INTEGRAL TREES OF ARBITRARILY LARGE DIAMETERS

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Abstract. In this paper we construct trees having only integer eigenvalues with arbitrarily large diameters. In fact, we prove that for every set S of positive integers there exists a tree whose positive eigenvalues are exactly the elements of S. If the set S is different from the set $\{1\}$ then the constructed tree will have diameter $2|S|$.

1. Introduction

An integral tree is a tree for which the eigenvalues of its adjacency matrix are all integers [9] . Many different classes of integral trees have been constructed in the past decades [2],[3],[4],[5],[10],[11],[12]. Most of these classes contain infinitely many integral trees, but till now only examples of trees of bounded diameters were known. The largest diameter of known integral trees was 10. In this paper we construct integral trees of arbitrarily large diameter. In fact, we prove the following much stronger theorem.

Theorem 1.1. *For every set S of positive integers there exists a tree whose positive eigenvalues are exactly the elements of S. If the set S is different from the set $\{1\}$ then the constructed tree will have diameter $2|S|$.*

Clearly, there is only one tree with set S of positive eigenvalues for $S = \{1\}$, the tree on two vertices with spectrum $\{-1, 1\}$ (and its diameter is 1).

The structure of this paper is the following. In the next section we will define a class of trees recursively. All trees belonging to this class will turn out to be *almost-integral*, i. e., all of their eigenvalues are squareroots of integers. We will find integral trees in this class of trees by special choice of parameters introduced later.

2. Construction of trees

Definition 2.1. For given positive integers $r_1, \dots, r_k$ we construct the trees $T_1(r_1), T_2(r_1, r_2), \dots, T_k = T_k(r_1, \dots, r_k)$ recursively as follows. We will consider the tree T_i as a bipartite graph with colorclasses A_{i-1}, A_i. The tree $T_1(r_1) = (A_0, A_1)$ consists of the classes of size $|A_0| = 1, |A_1| = r_1$ (so it is a star on $r_1 + 1$ vertices). If the tree $T_i(r_1, \dots, r_i) = (A_{i-1}, A_i)$ is defined then let $T_{i+1}(r_1, \dots, r_{i+1}) = (A_i, A_{i+1})$ be defined as follows. We connect each vertex of A_i with r_{i+1} new vertices of degree 1. Then for the resulting tree the colorclass A_{i+1} will have size $|A_{i+1}| = r_{i+1}|A_i| + |A_{i-1}|$, the colorclass A_i does not change.

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One should not confuse these trees with the balanced trees. These trees are very far from being balanced.

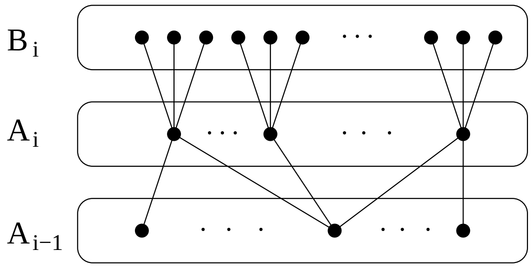

FIGURE 1. Let $A_{i+1} = A_{i-1} \cup B_i$ where each element of A_i has exacly r_{i+1} neighbors of degree 1 in B_i.

To analyse the trees $T_k(r_1, \dots, r_k)$ we will need the following definition.

Definition 2.2. Let us define the following sequence of expressions.

$$Q_0(.) = 1$$
$$Q_1(x_1) = x_1$$

and

$$Q_j(x_1, \dots, x_j) = x_j Q_{j-1}(x_1, \dots, x_{j-1}) + Q_{j-2}(x_1, \dots, x_{j-2})$$

for all $3 \leq j \leq k$. We will also use the convention $Q_{-1} = 0$. We will call these expression *continuants*. Sometimes if the $\underline{x} = (x_1, \dots, x_k)$ is well-understood then we will simply write Q_j instead of $Q_j(x_1, \dots, x_j)$.

