Tensor products of AC_* charges and AC Radon measures are not always AC_* charges

Zoltán Buczolich*, Department of Analysis, Eötvös Loránd University, Kecskeméti u. 10-12, Budapest, Hungary, 1053

August 17, 2000

Abstract

In this note we give an example of an AC_* charge, F, on $\mathbb{R}$ and an absolutely continuous Radon measure μ on $\mathbb{R}$ such that $F \otimes \mu$ is not an AC_* charge on $\mathbb{R}^2$.

1 Introduction

In [1] the tensor problem was stated for the tensor product of AC_* charges and the Lebesgue measure. Later W.F. Pfeffer suggested to look at possible generalizations of this problem. In this note we show that a more general version of this problem is false. We also give a version of the tensor theorem which seems to be true and will be the subject of a forthcoming paper of this author.

The generalized version of the tensor problem is the following (for the details of the definitions see the Preliminaries section of this paper or [1]):

Let μ be an absolutely continuous Radon measure in $\mathbb{R}^n$, and let F be a charge in $\mathbb{R}^m$ where m and n are positive integers. (In this paper absolute continuity is always considered with respect to the Lebesgue measure.) Given a bounded BV set $B \subset \mathbb{R}^{m+n}$, let

$$B^y = \{x \in \mathbb{R}^m : (x, y) \in B\}$$

and

$$(F \otimes \mu)(B) = \int_{\mathbb{R}^n} F(B^y) d\mu(y).$$

It is not difficult to see that $F \otimes \mu$ is a charge and and the question is whether $F \otimes \mu$ is AC_* in $E \times \mathbb{R}^n$ whenever F is AC_* in a locally BV set $E \subset \mathbb{R}^m$.

*This author was supported by Grants FKFP 0192/1999 and Hungarian National Foundation for Scientific Research Grant No. T 032042

1991 Mathematics Subject Classification: Primary 26B30 Secondary: 26B05, 46B28.

Keywords: non-absolute integrals, variational measure, tensor product.

This question is of interest because for nonabsolute integrals usually there is no Fubini's theorem and hence results concerning products are useful.

In this note we give a counterexample in the simplest possible setting. The charge F will be an AC_* charge on $\mathbb{R}$ (in fact, it will be supported on $[0,1]$) and μ will be an absolutely continuous Radon measure supported on $[0,1]$. Their tensor product $F \otimes \mu$ will not be an AC_* charge on $\mathbb{R}^2$. More sophisticated similar examples can be obtained in higher dimensions as well.

It is easy to see that if μ has an atomic part then the tensor conjecture is false. Our example shows that even for absolutely continuous measures one can obtain counterexamples. In our construction we give a μ such that its density function (its Radon-Nikodym derivative) ϕ is unbounded. It seems to us that the conjecture on tensor products is true if we assume that $\phi \in L^\infty(\mathbb{R}^n)$.

2 Preliminaries

We denote by $d(A)$, and $|A|$ the diameter and the Lebesgue measure of the set $A \subset \mathbb{R}^m$, respectively.

The open ball centered at x and of radius r will be denoted by $B(x,r)$.

Closed intervals in $\mathbb{R}^m$ are Cartesian products of closed one-dimensional intervals. Figures in $\mathbb{R}^m$ are finite unions of closed m-dimensional intervals.

The Lebesgue density points (see for example [2] Corollary 6.2.6) of a measurable set $A \subset \mathbb{R}^m$ form its essential interior, $\mathrm{int}^* A$. The Lebesgue dispersion points (zero-density points) of A form its essential exterior $\mathrm{ext}^* A$. The essential boundary of A equals $\partial^* A \overset{\mathrm{def}}{=} \mathbb{R}^m \setminus (\mathrm{int}^* A \cup \mathrm{ext}^* A)$.

The measurable set $A \subset \mathbb{R}^m$ is a BV set if its perimeter $||A|| \overset{\mathrm{def}}{=} \mathcal{H}^{m-1}(\partial^* A)$ is finite (here $\mathcal{H}^{m-1}$ denotes the $(m-1)$-dimensional Hausdorff measure).

