Typical Borel measures on $[0,1]^d$ satisfy a multifractal formalism

Zoltán Buczolich

‡

Department of Analysis, Eötvös Loránd University, Pázmány Péter Sétány 1/c, 1117 Budapest, Hungary

Stéphane Seuret

LAMA, CNRS UMR 8050, Université Paris-Est Créteil Val-de-Marne, 61 avenue du Général de Gaulle, 94 010 CRÉTEIL Cedex, France

Abstract. In this article, we prove that in the Baire category sense, measures supported by the unit cube of $\mathbb{R}^d$ typically satisfy a multifractal formalism. To achieve this, we compute explicitly the multifractal spectrum of such typical measures μ. This spectrum appears to be linear with slope 1, starting from 0 at exponent 0, ending at dimension d at exponent d, and it indeed coincides with the Legendre transform of the L^q-spectrum associated with typical measures μ.

Keywords: Borel measures, Hausdorff dimension, Multifractal analysis, Baire categories.

§

‡ Research supported by the Hungarian National Foundation for Scientific research K075242.

1. Introduction

Let $\mathcal{M}([0,1]^d)$ be the set of probability measures on $[0,1]^d$ endowed with the weak topology (which makes $\mathcal{M}([0,1]^d)$ a compact separable space). Recall that the local regularity of a positive measure $\mu \in \mathcal{M}([0,1]^d)$ at a given $x_0 \in [0,1]$ is quantified by a *local dimension* (or a *local Hölder exponent*) $h_\mu(x_0)$, defined as

$$h_\mu(x_0) = \liminf_{r \to 0^+} \frac{\log \mu(B(x_0, r))}{\log r}, \tag{1}$$

where $B(x_0, r)$ denotes the ball with center x_0 and radius r. In geometric measure theory $h_\mu(x_0)$ is called the lower local dimension of μ at x_0 and is denoted by $\underline{\dim}_{\rm loc}\mu(x_0)$. Then the singularity spectrum of μ is the map

$$d_\mu : h \geq 0 \mapsto \dim_{\mathcal{H}} E_\mu(h),$$

where

$$E_\mu(h) := \{x \in [0,1]^d : h_\mu(x) = h\}. \tag{2}$$

This spectrum describes the distribution of the singularities of the measure μ, and thus contains crucial information on the geometrical properties of μ. Most often, two forms of spectra are obtained for measures: either a spectrum with the classical concave shape (obtained as Legendre transform of some concave L^q-scaling function, for instance in the case of self-similar measures, Mandelbrot cascades and their extensions, see [2, 4, 8, 14, 10, 18, 19] for historical references, among many references), or a linear increasing spectrum (as in [1, 6, 12]).

These two distinct shapes arise in different contexts: On one hand, linear spectra are usually found for measures and functions which are infinite sums of mutually independent contributions, i.e. which are obtained from an *additive* procedure. Lévy subordinators, which are integrals of infinite sum of randomly distributed Dirac masses, and random wavelet series, where the wavelet coefficients are i.i.d. random variables, illustrate this fact. For such stochastic processes, the greatest Hölder exponent coincides with the almost sure exponent, meaning that at Lebesgue almost every point, the sample path of the process enjoys the highest possible local regularity. On the other hand, concave spectra are generally obtained for measures or functions built using a *multiplicative* or *hierarchical* scheme. As said above, Mandelbrot cascades are the archetypes of measures with a multiplicative structure and exhibit in full generality a concave spectrum. In such constructions, the strong local correlations make it possible the presence of points around which the local exponent is greater than the almost sure exponent. This constitutes a striking difference with additive processes, for which these more regular points do not exist.

Subsequently, the shape of the spectrum may reflect the structure of the object under consideration, and may reveal some properties of the physics underlying the signal, if any. Hence, it is very natural to investigate the structure of typical measures. Actually we will prove that typical measures tend to exhibit an *additive* structure, and the proof we develop will exploit this property.

Before stating our result, we recall the notion of L^q-spectrum for a probability measure $\mu \in \mathcal{M}([0,1]^d)$. If j is an integer greater than 1, then we set

$$\mathbb{Z}_j = \{0, 1, \cdots, 2^j - 1\}^d. \tag{3}$$

Then, let $\mathcal{G}_j$ be the partition of $[0,1)^d$ into dyadic boxes: $\mathcal{G}_j$ is the set of all cubes

$$I_{j,\mathbf{k}} \stackrel{\text{def}}{=} \prod_{i=1}^{d} [k_i 2^{-j}, (k_i+1)2^{-j}),$$

where $\mathbf{k} := (k_1, k_2, \cdots, k_d) \in \mathbb{Z}_j$.

The L^q-spectrum of a measure $\mu \in \mathcal{M}([0,1]^d)$ is the mapping defined for any $q \in \mathbb{R}$ by

$$\tau_\mu(q) = \liminf_{j \to \infty} -\frac{1}{j} \log_2 s_j(q) \quad \text{where} \quad s_j(q) = \sum_{Q \in \mathcal{G}_j, \ \mu(Q) \neq 0} \mu(Q)^q. \tag{4}$$

It is classical [8, 14] that the Legendre transform of τ_μ serves as upper bound for the multifractal spectrum d_μ: For every $h \geq 0$,

$$d_\mu(h) \leq (\tau_\mu)^*(h) := \inf_{q \in \mathbb{R}} (qh - \tau_\mu(q)). \tag{5}$$

A lot of work has been achieved to prove that for specific measures (like self-similar measures, ..., see all the references above) the upper bound in (5) turns out to be an equality. When (5) is an equality at exponent $h \geq 0$, the measure is said to *satisfy the multifractal formalism* at h. The validity of the multifractal formalism for given measures is a very important issue in Mathematics and in Physics, since when it is known to be satisfied, it makes it possible to estimate the singularity spectrum of real data through the estimation of the L^q-spectrum. Moreover, it gives important information on the geometrical properties (from the viewpoint of geometric measure theory) of the measure μ under consideration.

These considerations led us also to find out whether the validity of the multifractal formalism is *typical* (or *generic*). Recall that a property is said to be typical in a complete metric space E, when it holds on a residual set, i.e. a set with a complement of first Baire category. A set is of first Baire category if it is the union of countably many nowhere dense sets. Most often, including in this paper, one can verify that the residual set is dense G_δ, that is, a countable intersection of dense open sets in E.

A first result in this direction was found by Buczolich and Nagy, who proved in [9] that typical continuous probability measures on $[0,1]$ have a linear increasing spectrum with slope 1, and satisfy the formalism. Then Olsen studied the typical L^q-spectra of measures on general compact sets [17, 16], but did not compute the multifractal spectrum of typical measures.

In this paper, we are interested in the form of the multifractal spectrum of typical Borel measures in $\mathcal{M}([0,1]^d)$, and we investigate whether the multifractal formalism is typically satisfied for such measures.

