Graphs with Integer Matching Polynomial Roots

S. Akbari[a][*] **P. Csikvári**[b][†] **A. Ghafari**[a], **S. Khalashi Ghezelahmad**[c], **M. Nahvi**[a]

[a]Department of Mathematical Sciences, Sharif University of Technology, Tehran, Iran

[b]Department of Computer Science, Eötvös Loránd University, Budapest, Hungary

[c]School of Mathematics, Iran University of Science and Technology, Tehran, Iran

Abstract

In this paper, we study graphs with integer matching polynomial roots. We characterize all traceable graphs whose all matching polynomial roots are integer. We show that apart from $K_7 \setminus (E(C_3) \cup E(C_4))$ there is no connected k-regular graph ($k \geq 2$), whose all matching polynomial roots are integer. We also show that a graph with a perfect matching has a matching root in the interval $(0, 1]$. Finally, we describe all claw-free graphs.

1 Introduction

All graphs we consider are finite, simple and undirected. Let G be a graph. We denote the edge set and the vertex set of G by $E(G)$ and $V(G)$, respectively. By *order* and *size* of G, we mean the number of vertices and the number of edges of G, respectively. The maximum degree of G is denoted by $\Delta(G)$ (for the simplicity by Δ). The minimum degree of G is denoted by $\delta(G)$. In this paper, we denote the complete graph, the path and the cycle of order n, by K_n, P_n and C_n, respectively. The set of neighbors of a vertex v is denoted by $N(v)$. A *traceable graph*, is a graph with a Hamilton path. An r-*matching* in a graph G is a set of r pairwise non-incident edges. The number of r-matchings in G is denoted by $p(G, r)$. The *matching polynomial* of G is defined by

$$\mu(G, x) = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^r p(G, r) x^{n-2r},$$

where n is the order of G and $p(G, 0)$ is considered to be 1, see [7, 8, 9, 10, 11]. For instance the matching polynomial of the following graph

is $\mu(G, x) = x^5 - 5x^3 + 4x$. By the definition of $\mu(G, x)$, we conclude that every graph of odd order has a zero matching root. Furthermore, if θ is a matching root of a graph, then so is $-\theta$. We call a

---

graph, *matching integral* if all zeros of its matching polynomial are integer. A graph is said to be *integral* if eigenvalues of its adjacency matrix consist entirely of integers. Since 1974, integral graphs have been extensively studied by several authors, for instance see [2, 12]. It is worth mentioning that if T is a tree, then its characteristic polynomial and its matching polynomial are the same, see [6, Corollary 1.4, p.21]. Integral trees (so matching integral trees) have been investigated in [16].

In Section 2, we characterize all traceable graphs which are matching integral. In Section 3, we study matching integral regular graphs and show that for $k \geq 2$ there is only one connected matching integral k-regular graph, namely $K_7 \setminus (E(C_3) \cup E(C_4))$. In order to establish our results, first we need the following theorems:

Theorem A. [13] *For any graph G, the zeros of $\mu(G, x)$ are all real. If $\Delta > 1$, then the zeros lie in $(-2\sqrt{\Delta - 1}, \ 2\sqrt{\Delta - 1})$.*

Remark 1. Let G be a graph. Note that if $\sqrt{\Delta - 1}$ is not integer, then $\mu(G, x)$ contains at most $2\lfloor 2\sqrt{\Delta - 1} \rfloor + 1$ integer roots and if $\sqrt{\Delta - 1}$ is integer, then $\mu(G, x)$ has at most $4\sqrt{\Delta - 1} - 1$ integer roots.

Theorem B. [6, Corollary 1.3, p.97] *If G is a connected graph, then the largest zero of $\mu(G, x)$ is simple.*

Let $t(G)$ be the number of vertices of a longest path in the graph G.

Theorem C. [6, Theorem 4.5, p.107] *The maximum multiplicity of a zero of $\mu(G, x)$ is at most equal to the number of vertex-disjoint paths required to cover G. The number of distinct zeros of $\mu(G, x)$ is at least $t(G)$.*

Theorem D. [6] *If θ is a zero of $\mu(G, x)$ with multiplicity at least 2 then for any path P we have that θ is a zero of $\mu(G \setminus P, x)$, where $G \setminus P$ is the induced subgraph of G on the vertex set $V(G) \setminus V(P)$.*

Theorem D is not stated as a theorem in [6], but is used in the proof of Theorem 4.5 of Chapter 6 of [6]. Both Theorems C and D rely on the curious identity

$$\mu'(G, x)^2 - \mu(G, x)\mu''(G, x) = \sum \mu(G \setminus P, x)^2,$$

where the summation goes for all paths of G. For instance, if θ is a root of $\mu(G, x)$ with multiplicity at least 2 then it is a root of both $\mu(G, x)$ and $\mu'(G, x)$ so the left hand side is 0 at θ, but the right hand side is only 0 if all terms are 0.