One-dimensional BV sets coincide (modulo sets of zero Lebesgue measure) with finite unions of closed intervals, that is, with one-dimensional figures. Recalling that the zero-dimensional Hausdorff measure is the counting measure if the BV set $A \subset \mathbb{R}$ is equivalent to $\cup_{j=1}^k I_j$ (where the I_j's are disjoint closed intervals) then the essential boundary of A coincides with the endpoints of the intervals I_j and $||A|| = 2k$ (the number of the endpoints of these intervals).

A function F defined on m-dimensional BV-sets is an m-dimensional charge if it is

1. additive, that is $F(A \cup B) = F(A) + F(B)$ holds for any two nonoverlapping BV sets A and B (recall that A and B are nonoverlapping if $\mathrm{int}^* A \cap \mathrm{int}^* B = \emptyset$);

2. continuous, that is for every $\epsilon > 0$ there exists $\rho > 0$ such that $|F(A)| < \epsilon$ if $|A| < \rho$, $||A|| < 1/\epsilon$ and $A \subset B(0, 1/\epsilon)$.

Assume $E \subset \mathbb{R}^m$, $\delta : E \to [0, \infty)$ is a gage on E if $\{x : \delta(x) = 0\}$ is thin, that is, it is of σ-finite $(m-1)$-dimensional Hausdorff measure.

A system $\mathcal{P} = \{(A_i, x_i)\}_{i=1}^p$ is a partition if the A_i's are nonoverlapping BV sets and $x_i \in A_i$ for $i = 1, ..., p$. The partition $\mathcal{P}$ is anchored in E if $x_i \in E$ for all $i = 1, ..., p$. If a gage δ is given on E then $\mathcal{P}$ is δ-fine if $A_i \subset B(x_i, \delta(x_i))$ holds for all $i = 1, ..., p$.

The regularity, $r(A)$, of a BV-set $A \subset \mathbb{R}^m$ is the number $|A|/d(A)||A||$ if $d(A)||A|| \neq 0$, otherwise $r(A) \stackrel{\text{def}}{=} 0$. If $\eta > 0$ and $r(A) > \eta$ we say that A is η-regular. A partition $\mathcal{P}$ is η-regular if $r(A_i) > \eta$ for all $i = 1, ..., p$.

If a one-dimensional BV set $A \subset \mathbb{R}$ is η-regular then

$$d(A) \geq |A| > \eta d(A)||A|| \text{ and hence } 1/\eta \geq ||A||, \tag{1}$$

that is, A is equivalent to a finite union of less than $[1/2\eta]$ many closed intervals. Since $||A|| \geq 2$ we also have

$$|A| > \eta d(A)||A|| \geq 2\eta d(A). \tag{2}$$

Assume F is a charge defined on BV sets of $\mathbb{R}^m$, $\eta > 0$, $E \subset \mathbb{R}^m$ and $\delta : E \to [0, \infty)$ is a gage. Set

$$V_{\eta,\delta}F(E) = \sup_{\mathcal{P}} \sum_{i=1}^{p} |F(A_i)|,$$

where the supremum is taken for all η-regular, δ-fine partitions $\mathcal{P} = \{(A_i, x_i)\}_{i=1}^{p}$. The F-variational outer measure of E is defined as

$$V_*F(E) = \sup_{0<\eta} \inf_{\delta} V_{\eta,\delta}F(E),$$

where the infimum is taken for all δ gage functions defined on E. It is not difficult to show that V_*F is a metric outer measure.

We say that F is an AC_* (m-dimensional) charge if V_*F is absolutely continuous with respect to the Lebesgue measure (in $\mathbb{R}^m$).

3 The main result

Theorem 1. *There exists a charge F defined on BV sets of $\mathbb{R}$ and an absolutely continuous Radon measure μ in $\mathbb{R}$, such that $F \otimes \mu$ is not an AC_* charge in $\mathbb{R}^2$.*

Proof. Let C be the triadic Cantor-set. We denote by C_k the union of the 2^k many closed intervals of length 3^{-k} which are used in the definition of C at step k. We denote these intervals by $G_{1,k}, ..., G_{2^k,k}$. We denote by $G'_{1,k}, ..., G'_{2^k,k}$ the intervals which are concentric with $G_{1,k}, ..., G_{2^k,k}$, respectively, and which are of length

$$h_k = \frac{1}{k^2 2^{k+1} 3^{3k+2}}.$$

Observe that $C_{k+1} \cap \cup_{j=1}^{2^k} G'_{j,k} = \emptyset$.