Remark 2.3. The first few continuants are

$$Q_2(x_1, x_2) = 1 + x_1x_2, \quad Q_3(x_1, x_2, x_3) = x_1 + x_3 + x_1x_2x_3$$
$$Q_4(x_1, x_2, x_3, x_4) = 1 + x_1x_2 + x_1x_4 + x_3x_4 + x_1x_2x_3x_4$$

The expressions $Q_j(x_1, \dots, x_j)$ often show up in the study of some Euclidean type algorithms. For instance,

$$x_k + \cfrac{1}{x_{k-1} + \cfrac{1}{x_{k-2} + \cfrac{1}{\ddots + \cfrac{1}{x_1}}}} = \frac{Q_k(x_1, \dots, x_k)}{Q_{k-1}(x_1, \dots, x_{k-1})}$$

For more details on continuants see [8].

Lemma 2.4. *Let $T_k(r_1, \dots, r_k)$ be the constructed tree with colorclasses (A_{k-1}, A_k). Then $|A_{k-1}| = Q_{k-1}(r_1, \dots, r_k)$ and $|A_k| = Q_k(r_1, \dots, r_k)$.*

Proof. This is a trivial induction. □

Lemma 2.5. *If $r_1 \geq 2$ then the diameter of $T_k(r_1, \dots, r_k)$ is $2k$.*

Proof. Note that each vertex is at distance at most k from the only element v_0 of the set A_0. Thus the diameter is at most $2k$. On the other hand, if we go from v_0 to two different leaves through two different elements of A_1 which are at distance k from v_0 (so these are the elements of $A_k \backslash A_{k-2}$) then this two leaves must have distance $2k$. □

Remark 2.6. Note that $T_j(1, r_2, r_3, \ldots, r_j) = T_{j-1}(r_2+1, r_3, \ldots, r_j)$. Hence all constructed trees different from the tree on two vertices have a representation $T_k(r_1, \ldots, r_k)$ in which $r_1 \geq 2$.

The next lemma will be the main tool to determine the spectrum of the tree $T_k(r_1, \ldots, r_k)$. Before we state it we introduce the following notation.

Definition 2.7. Let $Sp(G)$ denote the spectrum of the graph G. Let $N_G(\lambda_i > 0)$ denote the number of positive eigenvalues of G and $N_G(\lambda = t)$ denote the multiplicity of the eigenvalue t.

Lemma 2.8. *Let $G = (A, B, E)$ be a bipartite graph with $\lambda \neq 0$ eigenvalue of multiplicity m. Let G' be obtained from G by joining each element of B with r new vertices of degree 1, so that the obtained graph has $|A| + (r+1)|B|$ vertices. Then $\pm\sqrt{\lambda^2 + r}$ are eigenvalues of G' of multiplicity m. Furthermore, the rest of the eigenvalues of the new graph are $\pm\sqrt{r}$ with multiplicity $|B| - N_G(\lambda_i > 0)$ and 0 with multiplicity $|A| + (r-1)|B|$ and there is no other eigenvalue.*

Proof. Since G and G' are both bipartite graphs we only need to deal with the non-negative eigenvalues. Let $0 < \mu \neq \sqrt{r}$ be an eigenvalue of the graph G' of multiplicity m. We prove that $\sqrt{\mu^2 - r}$ is an eigenvalue of G of multiplicity m. (Note that it means that $0 < \mu < \sqrt{r}$ cannot occur since the eigenvalues of a graph are real numbers.)

