, a_n + \delta_n] \cup [b_n - \delta_n, b_n)$ and we extend the definition of $f(x)$ by setting $F(x)$ using (35), or (36). It is clear that F is twice Peano differentiable on $H_0 \setminus H$. So assume $x_0 \in H$. If $x_0 \in \mathcal{E}$, then on the side(s) of x_0 which belong to a contiguous interval of H one can easily see that F is twice Peano differentiable and on the side of x_0 which is not in a contiguous interval an argument which is used for $x_0 \notin \mathcal{E}$ can be used with some simple modifications. Hence from now on we assume $x_0 \notin \mathcal{E}$ and $x \to x_0$, $x \in H_0$.

Since $F = f$ on H, it is enough to check differentiability of F when $x \in H_0 \setminus H$. We also assume that $x \in (a_n, b_n)$, and e denotes the endpoint of (a_n, b_n) with $x \in (e - \eta'(e), e + \eta'(e))$. Now we have two options, either x and x_0 are "relatively far" from each other; that is, $x_0 \notin (e - \eta'(e), e + \eta'(e))$, or x and x_0 are "close"; that is, $x_0 \in (e - \eta'(e), e + \eta'(e))$. In the first case the choice of δ_n, (43) and (44) imply that $F(x)$ is very close to $F(e) = f(e)$.

Here are the precise estimates:

$$\frac{\eta'(e)}{2} < \eta'(e) - \delta_n < |x_0 - e| - \delta_n < |x - x_0| < |x_0 - e| + \delta_n \tag{45}$$

by (44) we have

$$|F(x) - F(e)| < \frac{1}{n}(\eta'(e))^2 < \frac{4}{n}(x - x_0)^2 \tag{46}$$

and by (43) and $x_0 \notin (e - \eta'(e), e + \eta'(e))$

$$\begin{gathered} (1 - \frac{1}{n})|x_0 - e| < |x - x_0| < (1 + \frac{1}{n})|x_0 - e|, \text{ and} \\ |x - e| < \delta_n < \frac{1}{2n}(\eta'(e))^2 < \frac{2}{n}(x - x_0)^2. \end{gathered} \tag{47}$$

Next we need to separate several cases depending on whether $f_2(x_0) \in \mathbb{R}$ and whether e is of type I or II.

Case A. Now assume that $f_2(x_0) \in \mathbb{R}$ and $0 < \epsilon < 1$ is given.

Choose $\delta'(x_0) > 0$ such that $\delta'(x_0) < \min\{1, \delta(x_0)/2\}$ and for $x \in (x_0 - 2\delta'(x_0), x_0 + 2\delta'(x_0)) \cap H$ we have

$$|f(x) - f(x_0) - f_1(x_0)(x - x_0) - \frac{f_2(x_0)}{2!}(x - x_0)^2| < \epsilon|x - x_0|^2. \tag{48}$$

Since $x_0 \notin \mathcal{E}$, we can assume that $\delta'(x_0)$ is chosen so small that $(x_0 - \delta'(x_0), x_0 + \delta'(x_0)) \cap [a_n, b_n] = \emptyset$ for all $n \leq \max\{\frac{1}{\epsilon} + 1, |f_2(x_0)|, 20\}$.

Assume $x \in [a_n, b_n] \cap H_0 \subset (x_0 - \delta'(x_0), x_0 + \delta'(x_0))$. Then $\frac{1}{n-1} < \epsilon$, $n > |f_2(x_0)|$, and $2(b_n - a_n) < 2\delta'(x_0) < \delta(x_0)$.

Subcase AII. Suppose e is a type II endpoint.

By the choice of $\delta'(x_0)$ we have $x_0 \notin (e - \eta(e), e + \eta(e))$, which implies $x_0 \notin (e - \eta'(e), e + \eta'(e))$. Hence x_0 and x are "relatively far apart". By (48), (47), and $1/(n-1) < \epsilon$

$$\begin{aligned}|f(e) - f(x_0) - f_1(x_0)(e - x_0) - \frac{f_2(x_0)}{2!}(e - x_0)^2| &< \epsilon(e - x_0)^2 \\ &< \epsilon(\frac{1}{1 - \frac{1}{n}})^2(x - x_0)^2 < \epsilon(1 + \epsilon)^2(x - x_0)^2.\end{aligned}$$

Therefore, by using (46), (47), and $f(e) = F(e)$ we have

$$\begin{aligned}&|F(x) - f(x_0) - f_1(x_0)(x - x_0) - \frac{f_2(x_0)}{2!}(x - x_0)^2| \\ <&5\epsilon(1 + \epsilon)^2(x - x_0)^2 + |f_1(x_0)| \cdot |x - e| + \frac{|f_2(x_0)|}{2!}|(e - x_0)^2 - (x - x_0)^2| \\ <&5\epsilon(1 + \epsilon)^2(x - x_0)^2 + |f_1(x_0)| \cdot 2\epsilon(x - x_0)^2 \\ &+ \frac{|f_2(x_0)|}{2!}|e - x| \cdot (|e - x_0| + |x - x_0|) \\ <&\epsilon\Big(5(1 + \epsilon)^2 + 2|f_1(x_0)| + 2|f_2(x_0)|\Big)(x - x_0)^2.\end{aligned} \tag{49}$$

Subcase AI. Assume now that e is a type I endpoint.

If $x_0 \notin (e - \eta'(e), e + \eta'(e))$ we can argue as above. So suppose now that $x_0 \in (e - \eta'(e), e + \eta'(e)) \subset (e - \eta(e), e + \eta(e))$. Recall that $n \geq \max\{|f_2(x_0)|, 20\}$ and to choose $\eta(e)$ for $|s(e)| = 1$, Lemma 7 is used with $M' = \max\{20, n\}$. Now $\delta(x_0) > 2(b_n - a_n)$ and hence by Lemma 7 $|f_2(x_0)| > M' \geq n$, on the other hand $n \geq |f_2(x_0)|$ which is impossible. Therefore $s(e) = 0$ is the only possibility.

In this case by the definition of $\eta'(e)$ we can choose $x_1, x_2, x_3 \in H \setminus (e - n\eta'(e), e + n\eta'(e))$ such that (42) is satisfied. Observe that $x_j \notin (e - \eta'(e), e + \eta'(e))$, for $j = 1, 2, 3$ and hence (47) is applicable with $x = x_j$. Setting $T_e(t) = f(e) + f_1(e)(t - e) + \frac{f_2(e)}{2!}(t - e)^2$ and $T_0(t) = f(x_0) + f_1(x_0)(t - x_0) + \frac{f_2(x_0)}{2!}(t - x_0)^2$

by (41), (42), (47), and by (48) we have

$$\begin{aligned}|T_0(x_j) - f(x_j) - (T_e(x_j) - f(x_j))| &< \epsilon(x_j - e)^2 + \epsilon(x_j - x_0)^2 \\ &< 5\epsilon(x_j - e)^2 \text{ for } j = 1, 2, 3,\end{aligned} \tag{50}$$

where in the last inequality we used that $|x_j - e| \geq n\eta'(e) > 2\eta'(e)$, and $|x_0 - e| < \eta'(e)$ implies $|x_j - x_0| < 2|x_j - e|$.