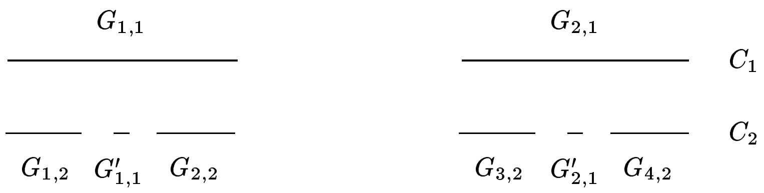


figure 1.

On $G'_{j,k}$ we set for each $j = 1, ..., 2^k$

$$\phi(x) = \frac{1}{h_k k^2 2^k}.$$

Observe that

$$\int_{\cup_j G'_{j,k}} \phi(x)dx = 2^k \frac{h_k}{h_k k^2 2^k} = \frac{1}{k^2}.$$

Set $\phi(x) = 0$ for $x \notin \cup_{k=1}^{\infty} \cup_{j=1}^{2^k} G'_{j,k}$.

Put $\mu(A) = \int_A \phi(x)dx$. Observe that μ is an absolutely continuous Radon measure with support in $[0, 1]$ and

$$\mu(G'_{j,k}) = \frac{1}{k^2 2^k}. \tag{3}$$

for each $k \in \mathbb{N}$ and $j = 1, ..., 2^k$.

To define the charge F we need an auxiliary Cantor type set E. We define this set as the intersection of some auxiliary nested closed sets E_k and each E_k will be a union of nonoverlapping closed intervals, each of length 3^{-k^2}.

Set $E_0 = I_{1,0} = [0, 1]$, $N_0 = 1$.

Assume $k \geq 0$, E_k is defined and equals the union of nonoverlapping closed intervals $I_{j,k}$, $j = 1, ..., N_k$, each of length 3^{-k^2}.

To define E_{k+1} divide each $I_{j,k}$ into $3^{-k^2}/3^{-(k+1)^2} = 3^{(k+1)^2-k^2}$ many nonoverlapping intervals, each of length $3^{-(k+1)^2}$, keep the first $3^{(k+1)^2-k^2} - 1$ of them, and delete the last one. Repeating this procedure in each $I_{j,k}$ we obtain the closed intervals $I_{j',k+1}$, $j' = 1, ..., N_{k+1}$. Where $N_{k+1} = (3^{(k+1)^2-k^2} - 1)N_k$. For $j = 1, ..., N_k$ denote by $J_{j,k}$ the last, deleted, subinterval of $I_{j,k}$. Set $E_{k+1} = \cup_{j'=1}^{N_{k+1}} I_{j',k+1}$.

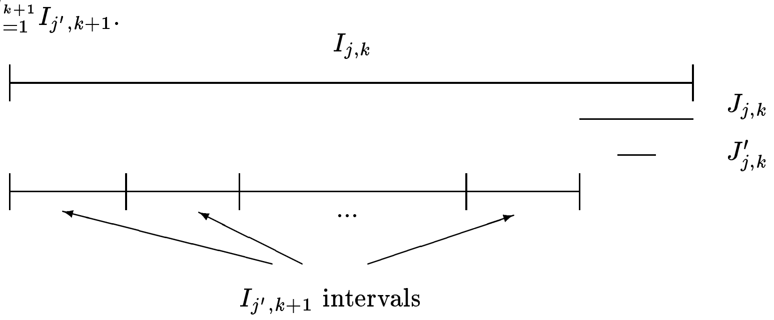


figure 2.

By induction we define E_k for all $k \in \mathbb{N}$ and set $E = \cap_{k=1}^{\infty} E_k$.

Observe that for $k \geq 1$

$$|E_k| = \prod_{j=1}^{k} \left(1 - 3^{(j-1)^2 - j^2}\right)$$

and hence $|E| > 0$.