2 Matching Integral Traceable Graphs

In this section, we show that there are finitely many matching integral traceable graphs and characterize all of them. In fact, we will characterize those graphs whose matching polynomial has only simple integer roots. By Theorem C we know that this is true for all traceable graph.

Theorem 2.1. *Let G be a connected graph whose matching polynomial has only simple integer roots. Then G is one of the following graphs: K_1, K_2, $K_7 \setminus (E(C_3) \cup E(C_4))$, G_1 or G_2, where*

$$G_1 : \qquad G_2 :$$

In particular, this is the list of matching integral traceable graphs.

Proof. Let n and m be the order and the size of G, respectively. It is enough to prove the first part of the theorem as the second part of the theorem indeed follows from the first one: since G is traceable, by Theorem C, the zeros of $\mu(G, x)$ are all distinct. Now, in order to prove the first part, we consider two cases:

Case 1. $n = 2k$, $k \geq 1$. Since G has even order and all roots are simple, every root of $\mu(G, x)$ is non-zero. Let $\theta_1, \ldots, \theta_k$ be the positive roots of $\mu(G, x)$. Hence

$$\mu(G, x) = \prod_{i=1}^{k} (x^2 - \theta_i^2) = x^{2k} - (\theta_1^2 + \cdots + \theta_k^2)x^{2k-2} + \cdots + (-1)^k \theta_1^2 \cdots \theta_k^2.$$

We have

$$m = \sum_{i=1}^{k} \theta_i^2 \geq \sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6} = \frac{n}{12}\left(\frac{n}{2} + 1\right)(n+1).$$

Thus for $n \geq 8$, $m > \binom{n}{2}$, a contradiction. Now, assume that $n \leq 6$. We consider three cases:

Case 1.1. $n = 2$. Hence $G = K_2$ and $\mu(G, x)$ has roots $-1, +1$.

Case 1.2. $n = 4$. Since $\Delta \leq 3$, Theorem A implies that the roots of $\mu(G, x)$ lie in $[-2, 2]$. Hence $\mu(G, x) = (x^2 - 1)(x^2 - 4) = x^4 - 5x^2 + 4$. Thus $m = 5$ and so $G = K_4 \setminus e$, for some edge e. But $\mu(K_4 \setminus e, x) = x^4 - 5x^2 + 2$, a contradiction.

Case 1.3. $n = 6$. Since $\Delta \leq 5$, by Theorem A the roots of $\mu(G, x)$ lie in $[-3, 3]$. Therefore $\mu(G, x) = (x^2 - 1)(x^2 - 4)(x^2 - 9)$. Hence $m = 14$ and $G = K_6 \setminus e$, for some edge e. Now, by [6, Theorem 1.1(b), p.2], we find that $\mu(G, x) = \mu(K_6, x) + \mu(K_4, x) = (x^6 - 15x^4 + 45x^2 - 15) + (x^4 - 6x^2 + 3)$, a contradiction.

Case 2. $n = 2k + 1$, $k \geq 0$. Let $0, \theta_1, \ldots, \theta_k$ be the non-negative roots of $\mu(G, x)$. Thus

$$\mu(G, x) = x \prod_{i=1}^{k} (x^2 - \theta_i^2) = x^{2k+1} - (\theta_1^2 + \cdots + \theta_k^2)x^{2k-1} + \cdots + (-1)^k \theta_1^2 \cdots \theta_k^2 x.$$

It follows that

$$m = \sum_{i=1}^{k} \theta_i^2 \geq \sum_{i=1}^{k} i^2 = \frac{n(n^2 - 1)}{24}.$$

Hence for $n \geq 13$, $m > \binom{n}{2}$, a contradiction. If $n \leq 3$, then clearly $G = K_1$. Now, we consider four cases:

Case 2.1. $n = 5$. Since $\Delta \leq 4$, by Theorem A the roots of $\mu(G, x)$ lie in $[-3, 3]$. Since $m = \theta_1^2 + \theta_2^2 \leq 10$, the positive roots of $\mu(G, x)$ are either $1, 3$ or $1, 2$. In the first case, $\mu(G, x) = x(x^2 - 1)(x^2 - 9)$. Hence $m = 10$ and so $G = K_5$. Therefore, $p(G, 2) = \frac{5!}{2!2^2} \neq 9$, a contradiction. In the second case, $\mu(G, x) = x(x^2 - 1)(x^2 - 4)$. Hence $m = 5$ and $p(G, 2) = 4$. It follows that G is one of the following graphs:

Case 2.2. $n = 7$. Since $\Delta \leq 6$, Theorem A implies that the roots of $\mu(G, x)$ lie in $[-4, 4]$. Since $m = \theta_1^2 + \theta_2^2 + \theta_3^2 \leq 21$, the positive roots of $\mu(G, x)$ are either $1, 2, 3$ or $1, 2, 4$. In the first case, $\mu(G, x) = x(x^2 - 1)(x^2 - 4)(x^2 - 9)$. So $m = 14$, $p(G, 2) = 49$ and $p(G, 3) = 36$. On the other hand, $|E(K_7 \backslash (E(C_3) \cup E(C_4)))| = 14$, $p(K_7 \backslash (E(C_3) \cup E(C_4)), 2) = 49$ and $p(K_7 \backslash (E(C_3) \cup E(C_4)), 3) = 36$. Since $K_7 \backslash (E(C_3) \cup E(C_4))$ is 4-regular, [6, Exercise 4, p.15] implies that G is 4-regular. Obviously, there are two non-isomorphic 4-regular graphs of order 7, $K_7 \backslash E(C_7)$ and $K_7 \backslash (E(C_3) \cup E(C_4))$. Therefore $G = K_7 \backslash (E(C_3) \cup E(C_4))$.

In the second case, $\mu(G, x) = x(x^2 - 1)(x^2 - 4)(x^2 - 16)$ and $m = 21$. Hence $G = K_7$ and so $p(G, 3) = \frac{7!}{3!2^3} \neq 64$, a contradiction.

Case 2.3. $n = 9$. Since $\Delta \leq 8$, by Theorem A the roots of $\mu(G, x)$ lie in $[-5, 5]$. Note that $m = \theta_1^2 + \theta_2^2 + \theta_3^2 + \theta_4^2 \leq 36$, so the positive roots of $\mu(G, x)$ should be $1, 2, 3, 4$. Thus $\mu(G, x) = x(x^2 - 1)(x^2 - 4)(x^2 - 9)(x^2 - 16)$. Hence $m = 30$ and $p(G, 2) = 273$. Furthermore, $p(G, 2) = \binom{30}{2} - \sum_{i=1}^{9} \binom{d_i}{2}$, where $d_1, \ldots, d_9$ is degree sequence of G. So we have the following:

$$\sum_{i=1}^{9} d_i = 60, \quad \text{and} \quad \sum_{i=1}^{9} d_i^2 = 384.$$

But this contradicts the Cauchy–Schwarz inequality:

$$60^2 = \left(\sum_{i=1}^{9} 1 \cdot d_i \right)^2 \leq \left(\sum_{i=1}^{9} 1^2 \right) \left(\sum_{i=1}^{9} d_i^2 \right) = 9 \cdot 384 < 60^2.$$

Case 2.4. $n = 11$. Since $\Delta \leq 10$, by Theorem A the roots of $\mu(G, x)$ lie in $[-5, 5]$. Thus $\mu(G, x) = x(x^2 - 1)(x^2 - 4) \cdots (x^2 - 25)$. Hence $m = 55$ and so $G = K_{11}$. Therefore $p(G, 5) = \frac{11!}{5!2^5} \neq 14400$, a contradiction. $\square$

3 Matching Integral Regular Graphs

In this section, we study matching integral regular graphs. We show the following theorem.