Let $\underline{x}$ be an eigenvector belonging to μ. We will construct an eigenvector $\underline{x'}$ to $\sqrt{\mu^2 - r}$ in the graph G. Let $v_i \in B$ and its new neighbors $w_{i1}, \ldots, w_{ir}$. Then

$$x(v_i) = \mu x(w_{i1}) = \mu x(w_{i2}) = \cdots = \mu x(w_{ir}).$$

Since $\mu \neq 0$ we have $x(w_{i1}) = \cdots = x(w_{ir})$. Moreover, for each $v_i \in B$ and $u_j \in A$ we have

$$\mu x(v_i) = \sum_{v_i \sim u_k} x(u_k) + r x(w_{i1})$$

and

$$\mu x(u_j) = \sum_{u_j \sim v_l} x(v_l).$$

Since $x(v_i) = \mu x(w_{i1})$ we can rewrite these equations as

$$(\mu^2 - r) x(w_{i1}) = \sum_{v_i \sim u_k} x(u_k)$$

and

$$\mu x(u_j) = \sum_{u_j \sim v_l} \mu x(w_{l1}).$$

In the second equation we can divide by μ since it is not 0. Hence it follows that

$$\sqrt{\mu^2 - r}(\sqrt{\mu^2 - r}\, x(w_{i1})) = \sum_{v_i \sim u_k} x(u_k)$$

and

$$\sqrt{\mu^2 - r}\, x(u_j) = \sum_{u_j \sim v_l} (\sqrt{\mu^2 - r}\, x(w_{l1})).$$

Thus the vector $\underline{x'}$ which is equal to $\sqrt{\mu^2 - r}x(w_{i1})$ on the vertices of B and $x(u_j)$ on the elements of A is an eigenvector of the graph G with eigenvalue $\sqrt{\mu^2 - r}$. Clearly, this vector is not $\underline{0}$, otherwise $\underline{x}$ should have been $\underline{0}$. It also implies that if the vectors $\underline{x}_1, \dots, \underline{x}_h$ are independent eigenvectors belonging to μ then the constructed eigenvectors $\underline{x'}_1, \dots, \underline{x'}_h$ are also independent. Note that this construction can be reversed if $\sqrt{\mu^2 - r} \neq 0$ implying that for $\mu \neq \sqrt{r}$ the multiplicity of μ in G' is the same as the multiplicity of $\sqrt{\mu^2 - r}$ in G.

We can easily determine the multiplicity of the eigenvalues 0 and $\sqrt{r}$ as follows:

$$e(G) + r|B| = e(G') = \sum_{\mu>0, \mu \in Sp(G')} \mu^2 = \sum_{\lambda>0, \lambda \in Sp(G)} (\lambda^2 + r) + N_{G'}(\mu = \sqrt{r})r =$$

$$e(G) + N_G(\lambda > 0)r + N_{G'}(\mu = \sqrt{r})r$$

Hence $N_{G'}(\mu = \sqrt{r}) = |B| - N_G(\lambda > 0)$. Finally the multiplicity of 0 as an eigenvalue of G' can be determined as follows:

$$N_{G'}(\mu = 0) = |A| + (r+1)|B| - 2N_{G'}(\mu > 0) =$$

$$|A| + (r+1)|B| - 2N_G(\lambda > 0) - 2N_{G'}(\mu = \sqrt{r}) = |A| + (r+1)|B| - 2|B|.$$

□

Theorem 2.9. *The set of different eigenvalues of the tree* $T_k(r_1, r_2, \dots, r_k)$ *is the set*

$$\{\pm\sqrt{r_k}, \pm\sqrt{r_k + r_{k-1}}, \pm\sqrt{r_k + r_{k-1} + r_{k-2}}, \dots, \pm\sqrt{r_k + \dots + r_1}, 0\}.$$

Furthermore, the multiplicity of 0 *is*

$$Q_k(r_1, \dots, r_k) - Q_{k-1}(r_1, \dots, r_{k-1})$$

and the multiplicity of the eigenvalues $\pm\sqrt{r_k + r_{k-1} + \dots + r_j}$ *are*

$$Q_{j-1}(r_1, \dots, r_{j-1}) - Q_{j-2}(r_1, \dots, r_{j-2})$$

where Q_i*'s are the continuants.*

Proof. We will use the short notation Q_j for $Q_j(r_1, \dots, r_j)$. We prove by induction on k. The statement is true for $k = 1$. Assume that the statement is true for $n = k - 1$. We need to prove it for $n = k$. By the induction hypothesis the tree $T_{k-1}(r_1, \dots, r_{k-1})$ has spectrum

$$\{\pm\sqrt{r_{k-1}}, \pm\sqrt{r_{k-1} + r_{k-2}}, \dots, \pm\sqrt{r_{k-1} + \dots + r_1}, 0\}.$$