Now, $P(t) = T_0(t) - T_e(t) = a_2(t-e)^2 + a_1(t-e) + a_0$ with suitable coefficients a_0, a_1, and a_2. Lemma 3 can be used with $k = 2$, x_j replaced by $(x_j - e)$ and ϵ replaced by 5ϵ. Hence, $|a_i| < 5\epsilon \cdot 3 \cdot 16 = 240\epsilon$ for $i = 0, 1, 2$, and $|P(e)| = |a_0| = |T_0(e) - f(e)| < \epsilon|e - x_0|^2$. On the other hand, $|P(x_0)| = |f(x_0) - T_e(x_0)| < \epsilon|e - x_0|^2$. Now,

$$\begin{aligned}|a_1| \cdot |x_0 - e| &< |P(x_0)| + |a_2| \cdot |x_0 - e|^2 + |a_0| \\ &< \epsilon|e - x_0|^2 + 240\epsilon|x_0 - e|^2 + \epsilon|e - x_0|^2 < 250\epsilon(e - x_0)^2.\end{aligned}$$

Therefore, $|a_1| < 250\epsilon|e - x_0|$. Observe that $F(x) = T_e(x)$. By $|x_0 - e| < \eta(e) < (b_n - a_n)/3$ it is clear that e is in the interval with endpoints x and x_0 and $|e - x_0| < |x - x_0|$. Thus,

$$\begin{aligned}&|F(x) - f(x_0) - f_1(x_0)(x - x_0) - \frac{f_2(x_0)}{2!}(x - x_0)^2| = |T_e(x) - T_0(x)| \\ &< |a_2|(x - e)^2 + |a_1| \cdot |x - e| + |a_0| < 500\epsilon(e - x_0)^2 < 500\epsilon(x - x_0)^2.\end{aligned} \tag{51}$$

This completes the list of inequalities we need for the proof of the twice Peano differentiability of F at $x_0 \in H$ when $|f_2(x_0)| < +\infty$.

Case B. Next assume $|f_2(x_0)| = +\infty$.

Without limiting generality we can suppose that $f_2(x_0) = +\infty$. Assume $M > \max\{20, 2|f_1(x_0)|\}$ is given and choose $\delta'(x_0) > 0$ such that $\delta'(x_0) < \min\{1, \delta(x_0)/2\}$ and for $x \in (x_0 - 2\delta'(x_0), x_0 + 2\delta'(x_0)) \cap H$ we have

$$\begin{aligned}&|f(x) - f(x_0) - f_1(x_0)(x - x_0)| < \frac{1}{M}|x - x_0| \text{ and} \\ &\frac{f(x) - f(x_0) - f_1(x_0)(x - x_0)}{(x - x_0)^2} > M.\end{aligned} \tag{52}$$

Since $x_0 \notin \mathcal{E}$, we can assume that $\delta'(x_0)$ is chosen so small that $(x_0 - \delta'(x_0), x_0 + \delta'(x_0)) \cap [a_n, b_n] = \emptyset$ for all $n \leq M + 1$.

Assume $x \in [a_n, b_n] \cap H_0 \subset (x_0 - \delta'(x_0), x_0 + \delta'(x_0))$. Then $n > M + 1$ and $2(b_n - a_n) < 2\delta'(x_0) < \delta(x_0)$.

Subcase BII. Suppose e is a type II endpoint. By the choice of $\delta'(x_0)$ we have $x_0 \notin (e - \eta(e), e + \eta(e))$ which implies $x_0 \notin (e - \eta'(e), e + \eta'(e))$. Hence x_0 and x are "relatively far apart". By (52), (47), and $n > M + 1$ we have

$$|f(e) - f(x_0) - f_1(x_0)(e - x_0)| < \frac{1}{M}|e - x_0| < \frac{1}{M}\Big(1 + \frac{1}{M}\Big)|x - x_0|,$$

and

$$f(e) - f(x_0) - f_1(x_0)(e - x_0) > M(e - x_0)^2 > M\Big(1 - \frac{1}{M}\Big)^2 (x - x_0)^2.$$

By using the above two inequalities, (46), (47), and $n > M + 1$ we obtain

$$\begin{aligned}
&|F(x) - f(x_0) - f_1(x_0)(x - x_0)| \\
<&\frac{1}{M}(1 + \frac{1}{M})|x - x_0| + |f_1(x_0)| \cdot |e - x| + \frac{4}{M}(x - x_0)^2 \\
<&\frac{1}{M}((1 + \frac{1}{M}) + |f_1(x_0)|2|x - x_0| + 4|x - x_0|)|x - x_0| \\
<&\frac{1}{M}(2 + |f_1(x_0)| + 4)|x - x_0|,
\end{aligned} \tag{53}$$

and

$$\begin{aligned}
&F(x) - f(x_0) - f_1(x_0)(x - x_0) \\
>&M(1 - \frac{1}{M})^2(x - x_0)^2 - |f_1(x_0)| \cdot |e - x| - \frac{4}{M}(x - x_0)^2 \\
>&M(1 - \frac{1}{M})^2(x - x_0)^2 - |f_1(x_0)| \cdot \frac{2}{M}(x - x_0)^2 - \frac{4}{M}(x - x_0)^2 \\
>&(M(1 - \frac{1}{M})^2 - 2)(x - x_0)^2
\end{aligned} \tag{54}$$

where at the last estimate we also used that $M > 2|f_1(x_0)|$.

Subcase BI. Now assume e is a type I endpoint.

If $x_0 \notin (e - \eta'(e), e + \eta'(e))$, we can argue as above. So assume now that $x_0 \in (e - \eta'(e), e + \eta'(e)) \subset (e - \eta(e), e + \eta(e))$.

Without limiting generality we assume that $x < x_0$. Since $\eta'(e) \le \eta(e) < (b_n - a_n)/3$, $e = b_n$ and $x_0 > b_n > x$.

Subsubcase BI0. Assume first that $s(e) = 0$, that is, $f_2(e) \in \mathbb{R}$.

By the first inequality of (40) used with $e = b_n$ we have

$$|f(x_0) - f(b_n) - f_1(b_n)(x_0 - b_n)| < \frac{1}{n}|x_0 - b_n| < \frac{1}{M}|x_0 - b_n|,$$

and by (52)

$$|f(b_n) - f(x_0) - f_1(x_0)(b_n - x_0)| < \frac{1}{M}|b_n - x_0|.$$

The above two inequalities imply

$$|f_1(b_n) - f_1(x_0)| < \frac{2}{M}. \tag{55}$$

Thus if $x \in (b_n - \eta'(b_n), b_n) \cap H_0$, then

$$\begin{aligned}
&|F(x) - f(x_0) - f_1(x_0)(x - x_0)| \\
=&|f(b_n) + f_1(b_n)(x - b_n) + \frac{f_2(b_n)}{2!}(x - b_n)^2 - f(x_0) - f_1(x_0)(x - x_0)| \\
\le&|f(b_n) - f(x_0) - f_1(x_0)(b_n - x_0)| + |f_1(b_n) - f_1(x_0)| \cdot |x - b_n| \\
&+ \frac{|f_2(b_n)|}{2!}|x - b_n| \cdot |x - b_n| \\
&\text{using (52) with } x = b_n \text{ and } n > M, \text{ together with (41)} \\
<&\frac{1}{M}|b_n - x_0| + \frac{2}{M}|x - b_n| + \frac{1}{M}|x - b_n| < \frac{4}{M}|x - x_0|.
\end{aligned} \tag{56}$$

One can use (56) to show that $F_1(x_0)$ exists and $F_1(x_0) = f_1(x_0)$.