Theorem 1.1. *There is a dense G_δ set $\mathcal{R}$ included in $\mathcal{M}([0,1]^d)$ such that for every measure $\mu \in \mathcal{R}$, we have*

$$\forall\ h \in [0,d], \qquad d_\mu(h) = h, \tag{6}$$

and $E_\mu(h) = \emptyset$ if $h > d$.

In particular, for every $q \in [0,1]$, $\tau_\mu(q) = d(q-1)$, and μ satisfies the multifractal formalism at every $h \in [0,d]$, i.e. $d_\mu(h) = \tau^*_\mu(h)$.

We note that there is a slight difference in notation in [9] since in (4) there is a negative sign in the definition of $\tau_\mu(q)$. Since we compute the multifractal spectrum of typical measures μ, using (5), we recover part of the result of Olsen [16], i.e. the value of $\tau_\mu(q)$ of $q \in [0,1]$, when the support of the measure is $[0,1]^d$.

We conjecture that similar properties hold on all compact sets of $\mathbb{R}^d$.

Conjecture 1.2. *For any compact set $K \subset \mathbb{R}^d$, there exists a constant $0 \le D \le d$ such that typical measures μ (in the Baire sense) in $\mathcal{M}(K)$ satisfy: for every $h \in [0,D]$, $d_\mu(h) = h$, and if $h > D$, $E_\mu(h) = \emptyset$.*

Whether D should be the Hausdorff dimension of K or the lower box dimension of K (or another dimension) is not obvious for us at this point.

In the rest of this work, pure atomic measures of the form (δ_x stands for the Dirac measure at $x \in [0,1]^d$)

$$\nu = \sum_{n \ge 0} r_n \cdot \delta_{x_n}, \tag{7}$$

will play a major role. For instance, the separability of $\mathcal{M}([0,1]^d)$ follows from the fact that measures ν of the form (7), where $(r_n)_{n\ge 0}$ are positive rational numbers such that $\sum_{n \ge 0} r_n = 1$, and where $(x_n)_{n\ge 0}$ are rational points of the cube $\mathcal{M}([0,1]^d)$, form a countable dense set in $\mathcal{M}([0,1]^d)$ for the weak topology. Atomic measures ν have been studied by many authors [1, 3, 5, 6, 7, 12, 14]. In particular, it is shown in [6, 7] that such measures always exhibit specific multifractal properties.

The paper is organized as follows. Section 2 contains the precise definitions and some known results on dimensions and multifractal spectra for Borel measures, as well as some recalls on the properties of $\mathcal{M}([0,1]^d)$. We also prove the seond part of Theorem 1.1, i.e. for generic measures, $\tau_\mu(q) = d(q-1)$ for every $q \in [0,1]$.

In Section 3, we build a dense G_δ set $\mathcal{R}$ of measures in $\mathcal{M}([0,1]^d)$ such that for every $\mu \in \mathcal{R}$, *for every $x \in [0,1]^d$*, $h_\mu(x) \le d$ and *for Lebesgue-almost every $x \in [0,1]^d$*, $h_\mu(x) = d$.

In Section 4 we prove that for every $\mu \in \mathcal{R}$, for every $h \in [0,d)$, $d_\mu(h) = h$. This implies Theorem 1.1.

2. Preliminary results

In $\mathbb{R}^d$ we will use the metric coming from the supremum norm, that is, for $x, y \in \mathbb{R}^d$, $\rho(x,y) = \max\{|x_i - y_i| : i = 1, ..., d\}$.

The open ball centered at x and of radius r is denoted by $B(x,r)$. The closure of the set $A \subset \mathbb{R}^d$ is denoted by $\overline{A}$, moreover $|A|$ and $\mathcal{L}_d(A)$ denote its diameter and d-dimensional Lebesgue measure, respectively.

2.1. Dimensions of sets and measures

We refer the reader to [10] for the standard definition of Hausdorff measures $\mathcal{H}^s(E)$ and Hausdorff dimensions $\dim_{\mathcal{H}}(E)$ of a set E.

For a Borel measure $\mu \in \mathcal{M}([0,1]^d)$, one defines the dimension of μ as

$$\dim_{\mathcal{H}}(\mu) := \sup\{s : h_\mu(x) \geq s \text{ for } \mu\text{-a.e. } x\}. \tag{8}$$

By Proposition 10.2 of [10]

$$\dim_{\mathcal{H}}(\mu) = \inf\{\dim_{\mathcal{H}}(E) : E \subset [0,1] \text{ Borel and } \mu(E) > 0\}.$$

The following property will be particularly relevant:

$$\begin{array}{c}\text{if } \dim_{\mathcal{H}}(\mu) \geq h, \text{ then for every Borel set } E \subset [0,1] \\ \text{of dimension strictly less than } h,\ \mu(E) = 0.\end{array} \tag{9}$$

We recall standard results on multifractal spectra of Borel probability measures.

Proposition 2.1. *Let $\mu \in \mathcal{M}([0,1]^d)$ and*

$$\widetilde{E_\mu}(h) = \{x \in [0,1]^d : h_\mu(x) \leq h\} \supset E_\mu(h). \tag{10}$$

For every $h \geq 0$, $d_\mu(h) = \dim_{\mathcal{H}} E_\mu(h) \leq \dim_{\mathcal{H}} \widetilde{E_\mu}(h) \leq \min(h,d)$.

This follows for instance from Proposition 2.3 of [11], where it is shown that for

$$\widetilde{E_\mu}(h) = \{x \in [0,1]^d : h_\mu(x) = \underline{\dim}_{\text{loc}}\mu(x) \leq h\}$$

we have $\dim_{\mathcal{H}} \widetilde{E_\mu} \leq h$. The rest follows from the embedding (10).

From this we deduce in Theorem 1.1 that for typical measures, $\tau_\mu(q) = d(q-1)$ for all $q \in [0,1]$. We prove it quickly for completeness.

Corollary 2.2. *Assume that (6) holds true for a probability measure μ on $[0,1]^d$. Then $\tau_\mu(q) = d(q-1)$ for all $q \in [0,1]$.*

Proof. Recall that τ_μ and $s_j(q)$ were defined in (4). Since $\mathcal{G}_j$ has 2^{dj} many cubes in $[0,1]^d$ by using Hölder's inequality for $0 < q < 1$

$$s_j(q) \leq (\sum_{Q \in \mathcal{G}_j} \mu(Q)^{q/q})^q (\sum_{Q \in \mathcal{G}_j} 1^{1/(1-q)})^{1-q} = 1 \cdot (2^{jd})^{1-q}.$$

This implies $\tau_\mu(q) \geq d(q-1)$. One could also notice that $\tau_\mu(0) = -d$, $\tau_\mu(1) = 0$ and τ_μ is a concave map on the interior of its support and hence $\tau_\mu(q) = d(q-1)$ for all $q \in [0,1]$.