Theorem 3.1. *If G is a matching integral k-regular graph $(k \geq 2)$ then it is disjoint union of $K_7 \backslash (E(C_3) \cup E(C_4))$.*

Let G be a graph of order n. Recall that $t(G)$ denotes the number of vertices of a longest path in the graph G. By Theorem A all zeros of a matching integral k–regular graph lie in the interval $(-2\sqrt{k-1}, 2\sqrt{k-1})$ and so the number of distinct zeros is at most $2\lfloor 2\sqrt{k-1} \rfloor + 1$. By the second claim of Theorem C this is an upper bound for $t(G)$, hence

$$t(G) \leq 2\lfloor 2\sqrt{k-1} \rfloor + 1.$$

On the other hand, $t(G) \geq k + 1$ for a k–regular graph simply by choosing the vertices of a path greedily. This already gives that $k \leq 14$. On the other hand one can improve on the bound $t(G) \geq k + 1$. The following lemma is practically an immediate consequence of a theorem of Dirac.

Lemma 3.2. *Let G be a k–regular connected graph on n vertices. Then*

$$t(G) \geq \min(2k + 1, n).$$

Before we prove this lemma let us deduce the following corollary.

Lemma 3.3. *Let G be a matching integral k–regular graph, where $k \geq 3$. Then G is a disjoint union of a few copies of $K_7 \setminus (E(C_3) \cup E(C_4))$.*

Proof. We can assume that G is connected since the set of matching zeros of a graph is the union of the set of matching zeros of the components of the graph. If $t(G) = n$, where n is the number of vertices then G is traceable and so by Theorem 2.1 it is $K_7 \setminus (E(C_3) \cup E(C_4))$. If $t(G) \neq n$ then $t(G) \geq 2k + 1$. Then

$$2k + 1 \leq t(G) \leq 2\lfloor 2\sqrt{k-1} \rfloor + 1$$

implies that $k \leq 2$ contradicting the condition $k \geq 3$. $\square$

Next we prove Lemma 3.2. We will use the following theorem of Dirac.

Lemma 3.4. *[5] Let $c(G)$ be the longest cycle of a connected graph G on n vertices. Assume that G is 2-connected and has minimum degree at least k. Then $c(G) \geq \min(2k, n)$.*

Proof of Lemma 3.2. If G is 2-connected then by Lemma 3.4 we have $c(G) \geq \min(2k, n)$. If $c(G) = n$ then clearly $t(G) = n$. If $c(G) \geq 2k + 1$ we are done again. If $c(G) = 2k < n$ then by connectedness of G there is a vertex v not in the cycle which is connected to some vertex of the cycle, but then there is a path of length at least $2k + 1$.

If G is not 2-connected then it contains a cut vertex v. From v let us build up two vertex disjoint paths in two different components of $G - v$ greedily. By concatenating the two paths we get a path of length at least $2k + 1$. $\square$

Finally for $k = 2$ the following lemma is an immediate consequence of Theorem 2.1.

Lemma 3.5. *For every positive integer n, C_n is not matching integral.*

4 Matching Integral Graphs with a Perfect Matching

In this section we study the roots of a matching polynomial of a graph with a perfect matching.

Theorem 4.1. *If a graph G has a perfect matching then its matching polynomial has a zero in the interval $(0, 1]$. If it has no zero in the interval $(0, 1)$ then it is the disjoint union of some K_2.*

Proof. Let G be a graph on $2n$ vertices. Since G has a perfect matching we have $p(G, n) \neq 0$, consequently

$$\mu(G, x) = \prod_{i=1}^{n} (x^2 - \theta_i^2),$$

where $\theta_i \neq 0$. Then

$$\frac{p(G, n-1)}{p(G, n)} = \sum_{i=1}^{n} \frac{1}{\theta_i^2}.$$

Next we show that

$$\frac{p(G, n-1)}{p(G, n)} \geq n.$$

Indeed, every perfect matching contains exactly n matchings of size $n - 1$, and every matching of size $n - 1$ can be extended to a perfect matching in at most 1 way. Hence

$$n \leq \frac{p(G, n-1)}{p(G, n)} = \sum_{i=1}^{n} \frac{1}{\theta_i^2}$$

implies that $\min_i \theta_i^2 \leq 1$, and if $\min_i \theta_i^2 = 1$ then $\theta_1^2 = \cdots = \theta_n^2 = 1$. Hence the graph has $\theta_1^2 + \cdots + \theta_n^2 = n$ edges which form a perfect matching, i. e., G is the disjoint union of some K_2. $\square$

We offer one more theorem in the same spirit. Let

$$f(t) = \begin{cases} t+1 & \text{if } t \neq 1, \\ 3 & \text{if } t = 1. \end{cases}$$

The proof of the following theorem is practically the same as the proof of Theorem 4.1. The only extra observation one needs is that a graph on $t + 2$ vertices without a 2–matching contains at most $f(t)$ edges.