Next we need an estimate showing that $F_2(x_0)$ exists and $F_2(x_0) = +\infty$. For x_0 by (52) we have

$$f(b_n) > f(x_0) + f_1(x_0)(b_n - x_0) + M(b_n - x_0)^2. \tag{57}$$

By (41) and $n > M > 20$ we also have

$$f(x_0) > f(b_n) + f_1(b_n)(x_0 - b_n) + \frac{f_2(b_n)}{2!}(x_0 - b_n)^2 - \frac{1}{M}(x_0 - b_n)^2. \tag{58}$$

Adding (57) and (58) and simplifying by $x_0 - b_n > 0$ we obtain

$$f_1(x_0) > f_1(b_n) + \left(\frac{f_2(b_n)}{2!} + M - \frac{1}{M}\right)(x_0 - b_n). \tag{59}$$

Using (41), (42) (with $e = b_n$), (43) and keeping in mind that

$$|x_1 - b_n| \ge n\eta'(b_n) > 4n\delta_n$$

we have

$$|x_0 - b_n| < \eta'(b_n) < |x_1 - b_n|/n < |x_1 - b_n|/M,$$

and

$$|f(x_1) - f(b_n) - f_1(b_n)(x_1 - b_n) - \frac{f_2(b_n)}{2!}(x_1 - b_n)^2| < \frac{1}{M}(x_1 - b_n)^2.$$

By (52) we also have

$$f(x_1) > f(x_0) + f_1(x_0)(x_1 - x_0) + M(x_1 - x_0)^2.$$

This implies

$$\begin{aligned}
&f(b_n) + f_1(b_n)(x_1 - b_n) + \frac{f_2(b_n)}{2!}(x_1 - b_n)^2 + \frac{1}{M}(x_1 - b_n)^2 \\
&> f(x_0) + f_1(x_0)(x_1 - x_0) + M(x_1 - x_0)^2;
\end{aligned}$$

that is,

$$\frac{f_2(b_n)}{2!}(x_1-b_n)^2 > f(x_0)-f(b_n)-f_1(b_n)(x_1-b_n)-\frac{1}{M}(x_1-b_n)^2$$
$$+ f_1(x_0)(x_1-x_0)+M(x_1-x_0)^2$$

by using (58)

$$\begin{aligned}
&> f_1(b_n)(x_0-b_n)+\frac{f_2(b_n)}{2!}(x_0-b_n)^2-f_1(b_n)(x_1-b_n)\\
&\quad -\frac{2}{M}(x_1-b_n)^2+f_1(x_0)(x_1-x_0)+M(x_1-x_0)^2\\
&= f_1(b_n)(x_0-x_1)+\frac{f_2(b_n)}{2!}(x_0-b_n)^2\\
&\quad -\frac{2}{M}(x_1-b_n)^2+f_1(x_0)(x_1-x_0)+M(x_1-x_0)^2\\
&= (f_1(x_0)-f_1(b_n))(x_1-x_0)+\frac{f_2(b_n)}{2!}(x_0-b_n)^2\\
&\quad -\frac{2}{M}(x_1-b_n)^2+M(x_1-x_0)^2
\end{aligned}$$

by (59)

$$\begin{aligned}
&> \left(\frac{f_2(b_n)}{2!}+M-\frac{1}{M}\right)(x_0-b_n)(x_1-x_0)+\frac{f_2(b_n)}{2!}(x_0-b_n)^2\\
&\quad -\frac{2}{M}(x_1-x_0)^2+M(x_1-x_0)^2.
\end{aligned}$$

Thus

$$\frac{f_2(b_n)}{2!}\left(1-\frac{(x_0-b_n)(x_1-x_0)}{(x_1-b_n)^2}-\frac{(x_0-b_n)^2}{(x_1-b_n)^2}\right) > \left(M-\frac{2}{M}\right)\frac{(x_1-x_0)^2}{(x_1-b_n)^2} > \frac{M}{2}.$$

Therefore,

$$f_2(b_n) > \frac{M}{2} > 0. \tag{60}$$

From (59) and $x < b_n < x_0$ it follows that

$$(f_1(b_n)-f_1(x_0))(x-b_n) > \left(\frac{f_2(b_n)}{2!}+M-\frac{1}{M}\right)(b_n-x_0)(x-b_n) > 0. \tag{61}$$

By using this and (57) we have

$$
\begin{aligned}
&F(x) - f(x_0) - f_1(x_0)(x - x_0) \\
=&f(b_n) + f_1(b_n)(x - b_n) + \frac{f_2(b_n)}{2!}(x - b_n)^2 - f(x_0) - f_1(x_0)(x - x_0) \\
>&M(b_n - x_0)^2 + (f_1(b_n) - f_1(x_0))(x - b_n) + \frac{f_2(b_n)}{2!}(x - b_n)^2 \qquad (62) \\
&\text{using (60), (61) and } x_0 < b_n \\
>&M(b_n - x_0)^2 + \frac{M}{4}(x - b_n)^2 > \frac{M}{16}(x - x_0)^2
\end{aligned}
$$

where in the last step we used that either $|b_n - x_0| \geq |x - x_0|/2$, or $|x - b_n| \geq |x - x_0|/2$.

Subsubcase BI1. Next we turn to the case when $|s(e)| = 1$. By $f_2(x_0) = +\infty$ and by Lemma 7 we can exclude $s(e) = -1$. Therefore $s(e) = 1$: that is, $f_2(e) = +\infty$. We have by (40) and $n \geq M + 1$

$$|f(x_0) - f(e) - f_1(e)(x_0 - e)| < \frac{1}{M}|x_0 - e| < \frac{1}{M}|x - x_0|, \tag{63}$$

and by (52)

$$|f(e) - f(x_0) - f_1(x_0)(e - x_0)| < \frac{1}{M}|e - x_0| < \frac{1}{M}|x - x_0|.$$

This also implies

$$|f_1(e) - f_1(x_0)| < \frac{2}{M}. \tag{64}$$

Hence, using (36)

$$
\begin{aligned}
&|F(x) - f(x_0) - f_1(x_0)(x - x_0)| \\
=&|f(e) + f_1(e)(x - e) + |x - e|^{3/2} - f(x_0) - f_1(x_0)(x - x_0)| \\
\leq&|f(e) - f(x_0) - f_1(x_0)(e - x_0)| + |x - e|^{3/2} + |f_1(e) - f_1(x_0)| \cdot |x - e| \qquad (65) \\
<&\frac{1}{M}|x - x_0| + |x - x_0|^{3/2} + \frac{2}{M}|x - x_0| = \left(\frac{3}{M} + |x - x_0|^{1/2}\right)|x - x_0|.
\end{aligned}
$$