Assume now that (6) holds true for μ. Proceeding towards a contradiction suppose that there exists $q' \in (0,1)$ such that $\tau_\mu(q') > d(q'-1)$. By concavity of $\tau_\mu(q)$ there exists $d' < d$ such that $\tau_\mu(q) > d'(q-1)$ for all $q \in (q',1)$. Hence for $d' < h < d$ by (5) and (6) we have

$$h = d_\mu(h) \leq \inf_{q\in\mathbb{R}}(qh - \tau_\mu(q)) \leq \inf_{q\in(q',1)}(qh - d'(q-1)) = \inf_{q\in(q',1)}(q(h-d') + d') < h,$$

a contradiction. This concludes the proof. □

2.2. Separability of $\mathcal{M}([0,1]^d)$

Let us denote by $\mathrm{Lip}([0,1]^d)$ the set of Lipschitz functions on $[0,1]^d$ with Lipschitz constant not exceeding 1. Recall that the weak topology on $\mathcal{M}([0,1]^d)$ is induced by the following metric: if μ and ν belong to $\mathcal{M}([0,1]^d)$, we set

$$\varrho(\mu,\nu) = \sup\left\{\left|\int f\,d\mu - \int f\,d\nu\right| : f \in \mathrm{Lip}([0,1]^d)\right\}. \tag{11}$$

As is mentioned in the introduction, $\mathcal{M}([0,1]^d)$ is a separable set. For our purpose, we specify a countable dense family of atomic measures. Indeed, the set of finite atomic measures of the form

$$\sum_{\mathbf{k}\in\mathbb{Z}_j} r_{j,\mathbf{k}} \cdot \delta_{\mathbf{k}2^{-j}}, \tag{12}$$

where $j \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$, $\mathbb{Z}_j$ was defined by (3) and $(r_{j,\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}_j}$ are (strictly) positive rational numbers such that

$$\sum_{\mathbf{k}\in\mathbb{Z}_j} r_{j,\mathbf{k}} = 1,$$

forms a dense set in $\mathcal{M}([0,1]^d)$ for the weak topology.

3. The construction of $\mathcal{R}$, our dense G_δ set in $\mathcal{M}([0,1]^d)$

We build a dense G_δ set $\mathcal{R}$ in $\mathcal{M}([0,1]^d)$. In this section we show that for every $\mu \in \mathcal{R}$, for every $x \in [0,1]^d$, $h_\mu(x) \leq d$, and for Lebesgue-almost every $x \in [0,1]^d$, $h_\mu(x) = d$.

Let us enumerate the measures of the form (12) as a sequence $\{\nu_1, \nu_2, ..., \nu_n, ..., \}$.

Let $n \geq 1$, and consider ν_n. We are going to construct another measure μ_n, close to ν_n in the weak topology, such that μ_n has a very typical behavior at a certain scale.

Let us write the measure ν_n as

$$\nu_n = \sum_{\mathbf{k}\in\mathbb{Z}_{j_n}} r_{j_n,\mathbf{k}} \cdot \delta_{\mathbf{k}2^{-j_n}}, \tag{13}$$

where j_n is the integer such that $r_{j_n,\mathbf{k}} > 0$ for all $\mathbf{k} \in \mathbb{Z}_{j_n}$ (j_n is necessarily unique since all the Dirac masses in measures in (12) have a strictly positive weight).

Set $\mathbf{e} = (1/2, ..., 1/2) \in \mathbb{R}^d$.

For every integer $j \geq 1$, let us introduce the measure π_j defined as

$$\pi_j = \sum_{\mathbf{k} \in \mathbb{Z}_j} 2^{-dj} \, \delta_{(\mathbf{k}+\mathbf{e})2^{-j}}. \tag{14}$$

This measure π_j consists of Dirac masses located at centers of the dyadic cubes of $[0,1]^d$ of generation j, and gives the same weight to each Dirac mass. For every integer $n \geq 1$, let

$$J_n = 2n(j_n)^2, \tag{15}$$

so that $J_n/n \geq 2j_n$. Finally, for every $n \geq 1$, we define

$$\mu_n = 2^{-J_n/n} \cdot \pi_{J_n} + (1 - 2^{-J_n/n}) \cdot \nu_n. \tag{16}$$

Obviously, for every $\mathbf{k} \in \mathbb{Z}_{J_n}$, we have

$$\mu_n(I_{J_n,\mathbf{k}}) \geq 2^{-J_n/n} \cdot \pi_{J_n}(I_{J_n,\mathbf{k}}) \geq 2^{-J_n/n} 2^{-dJ_n} = |I_{J_n,\mathbf{k}}|^{d+1/n} \tag{17}$$

where the last equality holds since we use the supremum metric.

Lemma 3.1. *For every* $n \geq 1$*,* $\varrho(\mu_n, \nu_n) \leq 2 \cdot 2^{-J_n/n}$*.*

Proof. Recall Definition (11) of the metric ϱ. Let $f \in \mathrm{Lip}([0,1]^d)$. We have

$$\begin{aligned} \left| \int f \, d\nu_n - \int f \, d\mu_n \right| &= 2^{-J_n/n} \left| \int f \, d\nu_n - \int f \, d\pi_{J_n} \right| \\ &= 2^{-J_n/n} \varrho(\nu_n, \pi_{J_n}) \leq 2 \cdot 2^{-J_n/n}. \end{aligned}$$

□

The density of the sequence $(\nu_n)_{n\geq 1}$ implies the density of $(\mu_n)_{n\geq 1}$, since the distance $\varrho(\mu_n, \nu_n)$ converges to zero as n tends to infinity.

Definition 3.2. *We introduce for every* $N \geq 1$ *the sets in* $\mathcal{M}([0,1]^d)$

$$\Omega_N^\varrho = \bigcup_{n \geq N} B(\mu_n, 2^{-(d+4)(J_n)^2}) \quad \text{and} \quad \mathcal{R} = \bigcap_{N \geq 1} \Omega_N^\varrho, \tag{18}$$

where the open balls are defined using the metric ϱ *defined by (11).*

Each set Ω_N^ϱ *is obviously a dense open set in* $\mathcal{M}([0,1]^d)$*, hence* $\mathcal{R}$ *is a dense* G_δ *set in* $\mathcal{M}([0,1]^d)$*.*

4. Upper bound for the local Hölder exponents of typical measures

We first prove that all exponents of typical measures μ are less than d, and then, in the last section, we compute the whole spectrum of μ.