Theorem 4.2. *Let G be a graph with at least one edge. Assume that the multiplicity of 0 as a root of the matching polynomial of the graph G is t. Then the interval $(0, \sqrt{f(t)}]$ contains a root of the matching polynomial of G.*

5 Matching Integral Claw-free Graphs

In this section we study matching integral claw-free graphs.

Theorem 5.1. *Let G be a connected matching integral claw-free graph. Then G is one of K_1, K_2 or G_2.*

Note that from the list of traceable matching integral graphs G_1 and $K_7 \setminus (E(C_3) \cup E(C_4))$ are not claw-free.

The proof of Theorem 5.1 is quite long, and so we summarize here the plan of the proof. First we show that a connected claw-free graph always contains a matching which avoids at most one vertex, so if G has even order then it contains a perfect matching, and if it has odd order then the largest matching avoids exactly one vertex. This settles the multiplicity of 0 as a root of the matching polynomial. If G contains a perfect matching then Theorem 4.1 already gives that G is K_2. Then just as in case of regular graphs we try to find long paths in the graph. We fix the largest degree Δ and as we gain more and more information about the length of the longest path we exclude the possibility of more and more values of Δ. By the time we stuck with the ideas of finding long paths we will have enough information about the structure of G so that we get many information about the matching polynomial. This way we can shrink the set of possible matching polynomials. Then we translate it back to structural information about the graph and we finish the proof.

Lemma 5.2. *Let G be a connected claw-free graph. Then it contains a matching which avoids at most one vertex.*

Proof. This is a well-known statement which can be found in the book of J. Akiyama and M. Kano [1]. See also [14, 15]. □

Next we start our hunting to long paths. Our main tool is the following lemma and its corollary.

Lemma 5.3. *Let H be a graph such that H has largest independent set of size at most 2. Then H has either a Hamiltonian cycle or there are two vertex-disjoint cliques in H covering all vertices of H.*

Proof. This lemma immediately follows from the following theorem of Chvátal and Erdős [4]. It can also be found in the book of Bondy and Murty [3] as Theorem 18.10.

Theorem. *Let G be a graph on at least 3 vertices. If, for some s, G is s-connected and contains no independent set of more than s vertices, then G has a Hamiltonian cycle.*

Clearly, if G is 2-connected then for $s = 2$ this immediately implies our statement. If G is not connected then it is easy to see that G must be the union of two disjoint cliques. While, if G has a cut vertex v, then $G - v$ must be the union of two disjoint cliques and v must be adjacent to all elements of at least one of the cliques. So we are done in this case too.

□

Lemma 5.4. *(a) Let G be a connected claw-free graph with a vertex v of largest degree Δ. Then for any $u \in N(v)$ there is a path P_u starting at u which covers all vertices of $N(v) \cup \{v\}$.*
(b) We have $t(G) \geq \Delta + 1$, and if $G \neq N(v) \cup \{v\}$ then $t(G) \geq \Delta + 2$.

Proof. Part (b) is an immediate consequence of part (a) since $|N(v) \cup \{v\}| = \Delta + 1$, and if $G \neq N(v) \cup \{v\}$ then by the connectedness of G there is a w adjacent to some $u \in N(v)$ so we can extend P_u with w.

To prove part (a), let H be the graph induced by the neighbors of v. Then H has largest independent set of size at most 2, otherwise there would be a claw in G with center v. Then by the previous lemma H has either a Hamiltonian cycle or there are two vertex-disjoint cliques in H covering all vertices of H. In both cases there is a path P_u starting at vertex u which covers $V(H) \cup \{v\}$.

□

For a positive integer r let $s(r)$ be the number of integers in the interval $(-2\sqrt{r-1}, 2\sqrt{r-1})$. Clearly, this is $2\lfloor 2\sqrt{r-1} \rfloor + 1$ if $r - 1$ is not a perfect square, and $4\sqrt{r-1} - 1$ if $r - 1$ is a perfect square. So far we know that

$$\Delta + 1 \leq t(G) \leq s(\Delta).$$

From this follows that $2 \leq \Delta \leq 14$. Furthermore, if $\Delta + 1 = s(\Delta)$ then $t(G) = \Delta + 1$ and then $G = N(v) \cup \{v\}$. In particular, it is traceable, but we already described all traceable claw-free graphs: K_1, K_2 and G_2 (and their largest degrees are $0, 1$ and 3). The following table shows the values of $s(\Delta)$ and $\Delta + 1$. The case $\Delta = 13$ contradicts $\Delta + 1 \leq s(\Delta)$. This table also shows that we can exclude the possibility of $\Delta \in \{2, 10, 12, 14\}$, because in these cases $\Delta + 1 = s(\Delta)$.