Assume that k is fixed and we choose a $j \in \{1, ..., N_k\}$. Then the interval $I_{j,k}$ contains a subinterval $J_{j,k}$ of length $3^{-(k+1)^2}$. Denote by $J'_{j,k}$ the subinterval of $J_{j,k}$ which is concentric with it and of length $(1/3)|J_{j,k}|$. Observe that $dist(J'_{j,k}, E_{k+1}) = (1/3)|J_{j,k}| = 3^{-(k+1)^2-1}$. Set

$$\kappa_k = k^4 2^{k^2+1} 3^{(k+1)^2}.$$

Divide $J'_{j,k}$ into κ_k many equal subintervals, $J_{i,j,k}$, $i = 1, ..., \kappa_k$. Then $|J_{i,j,k}| = 3^{-(k+1)^2-1}/\kappa_k$ for all i.

For $i = 1, ..., \kappa_k/2$ on $\mathrm{int}(J_{2i,j,k})$ set $f(x) = |J_{j,k}|/|J_{2i,j,k}|$; on $\mathrm{int}(J_{2i-1,j,k})$ set $f(x) = -|J_{j,k}|/|J_{2i-1,j,k}|$. If $x \notin \cup_k \cup_j \cup_i \mathrm{int} J_{i,j,k}$ then set $f(x) = 0$.

Observe that if $[a, b]$ is an arbitrary interval and $a, b \notin J_{j,k}$ then

$$\int_{[a,b] \cap J'_{j,k}} f(x)dx = 0.$$

If A is a one-dimensional BV set put

$$F(A) = \sum_{k=0}^{\infty} \sum_{j=1}^{N_k} \int_{A \cap J_{j,k}} f(x)dx.$$

By the above observation all but finitely many terms of the double sum in the definition of F equal zero, hence $F(A)$ is well-defined.

Next we need to verify that F is an AC_* charge.

Assume that the BV set A is equivalent (modulo a set of measure zero) to the disjoint closed intervals $Z_1, ..., Z_t$. We will denote this property later by $A \sim \cup_{s=1}^{t} Z_s$.

If no endpoint of an interval Z_s belongs to an interval of the form $J'_{j,k}$ then

$$F(Z_s) = \sum_{k=0}^{\infty} \sum_{j=1}^{N_k} \int_{Z_s \cap J_{j,k}} f = 0. \tag{4}$$

If at least one of the endpoints of Z_s belongs to an interval of the form J_{j_s,k_s} then, without limiting generality, we can assume that the other endpoint of Z_s either does not belong to an interval of the form $J'_{j,k} \subset J_{j,k}$, or if it belongs to such an interval then $k \geq k_s$. It is easy to see that in this case

$$|F(Z_s)| \leq 2|J_{j_s,k_s}|. \tag{5}$$

First we show that F is a charge. It is clearly additive. Assume $\epsilon > 0$ is given. Without limiting generality we assume $\epsilon < 1$. We need to choose $\rho > 0$ such that if the BV set $A \subset B(0, 1/\epsilon)$, $||A|| < 1/\epsilon$ and $|A| < \rho$ then $|F(A)| < \epsilon$.

Assume $A \sim \cup_{s=1}^{t} Z_s$. Then $||A|| < 1/\epsilon$ implies $t < 1/2\epsilon$. Choose k_0 such that

$$|J_{1,k_0}| = |J_{j,k_0}| < \epsilon^2/2, \qquad j = 1, ..., N_{k_0}. \tag{6}$$

Since f is bounded on $H_{k_0} \stackrel{\text{def}}{=} \cup_{k=1}^{k_0-1} \cup_{j=1}^{N_k} J_{j,k}$ we can choose $\rho_1 > 0$ such that if the interval $Z_s \subset H_{k_0}$ and $|Z_s| < \rho_1$ then $|\int_{Z_s} f| = |F(Z_s)| < \epsilon^2$.

If $\int_{Z_s \cap H_{k_0}} f \neq 0$ then there exists $k \in \{1, \dots, k_0 - 1\}$ and $j \in \{1, ..., N_k\}$ such that $Z_s \cap J'_{j,k} \neq \emptyset$. This implies that if we assume that $|Z_s| \leq |A| < \rho_2 = |J_{1,k_0}|/3$ then $Z_s \subset H_{k_0}$.