Recall that $x < e = b_n < x_0$. By (52) we have

$$f(e) > f(x_0) + f_1(x_0)(e - x_0) + M(e - x_0)^2 \tag{66}$$

and by (36)

$$F(x) = f(e) + f_1(e)(x - e) + |x - e|^{3/2}. \tag{67}$$

Using (40) and $n > M$ we also have

$$f(x_0) > f(e) + f_1(e)(x_0 - e) + M(x_0 - e)^2$$

which, added to (66), yields

$$0 > f_1(x_0)(e - x_0) + f_1(e)(x_0 - e) + 2M(x_0 - e)^2.$$

This by $x_0 > e = b_n$ implies $f_1(x_0) > f_1(e)$. Using this in (67) and taking into consideration (66) and $x < e$ we obtain

$$\begin{aligned} F(x) &> f(e) + f_1(x_0)(x - e) + |x - e|^{3/2} \\ &> f(x_0) + f_1(x_0)(e - x_0) + M(e - x_0)^2 + f_1(x_0)(x - e) + |x - e|^{3/2} \\ &> f(x_0) + f_1(x_0)(x - x_0) + \frac{1}{4} \min\left\{M, \frac{1}{\sqrt{|x - e|}}\right\}(x - x_0)^2 \qquad (68) \end{aligned}$$

where in the last estimate we used that either $|x_0 - e| \geq \frac{|x - x_0|}{2}$, or $|x - e| \geq \frac{|x - x_0|}{2}$.

Estimates (49), (51), (53), (54), (56), (62), (65), and (68) imply that in all cases F is twice Peano differentiable at x_0 with infinite values allowed. □

As a corollary to Theorems 5 and 6 we obtain.

Theorem 8. *Let $H \subset \mathbb{R}$ be a closed set and let $f : H \to \mathbb{R}$ be twice Peano differentiable on H with infinite values allowed for f_2. Then f_2 is in Baire class one on H.*

Next we prove Lemma 7.

Proof. Lemma 7. Without limiting generality we assume that $e = b$ and $s(b) = 1$, that is, $f_2(b) = +\infty$.

We need to deal with two cases.

Case A: we show that there is no sequence $x_n \in H$ such that $x_n \to b$, $\delta(x_n) > 2(b - a)$, and $s(x_n) = -1$.

Proceeding towards a contradiction we assume that there is such a sequence. Choose $\eta' > 0$ such that $\eta' < b - a$ and

$$f(y) - f(b) - f_1(b)(y - b) > 10(y - b)^2 \qquad (69)$$

holds for $y \in (b, b + \eta') \cap H$. Choose and fix one such y.

For any x_n we have

$$f(y) - f(x_n) - f_1(x_n)(y - x_n) < -(y - x_n)^2. \qquad (70)$$

By using (69) and (70) we also obtain

$$\begin{aligned} f(x_n) + f_1(x_n)(y - x_n) - (y - x_n)^2 &> f(y) \\ &> f(b) + f_1(b)(y - b) + 10(y - b)^2. \end{aligned}$$

Since $n \to \infty$, we have $x_n \to b$ and $f(x_n) \to f(b)$. Therefore using the above displayed inequality we can choose N such that for $n \geq N$

$$f_1(x_n)(y - x_n) > f_1(b)(y - x_n) + 10(y - b)(y - x_n)$$

and hence

$$f_1(x_n) > f_1(b) + 10(y - b). \tag{71}$$

Using the definition of $f_1(b)$ choose and fix $\eta'' > 0$ such that

$$|f(y') - f(b) - f_1(b)(y' - b)| < (y - b)(y' - b)$$

holds for any $y' \in (b, b + \eta'') \cap H$. When n is large this implies for $y' = x_n$

$$f(x_n) < f(b) + f_1(b)(x_n - b) + (y - b)(x_n - b).$$

On the other hand,

$$f(b) < f(x_n) + f_1(x_n)(b - x_n) - (b - x_n)^2.$$

Using the above two inequalities we obtain

$$0 < (f_1(b) - f_1(x_n))(x_n - b) - (b - x_n)^2 + (y - b)(x_n - b)$$

and keeping in mind that $b < x_n < y$ for large n we infer

$$f_1(x_n) < f_1(b) - (x_n - b) + (y - b) < f_1(b) + (y - b)$$

which contradicts (71).

This completes the proof of Case A.

Case B. We assumed that $M' > 20$ and proceeding towards a contradiction we suppose that there exists a sequence $x_n \in H$ such that $x_n \to b$, $\delta(x_n) > 2(b - a)$ and $-\infty < f_2(x_n) < M'$. Since $f_2(b) = +\infty$, we can assume that $y \in H \cap (b, b + (b - a))$ is so close to b that

$$f(y) > f(b) + f_1(b)(y - b) + 2M(y - b)^2. \tag{72}$$

By (38)

$$\left| f(y) - f(x_n) - f_1(x_n)(y - x_n) - \frac{f_2(x_n)}{2!}(y - x_n)^2 \right| < (y - x_n)^2 \tag{73}$$

which implies

$$f(y) < f(x_n) + f_1(x_n)(y - x_n) + \left(\frac{f_2(x_n)}{2!} + 1 \right)(y - x_n)^2. \tag{74}$$

Setting $M = \frac{M'}{2} + 1 > 11$ we have $f_2(x_n)/2! < M - 1$.

Now (74) and (72) imply

$$\begin{aligned} f(x_n) + f_1(x_n)(y - x_n) + \left(\frac{f_2(x_n)}{2!} + 1 \right)(y - x_n)^2 \\ > f(b) + f_1(b)(y - b) + 2M(y - b)^2; \end{aligned}$$

that is,

$$
\begin{aligned}
&(f_1(x_n) - f_1(b))(y - x_n) > f(b) - f(x_n) + f_1(b)(x_n - b) \\
&+ \left(2M - \frac{f_2(x_n)}{2!} - 1\right)(y - x_n)^2 + 2M((y - b)^2 - (y - x_n)^2) \\
&> \left(\frac{3}{2}M - \frac{f_2(x_n)}{2!} - 1\right)(y - x_n)^2
\end{aligned} \tag{75}
$$

where the last inequality holds for large values of n, when $|x_n - b|$ and hence $|f(b) - f(x_n)|$ are both small. Now, by using the definition of $f_1(b)$ choose and fix $y' \in (b, y) \cap H$ such that

$$f(y') < f(b) + f_1(b)(y' - b) + \frac{M}{4}(y - b)(y' - b).$$

For large values of n we also have by (38)

$$f(y') > f(x_n) + f_1(x_n)(y' - x_n) + \frac{f_2(x_n)}{2!}(y' - x_n)^2 - (y' - x_n)^2.$$

Therefore,

$$
\begin{aligned}
f(b) - f(x_n) >&(f_1(x_n) - f_1(b))(y' - x_n) + f_1(b)(b - x_n) \\
&+ \frac{-M}{4}(y - b)(y' - b) + \frac{f_2(x_n)}{2!}(y' - x_n)^2 - (y' - x_n)^2 \\
&\\
&\text{using (75) divided by } y - x_n > 0 \\
&> \left(\frac{3M}{2} - \frac{f_2(x_n)}{2!} - 1\right)(y - x_n)(y' - x_n) + f_1(b)(b - x_n) \\
&+ \frac{-M}{4}(y - b)(y' - b) + \frac{f_2(x_n)}{2!}(y' - x_n)^2 - (y' - x_n)^2 \stackrel{\text{def}}{=} A.
\end{aligned}
$$

If n is large and $f_2(x_n) \geq 0$, using $M - 1 > f_2(x_n)/2!$, $(b - x_n) \to 0$ and $(y' - x_n)^2 < (y - b)(y' - b)$ we have

$$A > \left(\frac{M}{8} - 1\right)(y - b)(y' - b)$$

which contradicts $f(b) - f(x_n) \to 0$.