r	2	3	4	5	6	7	8	9	10	11	12	13	14
$2\sqrt{r-1}$	2	2.82	3.46	4	4.47	4.89	5.29	5.65	6	6.32	6.63	6.92	7.21
$s(r)$	3	5	7	7	9	9	11	11	11	13	13	13	15
$s(r) - (r+1)$	0	1	2	1	2	1	2	1	0	1	0	-1	0

Next we show that it is not possible that $\Delta \in \{3, 5, 7, 9, 11\}$, and if $\Delta \in \{4, 6, 8\}$ then the multiplicity of some θ_i with $|\theta_i| > 1$ is at most 1.

Let $k(r) = s(r) - (r+1)$, it expresses how much longer the longest path can be than $\Delta + 1$ if $r = \Delta$. As before let v be a vertex of degree Δ and $N(v)$ denotes its set of neighbors. We will show that if $k(\Delta) = 1$ then $G \setminus (N(v) \cup \{v\})$ is an empty graph on some vertices, and if $k(\Delta) = 2$ then $G \setminus (N(v) \cup \{v\})$ is a union of K_1 and K_2.

For a vertex $u \in N(v)$ let G_u be the graph obtained from G by deleting $N(v) \cup \{v\}$ except u. Let C_u be the component of u in G_u. We show that if $k(\Delta) = 1$ then $|C_u| \leq 2$, and if $k(\Delta) = 2$ then $|C_u| \leq 3$. Let $u_1, \ldots, u_r$ be the neighbors of u in C_u. The vertices $u_1, \ldots, u_r$ have to form a clique otherwise there would be a claw u, v, u_i, u_j with center u for some non-adjacent u_i and u_j. This means that we can add a path $u_1 u_2 \ldots u_r$ to the path P_u covering $N(v) \cup \{v\}$. In particular, this means that in case of $k(\Delta) = 1$ we have $r \leq 1$, and in case of $k(\Delta) = 2$ we have $r \leq 2$. If $k(\Delta) = 1$ and $r = 1$ the vertex u_1 cannot have further neighbor in C_u so $|C_u| = 2$. Similarly, if $k(\Delta) = 2$ and $r = 2$ the vertices u_1 and u_2 cannot have further neighbors in C_u: if some w is adjacent to say u_1 then $w u_1 u_2 P_u$ be a path of length $\Delta + 4$. If $k(\Delta) = 2$ and $r = 1$ the vertex u_1 can have at most one further neighbor, but not more: if u_2 and u_3 are adjacent to u_1 then they are adjacent to each other too since otherwise we have a claw u_1, u, u_2, u_3 with center u_1, but then

u_2, u_3, u_1, P_u is again a path of length $\Delta + 4$. Finally, if $k(\Delta) = 2$ and $r = 1$, and the vertex u_1 has a further neighbor u_2, then u_2 cannot have further neighbors in C_u, so $|C_u| \leq 3$. This proves that if $k(\Delta) = 1$ then $|C_u| \leq 2$, and if $k(\Delta) = 2$ then $|C_u| \leq 3$. In particular, it shows that if $k(\Delta) = 1$ then $G \setminus (N(v) \cup \{v\})$ is a disjoint union of K_1, and if $k(\Delta) = 2$ then $G \setminus (N(v) \cup \{v\})$ is a disjoint union of K_1 and K_2.

Now let P be the path going through $N(v) \cup \{v\}$. If θ is a zero of $\mu(G, x)$ with multiplicity at least 2 then by Theorem D we have that θ is a zero of

$$\mu(G \setminus P, x) = \mu(K_1, x)^{\alpha_1} \mu(K_2, x)^{\alpha_2} = x^{\alpha_1}(x^2 - 1)^{\alpha_2},$$

where $\alpha_2 = 0$ if $k(\Delta) = 1$. If $k(\Delta) = 1$ then it means that $\theta = 0$, and if $k(\Delta) = 2$ then it means that $\theta \in \{-1, 0, 1\}$. Since in a connected claw-free graph there is always a matching which avoids at most 1 vertex we have that the multiplicity of 0 as a root is at most 1. This means that if $k(\Delta) = 1$ then all zeros are simple, but then by Theorem 2.1 we know the complete list of these graphs. In the list the only claw-free graphs were K_1, K_2, G_2.

