Integral trees of arbitrarily large diameters

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Introduction

An integral tree is a tree for which the eigenvalues of its adjacency matrix are all integers. Many different classes of integral trees have been constructed in the past decades. Most of these classes contain infinitely many integral trees, but till now only examples of trees of bounded diameters were known. The largest diameter of known integral trees was 10. The following question was a longstanding open problem.

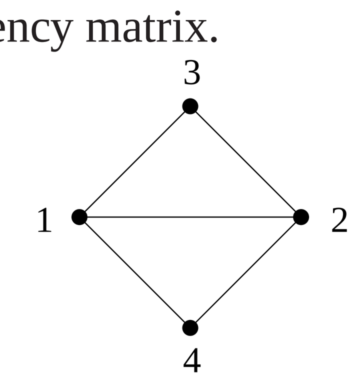
Question (1979): Are there integral trees of arbitrarily large diameters?

In this poster we answer this question affirmatively. In fact, we prove the following much stronger theorem.

Theorem: For every finite set S of positive integers there exists a tree whose positive eigenvalues are exactly the elements of S. If the set S is different from the set $\{1\}$ then the constructed tree will have diameter $2|S|$.

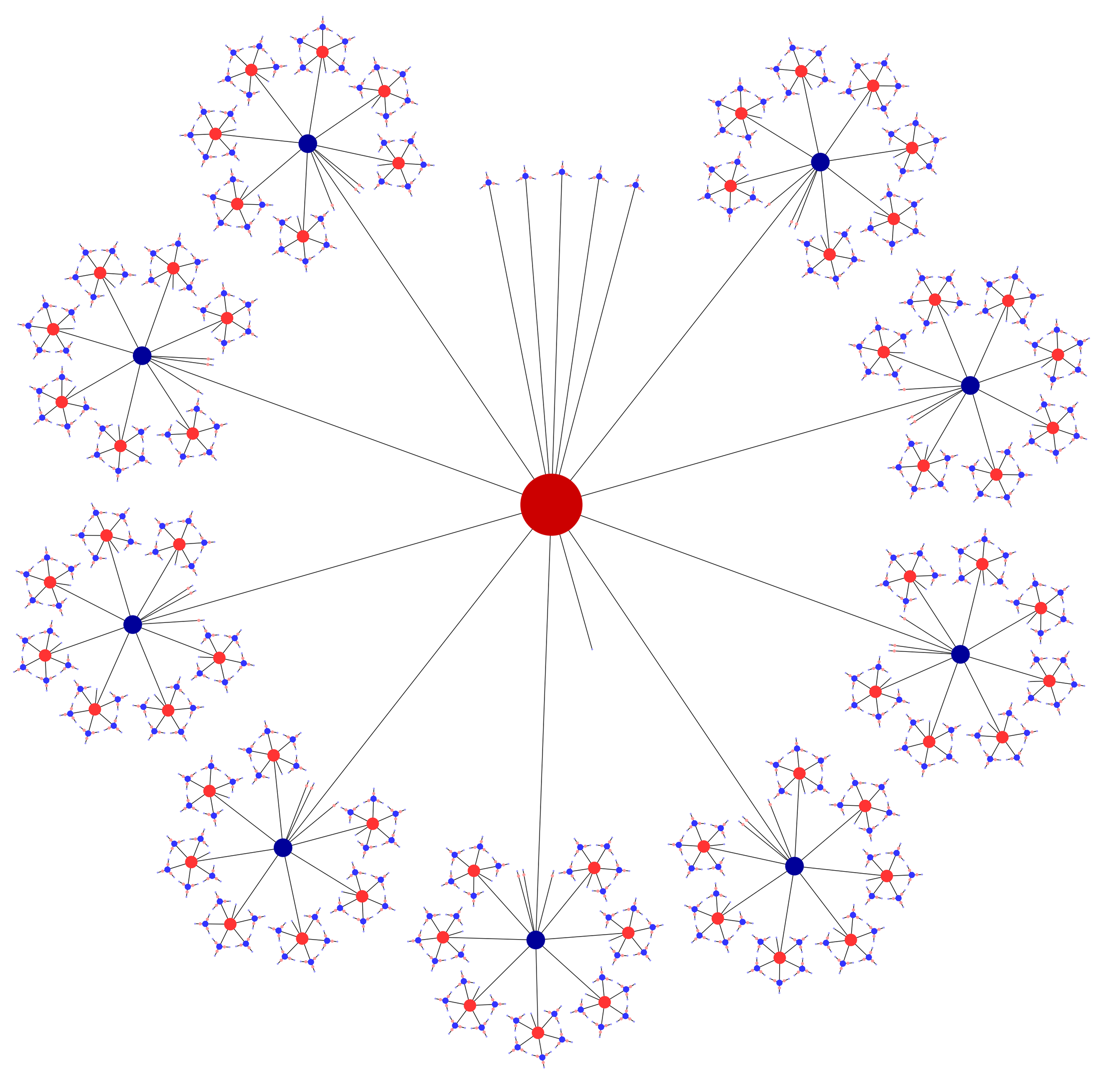
Adjacency matrix and graph spectra

The adjacency matrix is a $0-1$ matrix $A = (a_{ij})$ having as many rows and columns as the number of vertices of the graph. The element a_{ij} is 1 if the vertices i and j are adjacent, 0 otherwise. The spectrum of a graph is the set of eigenvalues of the adjacency matrix.



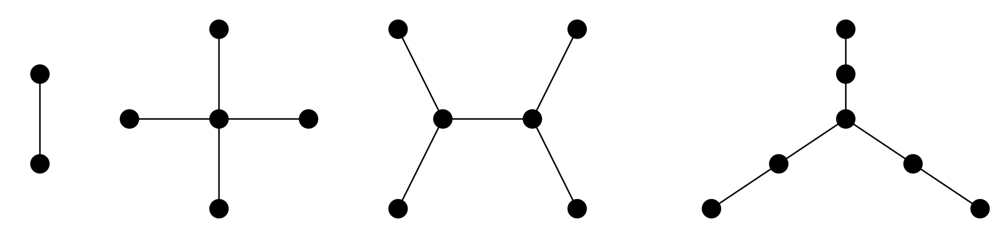
$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

The "diamond" graph with its adjacency matrix.



Small integral trees

One can find small integral trees easily.