If n is large and $f_2(x_n) < 0$, we have

$$\frac{f_2(x_n)}{2!}(y' - x_n)^2 > \frac{f_2(x_n)}{2!}(y - x_n)(y' - x_n)$$

and

$$A > M(y - b)(y' - b)$$

which again contradicts $f(b) - f(x_n) \to 0$. □

4 Counterexample for $k \geq 3$

Theorem 9. *Assume $k \geq 3$. There exists a non-empty perfect set $E \subset [0,1]$ and a function $f : E \to \mathbb{R}$ which is k-times Peano differentiable on E with infinite values allowed for f_k such that $f_k(x) = +\infty$ for any right endpoint of an interval contiguous to E and $f_k(x) = -\infty$ for any other point of E.*

Proof. We denote by $\mathcal{T}_n$ the set of zero-one sequences of length n and by $\mathcal{T}_\infty$ the set of infinite zero-one sequences.

By induction we define a sequence $d_m > 0$ converging very fast to 0. We put $d_0 = 1$. Later we make more assumptions about the speed at which d_m converges to 0. Here we have some initial ones. For now the reader should only observe that all the assumptions given in (76) can be satisfied if d_m converges to 0 sufficiently fast,

$$\begin{aligned} &13 m d_{m-1} < (d_{m-2} - 2d_{m-1})^k \text{ for } m = 2, 3, \dots, \\ &(m+1)d_{m+1} < (m+2)d_{m+1} < \frac{1}{1000} d_m < m d_m \text{ for } m = 1, 2, \dots, \\ &2^{k-2} d_m (k-1)(d_{m-2} - 2d_{m-1})^{k-2} < 1 \text{ for } m = 2, 3, \dots, \\ &25 \cdot 2^k \sum_{m'=m+1}^{\infty} (m'+2) d_{m'}^k < d_m^k \text{ for } m = 1, 2, \dots, \\ &|(m+1)d_m^k - (m+1)(d_m - d_{m+1})^k| < (d_m - d_{m+1})^k \text{ for } m = 1, 2, \dots, \\ &3(d_m - 2d_{m+1})^{k-1} > 2d_m^{k-1} \text{ for } m = 1, 2, \dots . \end{aligned} \tag{76}$$

We will define the Cantor set $E \subset E_0 = [0,1]$ so that it is the intersection of the closed sets E_m, consisting of 2^m closed base intervals, each of length d_m, so that if $I = [a,b]$ is a base interval of E_m, then $I \cap E_{m+1}$ consists of two base intervals of E_{m+1}; one of which contains a and the other contains b. To base intervals of E_m, $m \geq 1$ we assign a zero-one sequence $\tau \in \mathcal{T}_m$ in the usual way, 0 is assigned to the left interval, I_0 of E_1 in E_0, and 1 is assigned to the right interval, I_1 of E_1 in E_0. If $I_\tau = [a_\tau, b_\tau]$ is a base interval of E_m for a $\tau \in \mathcal{T}_m$, we denote by $I_{\tau 0}$ the left and by $I_{\tau 1}$ the right subinterval of $I_\tau \cap E_{m+1}$.

If $\tau \in \mathcal{T}_\infty \cup \cup_{k=1}^\infty \mathcal{T}_k$, having at least m terms, its mth term is denoted by τ_m and $\tau|m$ denotes the sequence consisting of the first m terms of τ. For $n \in \mathbb{N}$ if τ has at least n ones among its terms, let $m_n(\tau)$ be such that $\tau_{m_n(\tau)}$ is the nth one in τ. Otherwise set $m_n(\tau) = 0$. Set $\phi_n(\tau) = \tau|m_n(\tau)$ when $m_n(\tau) \neq 0$, and $\phi_n(\tau) = 0$ if $m_n(\tau) = 0$.

To $x \in E = \cap_{m=1}^\infty E_m$ we assign $\tau(x)$ such that it is the zero-one code of x; that is, $x \in I_{\tau(x)|m}$ for all $m \in \mathbb{N}$. To $\tau \in \mathcal{T}_\infty$ we assign $x(\tau) \in E$ so that $x(\tau) \in I_{\tau|m}$ for all $m \in \mathbb{N}$. Denote by $\mathcal{T}_\infty^1$ the set of those $\tau \in \mathcal{T}_\infty$ which contain finitely many ones and by $\mathcal{T}_\infty^{1,n}$ the set of those which contain exactly n ones.

Next we choose the sequence d_m. Choose d_1 satisfying (76). Suppose d_{m-1} is defined for an $m \geq 2$. Let $\tau \in \mathcal{T}_m$. For each $\tau \in \mathcal{T}_m$ we will choose a polynomial, g_τ of degree no more than $k-1$. If $\tau_m = 0$; that is, there is no $n \in \mathbb{N}$ such that $m_n(\tau) = m$, then we set $g_\tau = 0$. If $\tau_m = 1$, then there is an

$n \in \mathbb{N}$ such that $m_n(\tau) = m$. (When $m = 1$ we set $g_\tau = 0$ again.) If $n \geq 2$, set $J_m = I_{\tau|m_{n-1}(\tau)} = [\alpha_m, \beta_m]$, if $n = 1$ set $J_m = E_0 = [0,1] = [\alpha_m, \beta_m]$.

Set $\delta_m = \beta_m - \alpha_m$ and $\mu_m = m_{n-1}(\tau)$. By definition $\delta_m = d_{m_{n-1}(\tau)} = d_{\mu_m}$. Observe that $\tau_m = 1$ and all digits $\tau_k = 0$ for $\mu_m < k < m$ (or for $k < m$ when $J_m = [0,1]$). Hence $I_{\tau|m-1} = [\alpha_m, \alpha_m + d_{m-1}]$ and b_τ, the right endpoint of I_τ, will equal $\alpha_m + d_{m-1}$, the right endpoint of $I_{\tau|m-1}$. After we chose d_m the interval $I_\tau = [a_\tau, b_\tau]$ will have left endpoint $a_\tau = b_\tau - d_m$.