Proposition 4.1. *For every* $\mu \in \mathcal{R}$*, for every* $x \in [0,1]^d$*,* $h_\mu(x) \leq d$*.*

Proof. Let $\mu \in \mathcal{R}$. There is a sequence of positive integers $(N_p)_{p\geq 1}$ growing to infinity such that for every $p \geq 1$, $\varrho(\mu, \mu_{N_p}) \leq 2^{-(d+4)(J_{N_p})^2}$.

Suppose that $\mathbf{k} \in \mathbb{Z}_{J_{N_p}}$. We introduce an auxiliary function $f_{p,\mathbf{k}}$ defined as follows: First we set $f_{p,\mathbf{k}}((\mathbf{k}+\mathbf{e})2^{-J_{N_p}}) = 2^{-J_{N_p}-1}$ and $f_{p,\mathbf{k}}(x) = 0$ for $x \notin I_{J_{N_p},\mathbf{k}}$. Then we use an extension of $f_{p,\mathbf{k}}$ onto $I_{J_{N_p},\mathbf{k}}$ such that $f_{p,\mathbf{k}} \in \mathrm{Lip}([0,1]^d)$ and $0 \leq f_{p,\mathbf{k}} \leq 2^{-J_{N_p}-1}$.

First observe that

$$\int f_{p,\mathbf{k}}\, d\mu \leq 2^{-J_{N_p}-1}\mu(I_{J_{N_p},\mathbf{k}}). \tag{19}$$

Moreover, we have

$$\begin{aligned}
\int f_{p,\mathbf{k}}\, d\mu_{N_p} &\geq 2^{-J_{N_p}/N_p} \int f_{p,\mathbf{k}}\, d\pi_{J_{N_p}} \\
&\geq 2^{-J_{N_p}/N_p} 2^{-dJ_{N_p}} \int f_{p,\mathbf{k}}\, d\delta_{(\mathbf{k}+\mathbf{e})2^{-N_p}} \\
&\geq 2^{-J_{N_p}/N_p} 2^{-dJ_{N_p}} 2^{-J_{N_p}-1} \\
&= |I_{J_{N_p},\mathbf{k}}|^{d+1/N_p} \cdot 2^{-J_{N_p}-1} = 2^{-J_{N_p}(d+1+1/N_p)-1}.
\end{aligned} \tag{20}$$

We also have

$$\varrho(\mu, \mu_{N_p}) \leq 2^{-(J_{N_p})^2(d+4)} < 2^{-J_{N_p}(d+1+1/N_p)-2}. \tag{21}$$

Combining (19), (20) and (21), we deduce that

$$\begin{aligned}
2^{-J_{N_p}-1}\mu(I_{J_{N_p},\mathbf{k}}) &> 2^{-J_{N_p}(d+1+1/N_p)-1} - 2^{-J_{N_p}(d+1+1/N_p)-2} \\
&> 2^{-J_{N_p}(d+1+1/N_p)-2},
\end{aligned}$$

which leads to

$$\mu(I_{J_{N_p},\mathbf{k}}) > 2^{-J_{N_p}(d+1/N_p)-1} = |I_{J_{N_p},\mathbf{k}}|^{d+1/N_p+1/J_{N_p}} > |I_{J_{N_p},\mathbf{k}}|^{d+2/N_p}.$$

For any integer $j \geq 1$, let us denote by $I_j(x)$ the unique dyadic cube of generation j that contains x. Recalling the definition of the Hölder exponent (1) of μ at any $x \in [0,1]^d$, we obviously obtain

$$\begin{aligned}
h_\mu(x) = \liminf_{r\to 0^+} \frac{\log \mu(B(x,r))}{\log r} &\leq \liminf_{p\to+\infty} \frac{\log \mu(I_{J_{N_p}}(x))}{\log |I_{J_{N_p}}(x)|} \\
&\leq \liminf_{p\to+\infty} d + 2/N_p = d.
\end{aligned}$$

□

5. The multifractal spectrum of typical measures of $\mathcal{R}$

Let $\mu \in \mathcal{R}$, where $\mathcal{R}$ was defined by (18).

Hence, there is a sequence of integers $(N_p)_{p\geq 1}$ such that $\mu \in \Omega^{\varrho}_{N_p}$ for every $p \geq 1$. Equivalently, for every $p \geq 1$, $\varrho(\mu, \mu_{N_p}) \leq 2^{-(J_{N_p})^2(d+4)}$, where μ_{N_p} is given by (16).

We are going to prove that such a measure μ has necessarily a multifractal spectrum equal to $d_\mu(h) = h$, for every $h \in [0, d)$. Recall that we already have the upper bound

$d_\mu(h) \leq h$, hence it suffices to bound from below the Hausdorff dimension of each set $\{x \in [0,1]^d : h_\mu(x) = h\}$.

5.1. Sets $\mathcal{A}_{\theta,p}$ of points with given approximation rates by the dyadics

Let $p \geq 1$, and consider N_p and μ_{N_p}. As usual $\overline{B}(x,r)$ stands for the closed ball of centre x and radius r.

Definition 5.1. *Let us introduce, for every real number $\theta \geq 1$, the set of points*

$$\mathcal{A}_{\theta,p} = \bigcup_{\mathbf{k} \in \mathbb{Z}_{J_{N_p}}} \overline{B}((\mathbf{k}+\mathbf{e})2^{-J_{N_p}}, 2^{-\theta J_{N_p}}).$$

and then let us define

$$\mathcal{A}_\theta = \bigcap_{P \geq 1} \bigcup_{p \geq P} \mathcal{A}_{\theta,p} = \{x \in [0,1]^d : \ x \text{ belongs to infinitely many } \mathcal{A}_{\theta,p}\}.$$

Essentially, $\mathcal{A}_{\theta,p}$ consists of the points of $[0,1]^d$ which are located close to the Dirac masses of $\pi_{J_{N_p}}$ (and thus close to some of the Dirac masses of μ_{N_p}). The larger θ, the closer $\mathcal{A}_{\theta,p}$ to the dyadic points of generation J_{N_p}. Then $\mathcal{A}_\theta$ contains the points which are infinitely often close to some Dirac masses.