These trees have spectrum $\{\pm 1\}, \{\pm 2, 0^3\}, \{\pm 2, \pm 1, 0^2\}$ and $\{\pm 2, \pm 1^2, 0\}$, respectively. (The exponents are the multiplicities of the eigenvalues.) There are 28 integral trees on at most 50 vertices.

...and a bigger one

The tree in the large figure has 2431 vertices. Its spectrum is $\{\pm 5, \pm 4^8, \pm 3^{55}, \pm 2^{265}, \pm 1^{722}, 0^{329}\}$ and its diameter is 10. In fact, this tree is the tree $T(9,7,5,3,1)$ (see below the definition of this tree).

The general strategy: construction of almost-integral trees

Our general strategy to constuct integral trees is that we construct *almost-integral* trees, trees with spectrum consisting of square roots of integers; then we select the integral trees from this class of trees by special choice of parameters.

The construction of almost integral trees

For given positive integers $r_1, \dots, r_k$ we construct the sequence of trees $T_1(r_1), T_2(r_1, r_2), \dots, T_k = T_k(r_1, \dots, r_k)$ recursively as follows. We will consider the tree T_i as a bipartite graph with color classes (A_{i-1}, A_i). The tree $T_1(r_1) = (A_0, A_1)$ consists of the classes of size $|A_0| = 1, |A_1| = r_1$ (so it is a star on r_1+1 vertices). If the tree $T_i(r_1, \dots, r_i) = (A_{i-1}, A_i)$ is defined then let $T_{i+1}(r_1, \dots, r_{i+1}) = (A_i, A_{i+1})$ be defined as follows. We connect each vertex of A_i with r_{i+1} new vertices of degree 1. Then for the resulting tree the color class A_{i+1} will have size $|A_{i+1}| = r_{i+1}|A_i| + |A_{i-1}|$, the color class A_i does not change.