We choose the polynomial g_τ so that

$$\begin{aligned} g_\tau(\alpha_m) &= m(-d_{m-1})^k \\ g_\tau(b_\tau) &= g_\tau(\alpha_m + d_{m-1}) = m d_{m-1}^k \\ g_\tau(\beta_m + \delta_m) &= g_\tau(\alpha_m + 2\delta_m) = 2\mu_m (2\delta_m)^k. \end{aligned} \tag{77}$$

Specifically we define g_τ by

$$g_\tau(t) = m(-d_{m-1})^k + r(t-\alpha_m)^{k-1} + s(t-\alpha_m)^{k-2}$$

where conditions in (77) imply that

$$\begin{aligned} r =& \frac{8\mu_m\delta_m^2 - \frac{m(-d_{m-1})^k}{(2\delta_m)^{k-2}} - m(1+(-1)^{k+1})d_{m-1}^2}{2\delta_m - d_{m-1}} \\ s =& \frac{-d_{m-1}}{2\delta_m - d_{m-1}} 8\mu_m\delta_m^2 + \frac{2\delta_m}{2\delta_m - d_{m-1}} m(1+(-1)^{k+1})d_{m-1}^2 \\ &+ \frac{d_{m-1} m(-d_{m-1})^k}{(2\delta_m - d_{m-1})(2\delta_m)^{k-2}}. \end{aligned}$$

Using $1 \geq \delta_m \geq d_{m-1}$, (76) and $m d_{m-1} \leq \mu_m \delta_m$ one can obtain the following rough estimates of r, and s.

$$\begin{aligned} &|r| \leq 11\mu_m\delta_m \text{ and} \\ &|s| < 8\mu_m\delta_m d_{m-1} + 4m\delta_m d_{m-1} + m d_{m-1}^2 \leq 13\mu_m\delta_m d_{m-1}. \end{aligned} \tag{78}$$

Next we estimate g_τ. If $\mu_m = m_{n-1}(\tau) = m-1$, then $J_m = I_{\tau|m-1}$, $\delta_m = d_{m-1}$ and J_m does not contain any $y > b_\tau = \beta_m$. So if there exists $y > b_\tau$ with $y \in J_m$, then $\mu_m = m_{n-1}(\tau) \leq m-2$ and $\delta_m = d_{\mu_m}$ is much larger than d_{m-1}.We assume $x \in [\alpha_m, \alpha_m + d_{m-1}]$; in fact, after d_m is defined we assume $x \in [a_\tau, b_\tau] \subset [\alpha_m, \alpha_m + d_{m-1}]$.

Case A. Assume $y > b_\tau > \alpha_m$, $y \in J_m \cap E_{m-1}$. Then $d_{m-2} - d_{m-1} \leq y - \alpha_m \leq \delta_m$. Then $d_{m-2} - 2d_{m-1} \leq y - x \leq \delta_m$ and $y - x \leq y - \alpha_m < 2(d_{\mu_m} - 2d_{\mu_m - 1}) \leq 2(y-x)$. Using $\mu_m \leq m$ and $\delta_m \leq 1$ by (76) and (78) we have

$$\begin{aligned} |s| <& 13\mu_m\delta_m d_{m-1} < 13 m d_{m-1} \\ <& (d_{m-2} - 2d_{m-1})^k \leq (y-x)^k \leq (y-\alpha_m)^k. \end{aligned}$$

By (76) we also have

$$|m(-d_{m-1})^k| < 13md_{m-1} \leq (d_{m-2} - 2d_{m-1})^k \leq (y-x)^k \leq (y-\alpha_m)^k.$$

Hence

$$|g_\tau(y) - r(y-\alpha_m)^{k-1}| \leq 2(y-x)^k \leq 2(y-\alpha_m)^k. \tag{79}$$

Now by (76) we have $md_{m-1} < d_{m-2}/1000 \leq \delta_m/1000$ and this implies

$$3.99\mu_m\delta_m < \frac{7.99\mu_m\delta_m^2}{2\delta_m} < r < \frac{8.01\mu_m\delta_m^2}{1.99\delta_m} < 5\mu_m\delta_m < 5m\delta_m. \tag{80}$$

We also have

$$\begin{aligned}
&|r(y-\alpha_m)^{k-1} - r(y-x)^{k-1}| = r|(y-\alpha_m)^{k-1} - (y-x)^{k-1}| \\
\leq& r|(y-\alpha_m)^{k-1} - (y-\alpha_m-d_{m-1})^{k-1}| \leq rd_{m-1}(y-\alpha_m)^{k-2}(k-1) \\
\leq& 2^{k-2}rd_{m-1}(y-x)^{k-2}(k-1) \leq 2^{k-2}5m\delta_m d_{m-1}(k-1)(y-x)^{k-2} \\
&\text{(by using (76))} \\
<& 2^{k-2}\delta_m(k-1)(d_{m-2}-2d_{m-1})^k(y-x)^{k-2} \\
<& 2^{k-2}\delta_m(k-1)(\delta_{m-2}-2\delta_{m-1})^{k-2}(y-x)^k < (y-x)^k.
\end{aligned}$$

Therefore, by also using (79)

$$|g_\tau(y)| \leq r(y-x)^{k-1} + 3(y-x)^k < (5\mu_m+3)\delta_m(y-x)^{k-1}. \tag{81}$$

From this and (80) it also follows that

$$\begin{aligned}
3((\mu_m-1)\delta_m)(y-x)^{k-1} &< r(y-x)^{k-1} - 3(y-x)^k \leq g_\tau(y) \\
&\leq r(y-x)^{k-1} + 3(y-x)^k < 5(\mu_m+1)\delta_m(y-x)^{k-1}.
\end{aligned} \tag{82}$$

Case B. Recall that $J_m = I_{\tau|m_{n-1}(\tau)} = I_{\tau|\mu_m}$ and assume $y \notin J_m$ but $y \in E_{\mu_m}$. (This means that we also allow $y < x$ and no longer assume $\mu_m \leq m-2$.) Then $|y-x| > d_{\mu_m-1}/2 > \mu_m\delta_m > \delta_m \geq d_{m-1}$. Similarly, $d_{m-1} \leq \mu_m\delta_m < |y-\alpha_m|$, and

$$|y-x|/2 < |y-\alpha_m| < 2|y-x|.$$

Now we use estimates (78) to obtain

$$\begin{aligned}
|g_\tau(y)| &\leq \mu_m\delta_m^k + 11\mu_m\delta_m|y-\alpha_m|^{k-1} + 13\mu_m\delta_m d_{m-1}|y-\alpha_m|^{k-2} \\
&\leq \mu_m\delta_m^k + 11\mu_m\delta_m|y-\alpha_m|^{k-1} + 13\mu_m\delta_m|y-\alpha_m|^{k-1} \\
&\leq 25\cdot 2^{k-1}\mu_m\delta_m|y-x|^{k-1}.
\end{aligned} \tag{83}$$

Case C. Trival Estimates Finally we need a trivial/crude estimate of $g_\tau^{(\ell)}(t)$ for all $t \in J_m$ and $\ell = 0, \ldots, k-1$. In this estimate we will use

$$|m(-d_{m-1})^k| < md_{m-1} < 2\mu_m\delta_m \leq (k-1)!\mu_m\delta_m,$$

$$|(r(t-\alpha_m)^{k-1})^{(\ell)}| < |r|(k-1)! < 11\mu_m\delta_m(k-1)!,$$

and

$$|(s(t-\alpha_m)^{k-2})^{(\ell)}| < |s|(k-1)! < 13\mu_m\delta_m(k-1)!.$$

These imply

$$|g_\tau^{(\ell)}(t)| < (k-1)! \cdot 25\mu_m\delta_m = (k-1)! \cdot 25 m_{n-1}(\tau) d_{m_{n-1}(\tau)}. \tag{84}$$