Lemma 5.2. *Let $\varepsilon > 0$, then there exists an integer p_ε such that for every $p \geq p_\varepsilon$ and for every $x \in \mathcal{A}_{\theta,p}$*

$$\mu(B(x, 2 \cdot 2^{-\theta J_{N_p}})) \geq 2^{-d(1+2\varepsilon)J_{N_p}}. \tag{22}$$

Proof. Obviously, when $x \in \mathcal{A}_{\theta,p}$, the closed ball $\overline{B}(x, 2^{-\theta J_{N_p}})$ contains a Dirac mass of μ_{N_p} located at some element $(\mathbf{k}+\mathbf{e})2^{-J_{N_p}}$ (which is the location of a Dirac mass of $\pi_{J_{N_p}}$). Hence,

$$\mu_{N_p}(\overline{B}(x, 2^{-\theta J_{N_p}})) \geq 2^{-J_{N_p}/N_p} 2^{-dJ_{N_p}} = 2^{-dJ_{N_p}(1+1/(dN_p))}.$$

Hence, if p is large enough to have $\epsilon > 1/dN_p$, then

$$\mu_{N_p}(\overline{B}(x, 2^{-\theta J_{N_p}})) \geq 2^{-d(1+\varepsilon)J_{N_p}}. \tag{23}$$

As in Proposition 4.1 we use a specific function $f_{\theta,p} \in \text{Lip}([0,1]^d)$, that we define as follows: $f_{\theta,p}(z) = 2^{-\theta J_{N_p}}$ for $z \in \overline{B}(x, 2^{-\theta J_{N_p}})$, and $f_{\theta,p}(z) = 0$ if $z \notin B(x, 2 \cdot 2^{-\theta J_{N_p}})$. Otherwise choose an extension of $f_{\theta,p}$ onto $B(x, 2 \cdot 2^{-\theta J_{N_p}}) \setminus \overline{B}(x, 2^{-\theta J_{N_p}})$ such that $f_{\theta,p} \in \text{Lip}([0,1]^d)$ and

$$0 \leq f_{\theta,p} \leq 2^{-\theta J_{N_p}}. \tag{24}$$

Obviously by construction we have

$$2^{-\theta J_{N_p}} \mu(B(x, 2 \cdot 2^{-\theta J_{N_p}})) \geq \int f_{\theta,p} d\mu.$$

Then by (23)

$$\int f_{\theta,p}\, d\mu_{N_p} \geq 2^{-\theta J_{N_p}} 2^{-d(1+\epsilon)J_{N_p}}.$$

Recall also that $\varrho(\mu, \mu_{N_p}) \leq 2^{-(J_{N_p})^2(d+4)}$. If p is large enough to have

$$\frac{1}{2} 2^{-\theta J_{N_p}} 2^{-d(1+\epsilon)J_{N_p}} > 2^{-(J_{N_p})^2(d+4)},$$

then by (24) we obtain (using the same argument as in Proposition 4.1) that

$$\begin{aligned} 2^{-\theta J_{N_p}} \mu(B(x, 2\cdot 2^{-\theta J_{N_p}})) &\geq \int f_{\theta,p}\, d\mu \\ &\geq \int f_{\theta,p}\ d\mu_{N_p} - \varrho(\mu, \mu_{N_p}) \\ &\geq \frac{1}{2}\cdot 2^{-\theta J_{N_p}} 2^{-d(1+\epsilon)J_{N_p}}. \end{aligned}$$

This yields

$$\mu(B(x, 2\cdot 2^{-\theta J_{N_p}})) \geq \frac{1}{2} 2^{-d(1+\epsilon)J_{N_p}} > 2^{-d(1+2\varepsilon)J_{N_p}},$$

the last inequality being true when p is large. □

5.2. First results on local regularity and on the size of $\mathcal{A}_{\theta,p}$

Proposition 5.3. *If $\theta > 1$ and $x \in \mathcal{A}_\theta$, then $h_\mu(x) \leq d/\theta$.*

Proof. Let $x \in \mathcal{A}_\theta$. Then (22) is satisfied for an infinite number of integers p. In other words, there is a sequence of real numbers r_p decreasing to zero such that

$$\mu(B(x, 2r_p)) \geq (r_p)^{\frac{d}{\theta}(1+2\varepsilon)}.$$

This implies that $h_\mu(x) \leq \frac{d}{\theta}(1+2\varepsilon)$. Since this holds for any choice of $\varepsilon > 0$, the result follows. □

Proposition 5.4. *For every $\theta \geq 1$, $\dim_{\mathcal{H}} \mathcal{A}_\theta \leq d/\theta$.*

Proof. The upper bound is trivial for $\theta = 1$. Let $\theta > 1$, and let $s > d/\theta$.

Obviously $\mathcal{A}_\theta$ is covered by the union $\bigcup_{p\geq P} \mathcal{A}_{\theta,p}$, for any integer $P \geq 1$. Using this cover for large P's to bound from above the s-dimensional pre-measure of $\mathcal{A}_\theta$, we find for any $\delta > 0$

$$\begin{aligned} \mathcal{H}^s_\delta(\mathcal{A}_\theta) &\leq \mathcal{H}^s_\delta\Big(\bigcup_{p\geq P} \mathcal{A}_{\theta,p}\Big) \\ &\leq \sum_{p\geq P} \sum_{\mathbf{k}\in\mathbb{Z}_{J_{N_p}}} |B((\mathbf{k}+\mathbf{e})2^{-J_{N_p}}, 2^{-\theta J_{N_p}})|^s \\ &\leq C \sum_{p\geq P} \sum_{\mathbf{k}\in\mathbb{Z}_{J_{N_p}}} 2^{-s\theta J_{N_p}} \\ &\leq C \sum_{p\geq P} 2^{dJ_{N_p}}\ 2^{-s\theta J_{N_p}}, \end{aligned}$$

the last sum being convergent since $s\theta > d$. This sum converges to zero when $P \to \infty$, as a tail of a convergent series. Hence $\mathcal{H}^s_\delta(\mathcal{A}_\theta) = 0$ for every $s > d/\theta$ and $\delta > 0$. This implies $0 = \lim_{\delta\to 0} \mathcal{H}^s_\delta(\mathcal{A}_\theta) = \mathcal{H}^s(\mathcal{A}_\theta)$ and we deduce that $\dim_{\mathcal{H}} \mathcal{A}_\theta \leq d/\theta$. □

5.3. The lower bound for the dimension of $\mathcal{A}_\theta$

Theorem 5.5. *For every $\theta > 1$, there is a measure m_θ supported in $\mathcal{A}_\theta$ satisfying*

$$\text{for every Borel set } B \subset [0,1]^d, \quad m_\theta(B) \leq |B|^{d/\theta - \psi(|B|)}, \tag{25}$$

where $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a gauge function, i.e. a positive continuous increasing function such that $\psi(0) = 0$.

In particular, by (1), (8) and (25), $\dim_{\mathcal{H}} m_\theta \geq d/\theta$.

The proof of Theorem 5.5 is decomposed into two lemmas. Essentially we apply the classical method of constructing a Cantor set $\mathcal{C}_\theta$ included in $\mathcal{A}_\theta$ and simultaneously the measure m_θ supported by $\mathcal{A}_\theta$ which satisfies (25).