Note that in the cases $\Delta = 4, 6, 8$, all integers in the interval $(-2\sqrt{\Delta - 1}, 2\sqrt{\Delta - 1})$ must be a zero of $\mu(G, x)$ since otherwise the number of distinct zeros is at most $s(\Delta) - 2 \leq \Delta + 1$ (the missing root cannot be 0 since if G contains a perfect matching then we are done by Theorem 4.1) and so $t(G) = \Delta + 1$ and the graph is traceable, and we discussed this case already.

So we have the remaining cases for $\Delta = 4, 6, 8$:

$$\mu(G, x) = x(x^2 - 1)^{\alpha}(x^2 - 4)(x^2 - 9),$$

$$\mu(G, x) = x(x^2 - 1)^{\alpha}(x^2 - 4)(x^2 - 9)(x^2 - 16),$$

$$\mu(G, x) = x(x^2 - 1)^{\alpha}(x^2 - 4)(x^2 - 9)(x^2 - 16)(x^2 - 25).$$

Next we prove that the cases $\Delta = 6, 8$ are not possible and in case of $\Delta = 4$ there are 7 vertices of degree 4, all other vertices are of degree 1.

Let a_i be the number of vertices with degree i. Then $a_i = 0$ if $i > \Delta$ or $i = 0$ (G is connected), and $\sum a_i = n$ is the number of vertices; $\sum i a_i = 2m$, two times the number of edges; and $\sum i^2 a_i = \sum d(u)^2 = m(m + 1) - 2p(G, 2)$.

In particular, for $\Delta = 4$ we have

$$\sum_{i=1}^{4} a_i = 5 + 2\alpha, \quad \sum_{i=1}^{4} i a_i = 26 + 2\alpha, \quad \sum_{i=1}^{4} i^2 a_i = 110 + 2\alpha.$$

Then

$$0 \geq -2(a_2 + a_3) = \sum_{i=1}^{4}(i-1)(i-4)a_i = \left(\sum_{i=1}^{4} i^2 a_i\right) - 5\left(\sum_{i=1}^{4} i a_i\right) + 4\left(\sum_{i=1}^{4} a_i\right) = 0,$$

so $a_2 = a_3 = 0$ and $a_4 = 7$, $a_1 = 2(\alpha - 1)$.

For $\Delta = 6$ we have

$$\sum_{i=1}^{6} a_i = 7 + 2\alpha, \quad \sum_{i=1}^{6} i a_i = 58 + 2\alpha, \quad \sum_{i=1}^{6} i^2 a_i = 382 + 2\alpha.$$

Then

$$0 \geq \sum_{i=1}^{6}(i-1)(i-6)a_i = 18,$$

a contradiction.

For $\Delta = 8$ we have

$$\sum_{i=1}^{8} a_i = 9 + 2\alpha, \quad \sum_{i=1}^{8} ia_i = 108 + 2\alpha, \quad \sum_{i=1}^{8} i^2 a_i = 1032 + 2\alpha.$$

Then

$$0 \geq \sum_{i=1}^{8} (i-1)(i-8)a_i = 132,$$

again contradiction.

Next we eliminate the case $\Delta = 4$. In this case we know that 7 vertices have degree 4, and the rest of the vertices have degree 1. Clearly, two degree 1 vertices cannot be adjacent, because G is connected. One vertex of degree 4 cannot be adjacent to two vertices of degree 1 since there would be a claw in the graph. We can assume that there are indeed vertices of degree 1 since otherwise all roots would be simple. In this case we know from Theorem 2.1 that the only candidate is $K_7 \setminus E(C_3) \cup E(C_4)$ which is not claw-free. If we delete all degree 1 vertices we get a graph G' on 7 vertices where all degrees are 3 or 4. Both 3 and 4 should exist since we assumed the existence of degree 1 vertices, and it cannot occur that all degrees are 3 since the number of edges would be $7 \cdot 3/2 = 10.5$. Clearly, G' is still claw-free as it is an induced subgraph of G.