Therefore if $\int_{Z_s \cap H_{k_0}} f \neq 0$ then $|Z_s| \leq |A| \leq \rho \stackrel{\text{def}}{=} \min(\rho_1, \rho_2)$ implies that $|F(Z_s)| < \epsilon^2$.

If $\int_{Z_s \cap H_{k_0}} f = 0$, but $F(Z_s) \neq 0$ then an esimate of the form (5) holds with a $k_s \geq k_0$. Using (6) we obtain

$$|F(Z_s)| \leq 2|J_{j_s,k_s}| \leq 2|J_{1,k_0}| < \epsilon^2.$$

This implies

$$|F(A)| \leq \sum_{s=1}^{t} |F(Z_s)| \leq \epsilon^2/2\epsilon = \epsilon/2 < \epsilon.$$

Next we show that F is AC_*.

Since V_*F is a Borel measure, $V_*F(\mathbb{R} \setminus [0,1]) = 0$, and one can easily see that V_*F is absolutely continuous on each set $J_{j,k}$, $k = 0, 1, ...; j = 1, ..., N_k$ it is enough to show that it is absolutely continuous on E. Assume $S \subset E$, $|S| = 0$ and $\eta > 0$ is fixed and $\mathcal{P} = \{(A_i, x_i)\}_{i=1}^{p}$ is an η-regular partition, anchored in $S \subset E$. Assume, furthermore, that $A_i \sim \cup_{s=1}^{t_i} Z_{s,i}$. Then (1) implies $t_i \leq 1/2\eta$ and (2) implies $|A_i| > 2\eta d(A_i)$.

Now, if $F(Z_{s,i}) \neq 0$, that is, (4) does not hold then we can use (5) to obtain

$$|F(Z_{s,i})| \leq 2|J_{j_{s,i},k_{s,i}}|.$$

Since $x_i \in A_i$, $x_i \in S \subset E$, and $E \cap \operatorname{int}(J_{j_{s,i},k_{s,i}}) = \emptyset$ we also have $d(A_i) \geq |J_{j_{s,i},k_{s,i}}|/3$.

Hence, keeping in mind that $||A_i|| \geq 2$, we obtain

$$|A_i| \geq \eta d(A_i)||A_i|| \geq 2\eta |J_{j_{s,i},k_{s,i}}|/3 \geq \eta |F(Z_{s,i})|/3.$$

Since $t_i \leq 1/2\eta$ we obtain

$$|F(A_i)| \leq \sum_{s=1}^{t_i} |F(Z_{s,i})| \leq t_i \frac{3|A_i|}{\eta} \leq \frac{3|A_i|}{2\eta^2}.$$

Cover S by an open set G such that $|G| < \eta^2\epsilon/3$ and for $x \in H$ choose $\delta(x) > 0$ satisfying $B(x, \delta(x)) \subset G$. If $\mathcal{P}$ is δ-fine then $\cup_{i=1}^{p} A_i \subset G$ and hence

$$\sum_{i=1}^{p} |F(A_i)| \leq \sum_{i=1}^{p} \frac{3|A_i|}{2\eta^2} < 3|G|/\eta^2 < \epsilon.$$

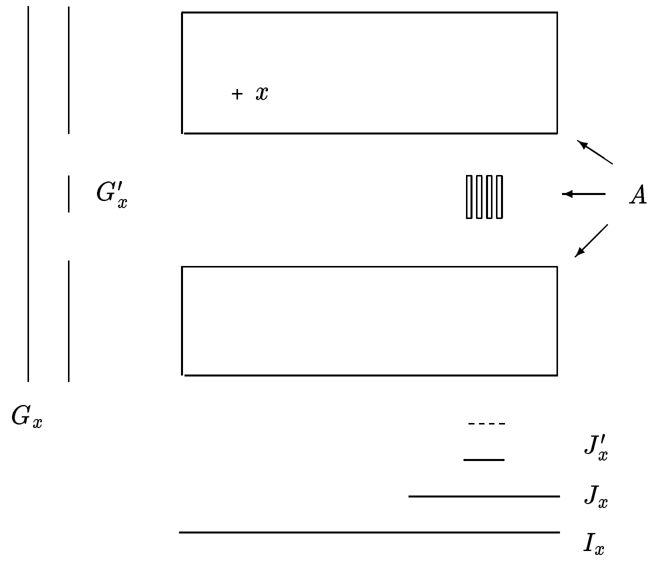
This shows that using this δ we have $V_{\eta,\delta}F(S) < \epsilon$. Using that for each $\epsilon > 0$ one can choose a suitable δ and one can repeat the above argument for all $\eta > 0$ we obtain that $V_*F(S) = 0$. This completes the proof of the fact that F is AC_*.