Now we are ready to fix the value of d_m. Using (77) and the continuity of g_τ we can assume that $d_m > 0$ will be chosen so small, that in addition to the above assumptions we also have

$$|g_\tau(t) - m(-d_{m-1})^k| < d_{m-1}^k \text{ for all } t \in [\alpha_m, \alpha_m + d_m] \tag{85}$$

and

$$|g_\tau(t) - m d_{m-1}^k| < d_{m-1}^k \text{ for all } t \in [b_\tau - d_m, b_\tau]. \tag{86}$$

Since there are only 2^m many τ's in $\mathcal{T}_m$, we have only finitely many new restrictions on d_m if we want to satisfy (85) and (86) for all $\tau \in \mathcal{T}_m$. Following the above steps for all $m \in \mathbb{N}$ and $\tau \in \mathcal{T}_m$ we define d_m and g_τ. Set $E = \cap_{m=1}^\infty E_m$.

Next we define our function $f(x)$ for $x \in E$. Set

$$f(x) = \sum_{n=1}^\infty g_{\phi_n(\tau(x))}(x) \tag{87}$$

and in general for $\ell = 0, \dots, k-1$ we also set

$$f_\ell(x) = \sum_{n=1}^\infty g^{(\ell)}_{\phi_n(\tau(x))}(x). \tag{88}$$

Since $x \in J_m$ when $m = m_n(\tau(x))$, we can use the rough estimates from (84) to show that the series in (87) and (88) converge.

Assume $x \in E$ and $\tau(x) \in \mathcal{T}_\infty^1$. Then $\tau(x) \in \mathcal{T}_\infty^{1,\nu}$ for some $\nu \in \{0, 1, \dots\}$ and x is the right endpoint of an interval contiguous to E. For all such right endpoints x, $\tau(x) \in \mathcal{T}_\infty^1$. Then $f(x) = \sum_{n=1}^\infty g_{\phi_n(\tau(x))}(x) = \sum_{n=1}^\nu g_{\phi_n(\tau(x))}(x)$, since $\phi_n(\tau(x)) = 0$ implies $g_{\phi_n(\tau(x))} = 0$ for $n > \nu$. We also have $f_\ell(x) = \sum_{n=1}^\nu g^{(\ell)}_{\phi_n(\tau(x))}(x)$ for $\ell = 0, \dots, k-1$. Since $g_{\phi_n(\tau(x))}(t)$ is a polynomial of degree $\le k-1$, we have

$$\sum_{n=1}^\nu g_{\phi_n(\tau(x))}(y) = \sum_{\ell=0}^{k-1} \sum_{n=1}^\nu \frac{g^{(\ell)}_{\phi_n(\tau(x))}(x)}{\ell!}(y-x)^\ell.$$

Since x is the right endpoint of an interval contiguous to E, to check Peano differentiability at x we can assume that $y \in I_{\tau(x)|m_\nu(\tau(x))}$ and $y \neq x$, which implies $y > x$ and $\phi_n(\tau(y)) = \phi_n(\tau(x))$ for $n = 1, \ldots, \nu$. Set

$$\begin{aligned} A &\stackrel{\text{def}}{=} f(y) - \sum_{\ell=0}^{k-1} \frac{f_\ell(x)}{\ell!}(y-x)^\ell \\ &= \sum_{n=1}^{\infty} g_{\phi_n(\tau(y))}(y) - \sum_{\ell=0}^{k-1} \frac{f_\ell(x)}{\ell!}(y-x)^\ell \\ &= g_{\phi_{\nu+1}(\tau(y))}(y) + \sum_{n=\nu+2}^{\infty} g_{\phi_n(\tau(y))}(y). \end{aligned}$$

Denote $m_{\nu+1}(\tau(y)) - 1$ by m_y. In this case $x, y \in I_{\tau(y)|m_y}$ but $x \notin I_{\tau(y)|m_y+1}$ and $d_{m_y}/2 < d_{m_y} - d_{m_y+1} \leq y - x \leq d_{m_y}$. By (86)

$$|g_{\phi_{\nu+1}(\tau(y))}(y) - (m_y + 1)d_{m_y}^k| < d_{m_y}^k.$$

By (76)

$$\begin{aligned} &|(m_y+1)d_{m_y}^k - (m_y+1)(y-x)^k| \\ \leq &|(m_y+1)d_{m_y}^k - (m_y+1)(d_{m_y} - d_{m_y+1})^k| < d_{m_y}^k \end{aligned}$$

and again by (86)

$$\Big|\sum_{n=\nu+2}^{\infty} g_{\phi_n(\tau(y))}(y)\Big| \leq \sum_{m=m_y+1}^{\infty} (m+1)d_m^k < d_{m_y}^k$$

where in the last step we used (76). Hence

$$|A - (m_y+1)(y-x)^k| < 3 \cdot d_{m_y}^k < 3 \cdot 2^k (y-x)^k. \tag{89}$$

Since $m_y \to \infty$ as $y \to \infty$, (89) implies $A \cdot k!/(y-x)^k \to +\infty$ as $y \to x$, $y \in E$. On the other hand, from (89) it also follows that $|A| \leq (m_y + 1 + 3 \cdot 2^k)(y-x)^k$. This implies

$$|A|/(y-x)^{k-1} < (m_y + 1 + 3 \cdot 2^k)(y-x) \leq (m_y + 1 + 3 \cdot 2^k)d_{m_y} \to 0$$

as $y \to x$, $y \in E$.
Thus f is k times Peano differentiable at x and $f_k(x) = +\infty$.

Now we assume that $x \in E$, $\tau(x) \notin \mathcal{T}_\infty^1$ and $y \to x$. Choose ν such that $y \in I_{\tau(x)|m_\nu(\tau(x))}$ but $y \notin I_{\tau(x)|m_{\nu+1}(\tau(x))}$. This implies $\tau(x)|m_\nu(\tau(x)) = \tau(y)|m_\nu(\tau(y))$. We will again use the notation

$$A \stackrel{\text{def}}{=} f(y) - \sum_{\ell=0}^{k-1} \frac{f_\ell(x)}{\ell!}(y-x)^\ell.$$

First we assume $y > x$. Set $m_y = m_{\nu+1}(\tau(y)) - 1$. In this case $x, y \in I_{\tau(y)|m_y} =$

$I_{\tau(x)|m_y}$ but $I_{\tau(x)|m_y+1}$ is the left, and $I_{\tau(y)|m_y+1}$ is the right "half" of $I_{\tau(y)|m_y}$ and $y - x > d_{m_y}/2$. Now we have

$$\begin{aligned}A =&\sum_{n=1}^{\infty} g_{\phi_n(\tau(y))}(y) - \sum_{n=1}^{\infty} g_{\phi_n(\tau(x))}(y)\\ =&g_{\phi_{\nu+1}(\tau(y))}(y) - g_{\phi_{\nu+1}(\tau(x))}(y) + \sum_{n=\nu+2}^{\infty} g_{\phi_n(\tau(y))}(y) - \sum_{n=\nu+2}^{\infty} g_{\phi_n(\tau(x))}(y)\\ =&A_1 - A_2 + A_3 - A_4.\end{aligned}$$