Furthermore, the multiplicity of the eigenvalues $\pm\sqrt{r_{k-1} + \dots + r_j}$ are $Q_{j-1} - Q_{j-2}$. Now let us apply Lemma 2.8 with $G = T_{k-1}(r_1, \dots, r_{k-1})$ and $r = r_k$. Then $G' = T_k(r_1, \dots, r_k)$ has spectrum

$$\{\pm\sqrt{r_k}, \pm\sqrt{r_k + r_{k-1}}, \pm\sqrt{r_k + r_{k-1} + r_{k-2}}, \dots, \pm\sqrt{r_k + \dots + r_1}, 0\}.$$

Furthermore, the multiplicity of the eigenvalues $\pm\sqrt{r_k + r_{k-1} + \dots + r_j}$ are $Q_{j-1} - Q_{j-2}$ for $j \leq k - 1$. The multiplicity of $\sqrt{r_k}$ is

$$Q_{k-1} - ((Q_{k-2} - Q_{k-3}) + (Q_{k-3} - Q_{k-4}) + \dots + (Q_0 - Q_{-1})) = Q_{k-1} - Q_{k-2}.$$

Finally, the multiplicity of 0 is

$$(r_k - 1)Q_{k-1} + Q_{k-2} = Q_k - Q_{k-1}.$$

□

Remark 2.10. Note that if $r_1 \geq 2$ then the tree $T_k(r_1, \dots, r_k)$ has $2k + 1$ different eigenvalues and diameter $2k$. Since the number of different eigenvalues is at least the diameter plus one for any graph [7] these trees have the largest possible diameter among graphs having restricted number of different eigenvalues.

Theorem 1.1 For every set S of positive integers there exists a tree whose positive eigenvalues are exactly the elements of S. If the set S is different from the set $\{1\}$ then the constructed tree will have diameter $2|S|$.

Proof. Let $S = \{n_1, n_2, \dots, n_{|S|}\}$ where $n_1 < n_2 < \cdots < n_{|S|}$. Then apply the previous theorem with

$$r_{|S|} = n_1^2, r_{|S|-1} = n_2^2 - n_1^2, \dots, r_1 = n_{|S|}^2 - n_{|S|-1}^2.$$

If the set is different from $\{1\}$ then $r_1 \geq 2$ and in this case the diameter of the tree is $2|S|$ by Lemma 2.5. □

Example : Let $S = \{1, 2, 4, 5\}$ then $r_4 = 1, r_3 = 3, r_2 = 12, r_1 = 9$. The resulting tree has 781 vertices and the spectrum is

$$\{-5, -4^8, -2^{100}, -1^{227}, 0^{109}, 1^{227}, 2^{100}, 4^8, 5\}.$$

Here the exponents are the multiplicities of the eigenvalues.

Example : Let $S = \{1, 2, 3, 4, 5, 6\}$ then $r_6 = 1, r_5 = 3, r_4 = 5, r_3 = 7, r_2 = 9, r_1 = 11$. The resulting tree has 27007 vertices and the spectrum is

$$\{\pm 6, \pm 5^{10}, \pm 4^{89}, \pm 3^{611}, \pm 2^{2944}, \pm 1^{8021}, 0^{3655}\}$$

The diameter of this tree is 12.

Remark 2.11. Recently Andries E. Brouwer (private communication) found a very elegant (and very short!) proof that $T(n_k^2 - n_{k-1}^2, n_{k-1}^2 - n_{k-2}^2, \dots, n_2^2 - n_1^2, n_1^2)$ are integral trees. It is really worth reading this proof. This proof is outlined on Brouwer's homepage [1] or a bit more detailed version of this proof can be found at [6].

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