Finally we show that $F \otimes \mu$ is not AC_*. Using Chapter 7 of [3] one can easily see that the Hausdorff dimension of $C \times E$ is $1 + \log 2/\log 3$, and hence $|C \times E| = 0$ but it is not of σ-finite one-dimensional Hausdorff measure, that is, it is not thin.

We show that $V_*F(C \times E) \neq 0$.

Assume η is sufficiently small (something like $\eta < 1/20$ is suitable). Since $C \times E$ is not thin if δ is an arbitrary two dimensional gage function then there exists an $x \in C \times E$ for which $\delta(x) > 0$. Choose and fix such an x.

Choose k such that $3^{-k^2} < \delta(x)/2$ and assume that G_x is the base interval of C_{k^2} which contains x and is of length 3^{-k^2}, and we denote by G_x' the corresponding middle third interval of G_x. Recall that G_x' is of length h_{k^2}, and it was used in the definition of ϕ and μ, furthermore by (3) we have $\mu(G_x') = k^{-4}2^{-k^2}$.



length of I_x is 3^{-k^2}

figure 3.

Similarly, assume that I_x is the base interval $I_{j,k}$ of length 3^{-k^2} which was used in the definition of E_k and J_x is the corresponding interval $J_{j,k}$ of length

$3^{-(k+1)^2}$. Furthermore, the middle third of J_x, $J'_{j,k}$ will be denoted by J'_x and the intervals $J_{i,j,k}$, used in the definition of f will be denoted by $J_{x,i}$ $(i = 1, ..., \kappa_k)$.

Observe that $C_{k^2+1} \cap G_x$ consists of two intervals, each of length $|G_x|/3$, and $G'_x \cap C_{k^2+1} = \emptyset$.

Set $A = (I_x \times (C_{k^2+1} \cap G_x)) \cup ((\cup_{i=1}^{\kappa_k/2} J_{x,2i}) \times G'_x)$.

Then $||A|| \leq 6 \cdot 3^{-k^2} + \kappa_k h_{k^2} < 7 \cdot 3^{-k^2}$, $d(A) < \sqrt{2} \cdot 3^{-k^2}$, and $|A| > (2/3)3^{-2k^2}$. This implies that A is η-regular. Since $A \subset G_x \times I_x \subset B(x, \delta(x))$, the one element partition $\mathcal{P} = \{(A, x)\}$ is η-regular and δ-fine.

On the other hand

$$(F \otimes \mu)(A) = \mu(C_{k^2+1} \cap G_x) F(I_x) + \mu(G'_x) \frac{\kappa_k}{2} \int_{J_{x,2}} f =$$

$$0 + k^{-4} 2^{-k^2} \frac{\kappa_k}{2} |J_{x,2}| \frac{|J_x|}{|J_{x,2}|} = \frac{\kappa_k}{k^4 2^{k^2+1} 3^{(k+1)^2}} = 1.$$

This implies $V_{\eta,\delta}(F \otimes \mu)(C \times E) \geq 1$ holds for all gage δ and hence $V_* F(C \times E) \geq 1$. Thus $V_*(F \otimes \mu)$ is not absolutely continuous. □

References

[1] Z. Buczolich, T. de Pauw, and W.F. Pfeffer, Charges, *BV* functions, and multipliers for generalized Riemann integrals, *Indiana Univ. Math. J.*, Vol. 48, No. 4, (1999), 1471-1512.

[2] D.L. Cohn, *Measure Theory*, Birkhäuser, Boston 1980.

[3] K. Falconer, *Fractal Geometry*, John Wiley & Sons, 1990.