We select and fix a sufficiently rapidly growing subsequence of $(N_p)_{p\geq 1}$, that, for ease of notation, we still denote by $(N_p)_{p\geq 1}$, such that $N_1 > 100$, and for every $p \geq 1$

$$J_{N_{p+1}} > \max(100 \cdot \theta J_{N_p}, p^2), \tag{26}$$

$$\text{and } 2^{dJ_{N_{p+1}}(1-1/(p+1))} \leq 2^{-d\theta J_{N_p}} 2^{dJ_{N_{p+1}}-2}. \tag{27}$$

Since $J_{N_p} \to \infty$ as $N_p \to \infty$ it is clear that (26) and (27) can be satisfied by choosing a suitable subsequence. We will need these assumptions to ensure that the next Cantor set generation used during the definition of $\mathcal{C}_\theta$ is much finer than the previous one and hence $\mathcal{C}_\theta$ is nonempty and we can use estimate (34) later.

Moreover, we also suppose that N_p is increasing so rapidly that for $p \geq 3$

$$2^{-dJ_{N_p}(1+1/p)} \leq \Big(\prod_{k=1}^{p} 2^{dJ_{N_k}}\Big)^{-1} \text{ and } \tag{28}$$

$$\Big(\prod_{k=1}^{p} 2^{dJ_{N_k}(1-1/k)}\Big)^{-1} \leq 2^{-dJ_{N_p}(1-2/p)}.$$

Then, the construction of $\mathcal{C}_\theta$ is achieved as follows:

- The first generation of cubes of $\mathcal{C}_\theta$ consists of all the balls of the form $\overline{B}((\mathbf{k}+\mathbf{e})2^{-J_{N_1}}, 2^{-\theta J_{N_1}-1}) \subset [0,1]^d$, where $\mathbf{k} \in \mathbb{Z}_{J_{N_1}}$. We call $\mathcal{F}_1$ the set of such cubes, and we set $\Delta_1 = \#\mathcal{F}_1$. Then, a measure m_1 is defined as follows: for every cube $I \in \mathcal{F}_1$, we set
$$m_1(I) = \frac{1}{\Delta_1}.$$
The probability measure m_1 gives the same weight to each dyadic cube of first generation. The measure m_1 can be extended to a Borel probability measure on the algebra generated by $\mathcal{F}_1$, i.e. on $\sigma(I : I \in \mathcal{F}_1)$.

- Assume that we have constructed the first $p \geq 1$ generations of cubes $\mathcal{F}_1$, $\mathcal{F}_2$, ..., $\mathcal{F}_p$ and a measure m_p on the algebra $\sigma(L : L \in \mathcal{F}_p)$. Then we choose the cubes of generation $p+1$ as those closed balls of the form $\overline{B}((\mathbf{k}+\mathbf{e})2^{-J_{N_{p+1}}}, 2^{-\theta J_{N_{p+1}}-1})$ which are entirely included in one (and, necessarily, in only one) cube I of generation p and $\mathbf{k} \in \mathbb{Z}_{J_{N_{p+1}}}$. We call $\mathcal{F}_{p+1}$ the set consisting of them. We also set for every $I' \in \mathcal{F}_p$,
$$\Delta_{p+1}^{I'} = \#\{I \in \mathcal{F}_{p+1} : I \subset I'\}.$$
Then we define the measure m_{p+1}: For every cube $I \in \mathcal{F}_{p+1}$, we set
$$m_{p+1}(I) = m_p(I')\frac{1}{\Delta_{p+1}^{I'}},$$
where I' is the unique cube of generation p in $\mathcal{F}_p$ such that $I \subset I'$.

  The probability measure m_{p+1} can be extended to a Borel probability measure on the algebra $\sigma(L : L \in \mathcal{F}_{p+1})$ generated by $\mathcal{F}_{p+1}$.

  Finally, we set
$$\mathcal{C}_\theta = \bigcap_{p\geq 1} \bigcup_{I\in\mathcal{F}_p} I.$$

By the Kolmogorov extension theorem, $(m_p)_{p\geq 1}$ converges weakly to a Borel probability measure m_θ supported on $\mathcal{C}_\theta$ and such that for every $p \geq 1$, for every $I \in \mathcal{F}_p$, $m_\theta(I) = m_p(I)$.

5.4. Hausdorff dimension of $\mathcal{C}_\theta$ and m_θ

We first prove that m_θ has uniform behavior on the cubes belonging to $\bigcup_p \mathcal{F}_p$.

Lemma 5.6. *When p is sufficiently large, for every cube $I \in \mathcal{F}_p$,*
$$2^{-dJ_{N_p}(1+1/p)} \leq m_\theta(I) \leq 2^{-dJ_{N_p}(1-2/p)} \tag{29}$$
and
$$|I|^{\frac{d}{\theta}+\frac{1}{|\log|I||}} \leq m_\theta(I) \leq |I|^{\frac{d}{\theta}-\frac{1}{|\log|I||}}. \tag{30}$$

Proof: Obviously,
$$\Delta_1 = 2^{dJ_{N_1}}. \tag{31}$$
Using $J_{N_1} > N_1 > 100$
$$2^{dJ_{N_1}(1-1/2)} \leq \frac{1}{2}\cdot 2^{dJ_{N_1}} \leq \Delta_1. \tag{32}$$

Let I be a cube of generation $p \geq 1$ in the Cantor set $\mathcal{C}_\theta$. The subcubes of I are of the form $\overline{B}((\mathbf{k}+\mathbf{e})2^{-J_{N_{p+1}}}, 2^{-\theta J_{N_{p+1}}-1})$ and are regularly distributed. Next, when calculating the number of these subcubes in I on the right-handside of the inequality in

(33) a factor 1/2 will take care of the fact that for a few $(\mathbf{k}+\mathbf{e})2^{-J_{N_{p+1}}}$ on the frontier of I, we do not have $\overline{B}((\mathbf{k}+\mathbf{e})2^{-J_{N_{p+1}}}, 2^{-\theta J_{N_{p+1}}-1}) \subset I$. we deduce that

$$\begin{aligned} \Delta^I_{p+1} \quad & \geq \frac{1}{2}(|I|)^d 2^{dJ_{N_{p+1}}} = \frac{1}{2}2^{-d\theta J_{N_p}} \cdot 2^{dJ_{N_{p+1}}} \\ \text{and} \quad \Delta^I_{p+1} &= \#\{I' \in \mathcal{F}_{p+1} : I' \subset I\} \ \leq \ 2(|I|)^d \, 2^{dJ_{N_{p+1}}}. \end{aligned} \tag{33}$$

Using (27), and the fact that $2(|I|)^d \leq 1$, we obtain

$$2^{dJ_{N_{p+1}}(1-1/(p+1))} \leq \Delta^I_{p+1} \leq 2^{dJ_{N_{p+1}}}. \tag{34}$$

Recalling that $I \in \mathcal{F}_p$, for $k \leq p$ denote by I_k the unique cube in $\mathcal{F}_k$ containing I. Set $I_0 = [0,1]^d$ and $\Delta^{I_0} = \Delta_1$. We obtain

$$(\prod_{k=1}^{p} \Delta_k^{I_{k-1}})^{-1} = m_\theta(I). \tag{35}$$

The key property is that in (34), the bounds are uniform in $I \in \mathcal{F}_p$. Hence,

$$\Big(\prod_{k=1}^{p} 2^{dJ_{N_k}}\Big)^{-1} \leq m_\theta(I) \leq \Big(\prod_{k=1}^{p} 2^{dJ_{N_k}(1-1/k)}\Big)^{-1}.$$

By (28) we have (29) when $p \geq 3$.