We show that G' contains a Hamiltonian cycle. G' is clearly connected since G is connected. We show that G' is also 2-connected. Indeed, if there were a cut-vertex u then by the minimal degree at least 3 condition the only possibility is that $G - u$ has 2 components of size 3 and every vertex is connected to u, but then u has degree 6. Then again we use Dirac's theorem, Lemma 3.4: the graph contains a cycle C of length at least 6. If the length is 7 then we are done. If the length of C is 6, then let the vertices of C be $v_1 \ldots v_6$ and v_7 be the remaining vertex. The degree of v_7 is at least 3, if it is adjacent to 2 neighboring vertices of C then we can extend C to a Hamiltonian cycle. Since the degree of v_7 is at least 3, the only case we have to consider is when v_7 is adjacent to vertices of even indices, or those of odd indices. By symmetry we can assume that v_7 is adjacent to v_1, v_3, v_5. Next observe that if any 2 of v_2, v_4, v_6 are adjacent then there is a Hamiltonian cycle: if v_2 and v_6 are adjacent then $v_1 v_2 v_6 v_5 v_4 v_3 v_7 v_1$ is a Hamiltonian cycle, the other two cases are symmetric to this one. Because of the minimum degree at least 3 condition v_2 has to be adjacent to v_5, v_4 has to be adjacent to v_1, and v_6 has to be adjacent to v_3. The obtained graph is actually a $K_{3,4}$ with classes v_1, v_3, v_5 and v_2, v_4, v_6, v_7. This graph doesn't contain a Hamiltonian cycle, but it contains a lot of claws, for instance v_1, v_2, v_4, v_6. If we add one more edge which we can assume to be $v_2 v_4$ by symmetry then it will contain a Hamiltonian cycle: $v_1 v_7 v_3 v_6 v_5 v_4 v_2 v_1$.

Now we can show a path consisting of 8 vertices in G: start at some pendant vertex, and after jumping to its unique neighbor go through a Hamiltonian path of G'. Hence $t(G) = 8 > 7 = s(4)$, so G cannot be matching integral. Hence we eliminated the case $\Delta = 4$ too. We are done.

We close the paper with the following question:

Question. Is it true that there are finitely many matching integral 2-connected graphs?

References

[1] J. Akiyama and M. Kano, Factors and factorizations of graphs, Springer 2011.

[2] K.T. Balińska, D. Cvetković, Z. Radosavljević, S.K. Simić and D. Stevanović, A survey on integral graphs, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat., 13 (2002), 42-65.

[3] J.A. Bondy and U.S.R. Murty, Graph theory, Springer 2008.

[4] V. Chvátal and P. Erdős, A note on Hamiltonian-circuits, Discrete Mathematics, 2 (1972), 111-113.

[5] G.A. Dirac, Some theorems on abstract graphs, Proc. London Math. Soc., 2 (1952), 69–81.

[6] C.D. Godsil, Algebraic Combinatorics, Chapman and Hall, Inc., 1993.

[7] C.D. Godsil and I. Gutman, On the theory of the matching polynomial, J. Graph Theory, 5(2) (1981), 137-144.

[8] I. Gutman, Characteristic and matching polynomial of some compound graphs, Publications De L'Institut Mathematique, 27 (41) (1980), 61-66.

[9] I. Gutman, The matching polynomial, MATCH Commun. Math. Comput. Chem., 6 (1979), 75-91.

[10] I. Gutman, Uniqueness of the matching polynomial, MATCH Commun. Math. Comput. Chem., 55 (2006), 351-358.

[11] I. Gutman and F. Harary, Generalizations of the matching polynomial, Utilitas Math., 24 (1983), 97-106.

[12] F. Harary and A.J. Schwenk, Which graphs have integral spectra? Graphs and Combinatorics., Lecture Notes in Math., Springer-Verlag, Berlin, 406 (1974), 45-51.

[13] O.J. Heilmann and E.H. Lieb, Theory of monomer-dimer systems, Commun. Math. Physics, 25 (1972), 190-232.

[14] M. Las Vergnas, A note on matchings in graphs, Cahiers Centre Etudes Reech. Opér., 17 (1975), 257-260.

[15] D.P. Sumner, 1-factors and antifactor sets, J. London Math. Soc., 22 (1976), 351-359.

[16] L. Wang, Integral trees and Integral graphs, Ph.D thesis, Printed by Wohrmann Print Service, The Netherlands, 2005.