Clearly by (86) and (76)

$$|A_3| \leq \sum_{m=m_y+1}^{\infty} (m+1)d_m^k < d_{m_y}^k.$$

By (86)

$$|A_1 - (m_y+1)d_{m_y}^k| = |g_{\phi_{\nu+1}(\tau(y))}(y) - (m_y+1)d_{m_y}^k| < d_{m_y}^k$$

and by (82) used with $\mu_m = m_\nu(\tau(x))$, $\delta_m = d_{m_\nu(\tau(x))}$

$$|A_2| = |g_{\phi_{\nu+1}(\tau(x))}(y)| \leq 5(m_\nu(\tau(x))+1)d_{m_\nu(\tau(x))}|y-x|^{k-1} \tag{90}$$

and

$$\begin{aligned}A_2 = g_{\phi_{\nu+1}(\tau(x))}(y) >&3(m_\nu(\tau(x))-1)d_{m_\nu(\tau(x))}|y-x|^{k-1}\\ \geq&(m_\nu(\tau(x))-1)d_{m_\nu(\tau(x))}3(d_{m_y} - 2d_{m_y+1})^{k-1} \qquad (91)\\ &\text{ using (76)}\\ >&2(m_\nu(\tau(x))-1)d_{m_\nu(\tau(x))}d_{m_y}^{k-1}.\end{aligned}$$

If $n \geq \nu + 2$, then $y - x > d_{m_{\nu+1}(\tau(x))} \geq d_{m_{n-1}(\tau(x))}$, $y \notin I_{\tau(x)|m_{n-1}(\tau(x))}$ and hence by (83) and (76)

$$|A_4| \leq 25 \cdot 2^{k-1}(y-x)^{k-1} \sum_{m=m_y+1}^{\infty} m \cdot d_m < d_{m_y}^k.$$

Using the above estimates we obtain

$$|A| = |A_1 - A_2 + A_3 - A_4| < (m_y+4)d_{m_y}^k + 5(m_\nu(\tau(x))+1)d_{m_\nu(\tau(x))}d_{m_y}^{k-1}$$

and hence

$$\frac{|A|}{|y-x|^{k-1}} < 2^{k-1} \cdot (m_y+4)d_{m_y} + 2^{k-1} \cdot 5(m_\nu(\tau(x))+1)d_{m_\nu(\tau(x))}.$$

Since $\nu \to \infty$ as $y \to x$, this implies that

$$\frac{A}{|y-x|^{k-1}} \to 0 \text{ if } y > x,\ y \to x,\ y \in E.$$

By using (91) we have

$$\begin{aligned}
A_1 - A_2 <&(m_y+2)d_{m_y}^k - 2(m_\nu(\tau(x))-1)d_{m_\nu(\tau(x))}d_{m_y}^{k-1}\\
<&\Big(-2(m_\nu(\tau(x))-1)d_{m_\nu(\tau(x))} + (m_y+2)d_{m_y}\Big)d_{m_y}^{k-1}\\
&\text{by (76) and } m_\nu(\tau(x)) \le m_y\\
<&\Big(-2(m_\nu(\tau(x))-1)d_{m_\nu(\tau(x))} + (m_\nu(\tau(x))+2)d_{m_\nu(\tau(x))}\Big)d_{m_y}^{k-1}\\
<&(-m_\nu(\tau(x))+4)d_{m_\nu(\tau(x))}d_{m_y}^{k-1}\\
<&-0.5\cdot m_\nu(\tau(x))d_{m_y}^k < -0.5\cdot m_\nu(\tau(x))(y-x)^k.
\end{aligned}$$

Therefore,

$$\frac{A\cdot k!}{(y-x)^k} < -0.5k!\cdot m_\nu(\tau(x)) + 2\cdot 2^k \to -\infty$$

as $\nu \to \infty$; that is, as $y \to x$, $y > x$, $y \in E$.

Next we assume $y < x$ and $y \to x$, $y \in E$. Set $m_y = m_{\nu+1}(\tau(x)) - 1$. In this case $x, y \in I_{\tau(y)|m_y} = I_{\tau(x)|m_y}$ but $I_{\tau(x)|m_y+1}$ is the right, and $I_{\tau(y)|m_y+1}$ is the left "half" of $I_{\tau(y)|m_y}$ and $d_{m_y}/2 < x - y < d_{m_y}$. Now we have

$$\begin{aligned}
A =&\sum_{n=1}^\infty g_{\phi_n(\tau(y))}(y) - \sum_{n=1}^\infty g_{\phi_n(\tau(x))}(y)\\
=&-g_{\phi_{\nu+1}(\tau(x))}(y) + \sum_{n=\nu+1}^\infty g_{\phi_n(\tau(y))}(y) - \sum_{n=\nu+2}^\infty g_{\phi_n(\tau(x))}(y)\\
=&-B_1 + B_2 - B_3.
\end{aligned}$$

Clearly, by (86) and (76)

$$|B_2| \le \sum_{m=m_y+1}^\infty (m+1)d_m^k < d_{m_y}^k.$$

If $n \ge \nu+2$, then $x - y > d_{m_{\nu+1}(\tau(x))} \ge d_{m_{n-1}(\tau(x))}$, $y \notin I_{\tau(x)|m_{n-1}(\tau(x))}$ and hence by (83) and (76)

$$|B_3| \le 25\cdot 2^{k-1}|y-x|^{k-1}\sum_{m=m_y+1}^\infty m\cdot d_m < d_{m_y}^k.$$

By $m_y + 1 = m_{\nu+1}(\tau(x))$ and (85)

$$|B_1 - (m_y+1)(-d_{m_y})^k| = |g_{\phi_{\nu+1}(\tau(x))}(y) - (m_y+1)(-d_{m_y})^k| < d_{m_y}^k.$$

Hence

$$|A + (m_y + 1)(-d_{m_y})^k| < 3d_{m_y}^k \tag{92}$$

and

$$|A| < (m_y + 4)d_{m_y}^k < (m_y + 4)2^{k-1}d_{m_y}|y - x|^{k-1}.$$

This implies

$$\left|\frac{A}{(y-x)^{k-1}}\right| < (m_y + 4)2^{k-1}d_{m_y} \to 0$$

since $m_y \to \infty$ as $y \to x$, $y < x$, $y \in E$.
On the other hand, by (92) (used with a little caution when k is odd)

$$\begin{aligned}\frac{A}{(y-x)^k} &< \frac{-(m_y + 1)(-d_{m_y})^k + (-1)^k 3d_{m_y}^k}{(y-x)^k} \\ &< \frac{-(m_y - 2)d_{m_y}^k}{(x-y)^k} < -(m_y - 2)\end{aligned}$$

which implies

$$\frac{A \cdot k!}{(y-x)^{k-1}} \to -\infty$$

as $y \to x$, $y < x$, $y \in E$.

□

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