This means that, the measure m_θ is almost uniformly distributed on the cubes of the same generation. Since these cubes $I \in \mathcal{F}_p$ have the same diameter which is $2^{-\theta J_{N_p}}$, we obtain for large p's that

$$|I|^{\frac{d}{\theta}(1+1/p)} \leq m_\theta(I) \leq |I|^{\frac{d}{\theta}(1-2/p)}. \tag{36}$$

Finally, we remark that by (26), $p = o(|\log|I||)$ when $I \in \mathcal{F}_p$ is arbitrary, hence (36) yields (30). □

Now we extend (30) and Lemma 5.6 to all Borel subsets of $[0,1]$.

Lemma 5.7. *There is a continuous increasing mapping $\psi : \mathbb{R}^+ \to \mathbb{R}^+$, satisfying $\psi(0) = 0$, and there is $\eta > 0$, such that for any Borel set $B \subset [0,1]$ with $|B| < \eta$ we have*

$$m_\theta(B) \leq |B|^{\frac{d}{\theta} - \psi(|B|)}. \tag{37}$$

Proof. Fix $\varepsilon_1 = 2^{-1}$, a Borel set $B \subset [0,1]$ with $|B| < 2^{-J_{N_1}} = \eta_0$. Let $p \geq 2$ be the unique integer such that

$$2^{-J_{N_p}} \leq |B| < 2^{-J_{N_{p-1}}}. \tag{38}$$

Let us distinguish two cases:

• $\mathbf{2^{-\theta J_{N_{p-1}}} \leq |B| < 2^{-J_{N_{p-1}}}}$: By (38), B intersects at most 2^d cubes $I' \in \mathcal{F}_{p-1}$. If there is no such cube then $m_\theta(B) = 0$. Otherwise, denoting by I' one of these cubes, using (29) and (36) we find that

$$\begin{aligned} m_\theta(B) &\leq 2^d \cdot m_\theta(I') \ \leq \ 2^d \cdot 2^{-dJ_{N_{p-1}}(1-\frac{2}{p-1})} \\ &\leq C \cdot |B|^{\frac{d}{\theta}(1-\frac{2}{p-1})} < |B|^{\frac{d}{\theta}-\varepsilon_1}. \end{aligned}$$

when p is sufficiently large. Recall that p is related to the diameter of B: the smaller $|B|$ is, the larger p becomes.

• $\mathbf{2^{-J_{N_p}}} < |\mathbf{B}| < \mathbf{2^{-\theta J_{N_{p-1}}}}$: We will determine a sufficiently small $\eta_1 \in (0, \eta_0)$ later and will suppose that $|B| < \eta_1$. For small $|B|$'s, that is, for large p's, B intersects at most one cube $I' \in \mathcal{F}_{p-1}$. If there is no such cube then $m_\theta(B) = 0$. Hence we need to deal with the case when such a cube I' exists. Obviously, $|B| < |I'| = 2^{-\theta J_{N_{p-1}}}$. The mass $m_\theta(I')$ is distributed evenly on the cubes $\overline{B}((\mathbf{k}+\mathbf{e})2^{-J_{N_p}}, 2^{-\theta J_{N_p}-1}) \subset I'$. By (26), $J_{N_p} > 100 \cdot \theta J_{N_{p-1}}$. On one hand, we saw in (33) that $\Delta_p^{I'} > \frac{1}{2} 2^{-d\theta J_{N_{p-1}}} 2^{dJ_{N_p}}$. We deduce that the mass of a ball I in $\mathcal{F}_p$ included in I' has m_θ-mass which satisfies

$$m_\theta(I) = m_\theta(I') \frac{1}{\Delta_p^{I'}} \leq m_\theta(I') 2^{1+d\theta J_{N_{p-1}} - dJ_{N_p}}. \tag{39}$$

On the other hand, since B is within a cube of side length $2|B|$ the number of cubes of generation p (i.e. of the form $\overline{B}((\mathbf{k}+\mathbf{e})2^{-J_{N_p}}, 2^{-\theta J_{N_p}-1})$) intersecting B is less than $4^d |B|^d 2^{dJ_{N_p}}$.

Hence, combining (29), (39) and $|B|^{-1/\theta} > 2^{J_{N_{p-1}}}$, we find that

$$\begin{aligned} m_\theta(B) &\leq \sum_{I \in \mathcal{F}_p : I \cap B \neq \emptyset} m_\theta(I) \\ &\leq 4^d |B|^d 2^{dJ_{N_p}} m_\theta(I') 2^{1+d\theta J_{N_{p-1}} - dJ_{N_p}} \leq C |B|^d 2^{d\theta J_{N_{p-1}}} m_\theta(I') \\ &\leq C |B|^d 2^{d\theta J_{N_{p-1}}} 2^{-dJ_{N_{p-1}}(1-\frac{2}{p-1})} \leq C |B|^d \cdot 2^{dJ_{N_{p-1}}(\theta - 1 + \frac{2}{p-1})} \\ &\leq C |B|^d \cdot |B|^{-\frac{1}{\theta} d(\theta - 1 + \frac{2}{p-1})} \leq |B|^{\frac{d}{\theta}(1-\frac{2}{p-1})} \leq |B|^{\frac{d}{\theta} - \varepsilon_1}, \end{aligned}$$

the last inequality being true for large p, i.e. for Borel sets B of diameter small enough (by the same argument as above).

We can thus choose $\eta_1 \in (0, \eta_0)$ so that

$$\text{when } |B| \leq \eta_1, \quad m_\theta(B) \leq |B|^{\frac{d}{\theta} - \varepsilon_1}.$$

Fix now $\varepsilon_2 = 2^{-2}$. By the same method as above, we find $0 < \eta_2 < \eta_1$ such that if $|B| \leq \eta_2$,

$$m_\theta(B) \leq |B|^{\frac{d}{\theta} - \varepsilon_2}.$$

We iterate the procedure: $\forall\ p > 1$, there is $0 < \eta_p < \eta_{p-1}$ such that

$$\text{if } |B| \leq \eta_p, \quad m_\theta(B) \leq |B|^{\frac{d}{\theta} - \varepsilon_p}, \qquad \text{where } \varepsilon_p = 2^{-p}.$$

In order to conclude, we consider a map ψ built as an increasing continuous interpolation function which goes through the points $(\eta_{p+1}, \varepsilon_p)_{p \geq 1}$ and (η_1, ε_1). The shift in the indices in the sequence is introduced so that $\varepsilon_p \leq \psi(x) \leq \varepsilon_{p-1}$ holds for $x \in [\eta_{p+1}, \eta_p]$. Hence (37) holds true for every Borel set B satisfying $|B| \leq \eta := \eta_1$. □

We can now conclude our results on the values of the spectrum of μ.