Which almost-integral trees are integral?

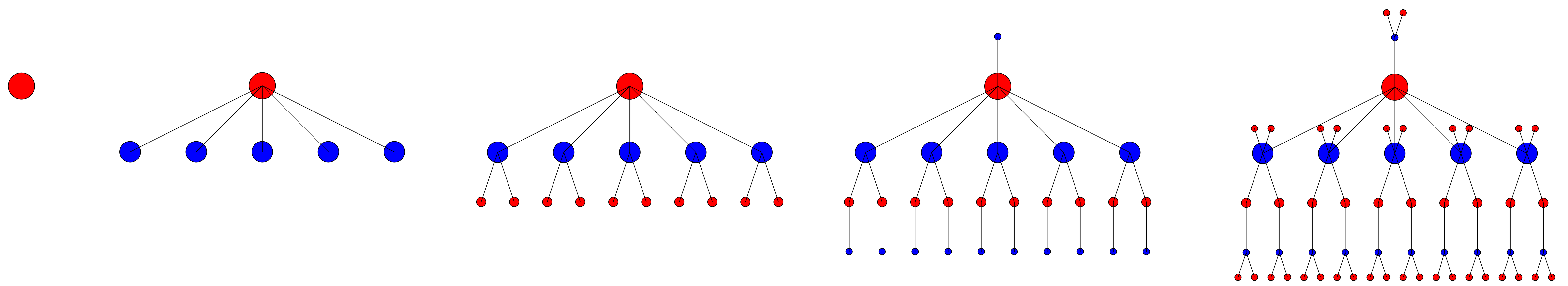
The set of different eigenvalues of the tree $T_k(r_1, r_2, \dots, r_k)$ $(r_1 \geq 2)$ is the set

$$\{\pm\sqrt{r_k}, \pm\sqrt{r_k + r_{k-1}}, \pm\sqrt{r_k + r_{k-1} + r_{k-2}}, \dots, \pm\sqrt{r_k + \dots + r_1}, 0\}.$$

Hence the tree

$$T(n_{|S|}^2 - n_{|S|-1}^2, n_{|S|-1}^2 - n_{|S|-2}^2, \dots, n_2^2 - n_1^2, n_1^2)$$

has different eigenvalues $S = \{\pm n_1, \pm, n_2, \dots, \pm n_{|S|}, 0\}$ if $n_1 < n_2 < \dots < n_{|S|}$ and $n_{|S|} > 1$. In this case the diameter of the tree is $2|S|$.



The trees in the figure

In this figure you can see $T_0 = T(), T_1 = T(5), T_2 = T(5,2), T_3 = T(5,2,1)$ and $T_4 = T(5,2,1,2)$. The spectrum of these trees are the following

$$Spec(T_0) = \{0\},\ \ Spec(T_1) = \{\sqrt{5}, 0^4, -\sqrt{5}\},\ \ Spec(T_2) = \{\pm\sqrt{7}, \pm\sqrt{2}^4, 0\},\ \ Spec(T_3) = \{\pm\sqrt{8}, \pm\sqrt{3}^4, \pm 1^6, 0^5\} \text{ and } Spec(T_4) = \{\pm\sqrt{10}, \pm\sqrt{5}^4, \pm\sqrt{3}^6, \pm\sqrt{2}^5, 0^{27}\}.$$

The exponents are the multiplicities of the eigenvalues.

Bibliography

Péter Csikvári: Integral trees of arbitrarily large diameters, Journal of Algebraic Combinatorics, to appear

Did you know?

The Integral Trees is a 1984 science fiction novel by Larry Niven (first published as a serial in *Analog* in 1983). It won the 1985 Locus Award for science fiction novel. This work has no relationship with this novel.