Proposition 5.8. *For any $h \in [0, d)$, $d_\mu(h) = h$.*

Proof. Let $h \in (0,d)$, and let $\theta = d/h > 1$. Recall that

$$\widetilde{E_\mu}(h) = \{x \in [0,1]^d : h_\mu(x) \le h\} = \bigcup_{h' \le h} E_\mu(h').$$

By Proposition 5.3, $\mathcal{A}_\theta \subset \widetilde{E_\mu}(h)$, hence by Proposition 2.1 we have $\dim \mathcal{A}_\theta \le \dim \widetilde{E_\mu}(h) \le h$.

Let us write

$$\mathcal{A}_\theta = \Big(\mathcal{A}_\theta \cap E_\mu(h)\Big) \bigcup \Big(\bigcup_{n \ge 1} \mathcal{A}_\theta \cap \widetilde{E_\mu}(h - 1/n)\Big).$$

Now, consider the measure m_θ provided by Theorem 5.5. Since the Cantor set $\mathcal{C}_\theta$ is the support of m_θ and since it is included in $\mathcal{A}_\theta$, we have $m_\theta(\mathcal{A}_\theta) \ge m_\theta(\mathcal{C}_\theta) > 0$.

For any $n \ge 1$, $\dim_{\mathcal{H}} (\mathcal{A}_\theta \cap \widetilde{E_\mu}(h-1/n)) \le \dim_{\mathcal{H}} \widetilde{E_\mu}(h-1/n) \le h - 1/n < h = d/\theta$ again by Proposition 2.1. Since we proved in Theorem 5.5 that $\dim_{\mathcal{H}} m_\theta \ge d/\theta$, by Property (9) we deduce that $m_\theta(\mathcal{A}_\theta \cap \widetilde{E_\mu}(h-1/n)) = 0$.

Combining the above, we see that $m_\theta(\mathcal{A}_\theta) = m_\theta\Big(\mathcal{A}_\theta \cap E_\mu(h)\Big) > 0$, hence $\dim_{\mathcal{H}} E_\mu(h) \ge \dim_{\mathcal{H}} \mathcal{A}_\theta \cap E_\mu(h) \ge d/\theta = h$. We already had the corresponding upper bound, hence the result for $h \in (0,d)$.

It remains us to treat the case $h = 0$.

For $h = 0$, we consider the set $\mathcal{A}_\infty = \bigcap_{\theta > 1} \mathcal{A}_\theta = \bigcap_{N \ge 1} \mathcal{A}_N$. This set is non-empty and uncountable, since each $\mathcal{A}_N$ contains a dense G_δ subset of $[0,1]^d$, namely $\cap_{P \ge 1} \cup_{p \ge P} \operatorname{int}(\mathcal{A}_{N,p})$ and the countable intersection of dense G_δ sets is still dense G_δ. Moreover, by Proposition 5.3, every $x \in \mathcal{A}_\infty$ has exponent 0 for μ. Hence, $\mathcal{A}_\infty \subset E_\mu(0)$ and $\dim_{\mathcal{H}} E_\mu(0) = 0$. □

[1] V. Aversa, C. Bandt, *The multifractal spectrum of discrete measures.* 18th Winter School on Abstract Analysis (1990), Acta Univ. Carolin. Math. Phys. **31(2)** (1990), 5–8.

[2] J. Barral, *Continuity of the multifractal spectrum of a random statistically self-similar measures,* J. Theor. Probab. **13**, 1027–1060 (2000).

[3] J. Barral, S. Seuret, *Combining multifractal additive and multiplicative chaos*, Commun. Math. Phys., **257(2)** (2005), 473–497.

[4] J. Barral, B.M. Mandelbrot, *Random multiplicative multifractal measures*, Fractal Geometry and Applications, Proc. Symp. Pure Math., AMS, Providence, RI, 2004.

[5] J. Barral, S. Seuret, *The multifractal nature of heterogeneous sums of Dirac masses*,Math. Proc. Cambridge Philos. Soc., 144(3) 707-727, 2008.

[6] J. Barral, S. Seuret, *Threshold and Hausdorff dimension of a purely discontinuous measure*, Real Analysis Exchange, 32(2), 455-472, 2007.

[7] J. Barral, S. Seuret, *Information parameters and large deviations spectrum of discontinuous measures*, Real Analysis Exchange, 32(2), 429-454, 2007.

[8] G. Brown, G. Michon, J. Peyrière, *On the multifractal analysis of measures*, J. Stat. Phys. **66** (1992), 775–790.

[9] Z. Buczolich, J. Nagy, *Hölder spectrum of typical monotone continuous functions*, Real Anal. Exchange **26** (2000/01), 133–156.

[10] K.J. Falconer, *Fractal Geometry*, John Wiley, Second Edition, 2003.

[11] K.J. Falconer, *Techniques in Fractal Geometry,* John Wiley & Sons, 1997.

[12] S. Jaffard, *Old friends revisited: the multifractal nature of some classical functions*, J. Fourier Anal. Appl. **3(1)** (1997), 1–21.

[13] S. Jaffard, *The multifractal nature of Lévy processes*, Probab. Theory Relat. Fields **114** (1999), 207–227.

[14] L. Olsen, *A multifractal formalism*, Adv. Math. (1995) **116**, 92–195.

[15] L. Olsen, N. Snigireva, *Multifractal spectra of in-homogenous self-similar measures.* Indiana Univ. Math. J. 57 (2008), no. 4, 1789–1843.

[16] L. Olsen, *Typical L^q-dimensions of measures.* Monatsh. Math. 146 (2005), no. 2, 143–157.

[17] L. Olsen, *Typical upper L^q-dimensions of measures for $q \in [0,1]$*. Bull. Sci. Math. 132 (2008), no. 7, 551–561.

[18] Y. Pesin, *Dimension theory in dynamical systems: Contemporary views and applications,* Chicago lectures in Mathematics, The University of Chicago Press, 1997.

[19] D. Rand, *The singularity spectrum $f(\alpha)$ for cookie-cutters, Ergod. Th. Dynam. Sys.* (1989), **9**, 527–541.

[20] R. Riedi, B.B. Mandelbrot, *Exceptions to the multifractal formalism for discontinuous measures*, Math. Proc. Cambridge Philos. Soc., **123** (1998), 133–157.