SHORT NOTE ON THE INTEGRALITY OF SOME TREES

PÉTER CSIKVÁRI

ABSTRACT. In this note we prove that the trees $T(n_k^2 - n_{k-1}^2, n_{k-1}^2 - n_{k-2}^2, \dots, n_2^2 - n_1^2, n_1^2)$ described in "Péter Csikvári: Integral trees of arbitrarily large diameters" are integral. The short proof given here is due to Andries E. Brouwer.

I also propose some open questions.

1. Introduction

In [2] we constructed integral trees of arbitrarily large diameters. In fact, we defined a large class of trees and we determined the spectrum of each tree belonging to this class. Then we selected the trees with spectrum consisting of only integers. In this paper we describe Andries Brouwer's ingenious proof that these selected trees are indeed integral without determining the spectrum.

The structure of this paper is the following. In the next section we define the trees $T(r_1, \dots, r_k)$ and we give those fundamental facts which we need later. In Section 3. we prove the integrality of the trees $T(n_k^2 - n_{k-1}^2, n_{k-1}^2 - n_{k-2}^2, \dots, n_2^2 - n_1^2, n_1^2)$.

2. The trees $T(r_1, r_2, \dots, r_k)$

Definition 2.1. We will define the trees $T_k = T_k(r_1, \dots, r_k)$ recursively as follows. We will consider the tree T_k as a bipartite graph with colorclasses A_{k-1}, A_k. The tree $T_1(r_1) = (A_0, A_1)$ consists of the classes of size $|A_0| = 1, |A_1| = r_1$ (so it is a star on $r_1 + 1$ vertices). If the tree $T_i(r_1, \dots, r_i) = (A_{i-1}, A_i)$ is defined then let $T_{i+1}(r_1, \dots, r_{i+1}) = (A_i, A_{i+1})$ be defined as follows. We connect each vertex of A_i with r_{i+1} new vertices of degree 1. Then for the resulting tree the colorclass A_{i+1} will have size $|A_{i+1}| = r_{i+1}|A_i| + |A_{i-1}|$, the colorclass A_i does not change. Note that the only element of $v_0 \in A_0$ has a special role; when we consider the tree $T_k(r_1, \dots, r_k)$ as a rooted tree then we select v_0 to be the root of the tree.

We need one more definition.

Definition 2.2. Let (T_1, x) and (T_2, y) be rooted trees with roots x and y, respectively. Then the tree $T_1 \sim T_2$ obtained from $T_1 \cup T_2$ by joining the vertices x and y. We obtain the tree $T_1 \sim mT_2$ by taking T_1 and m copies of T_2 and we join x to each copies of y.

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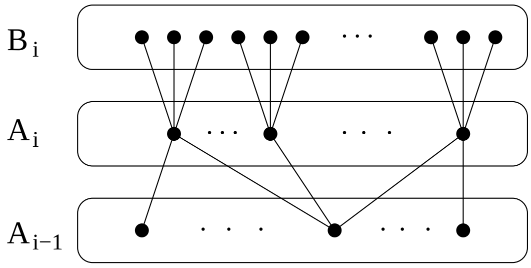

FIGURE 1. Let $A_{i+1} = A_{i-1} \cup B_i$ where each element of A_i has exacly r_{i+1} neighbors of degree 1 in B_i.

Lemma 2.3. *We have the following relations for the constructed trees*

$$T_k(r_1, \dots, r_k) = T_{k-2}(r_3, r_4, \dots r_k) \sim r_1 T_{k-1}(r_2, r_3, \dots, r_k)$$

and

$$T_{k-1}(r_1 + r_2, r_3, \dots, r_k) = T_{k-1}(r_2, r_3, \dots, r_k) \sim r_1 T_{k-2}(r_3, r_4, \dots, r_k).$$

Proof. Both statement are clear from the definition. □

3. Proof of the integrality

Definition 3.1. Let $\varphi_G(z)$ be the characteristic polynomial of the adjacency matrix of the graph G.

Let $M_G(z)$ be the matching polynomial of the graph G, i.e. ,

$$M_G(z) = z^n - m_1(G)z^{n-2} + m_2(G)z^{n-4} + \dots + (-1)^{[n/2]} m_{[n/2]} z^{n-2[n/2]}$$

where $m_r(G)$ is the number of r-matchings.

We use the following well-known and easy statement about the characteristic polynomials of trees.

Lemma 3.2. *For any tree T we have*

$$\varphi_T(z) = M_T(z).$$

Lemma 3.3. *Let $T = T_1 \sim T_2$ where the roots of T_1 and T_2 are x and y, respectively. Furthermore let $T_m = T_1 \sim mT_2$. Then*

$$\varphi_T(z) = \varphi_{T_1}(z)\varphi_{T_2}(z) - \varphi_{T_1-x}(z)\varphi_{T_2-y}(z).$$

Furthermore,

$$\varphi_{T_m}(z) = \varphi_{T_2}^m(z)(\varphi_{T_1}(z)\varphi_{T_2}(z) - m\varphi_{T_1-x}(z)\varphi_{T_2-y}(z)).$$

Proof. Let G be a graph and $e = (x, y) \in E(G)$ then for the matching polynomial we have the identities

$$M(G, z) = M(G - e, z) - M(G - \{x, y\}, z)$$

and

$$M(G_1 \cup G_2, z) = M(G_1, z)M(G_2, z).$$

Thus the first statement of Lemma 3.3 follows from Lemma 3.2 and from these identities.

The second statement is an easy induction on m. □

Corollary 3.4. *Let $T = T_1 \sim T_2$ where the roots of T_1 and T_2 are x and y, respectively. If the trees $T_1, T_2, T_1 \sim mT_2$ are all integral then so the tree $T_2 \sim mT_1$.*

Proof. Using Lemma 3.3 observe the symmetry between the characteristic polynomials of $T_1 \sim mT_2$ and $T_2 \sim mT_1$. □

Now we are ready to prove that the integrality of the introduced trees.

Theorem 3.5. *All eigenvalus of the trees*

$$T_k(n_k^2 - n_{k-1}^2, n_{k-1}^2 - n_{k-2}^2, \dots, n_2^2 - n_1^2, n_1^2)$$

are integers.

Proof. We prove by induction on k. Clearly, the trees $T_1(n^2)$ are integral with spectrum $\{\pm n, 0^{n^2-1}\}$. Assume that we already know the statement for $n = k-1$. We need to prove it for $n = k$. By induction we have that the trees

$$T_1 = T_{k-1}(n_{k-1}^2 - n_{k-2}^2, n_{k-2}^2 - n_{k-3}^2, \dots, n_2^2 - n_1^2, n_1^2)$$

and

$$T_2 = T_{k-2}(n_{k-2}^2 - n_{k-3}^2, n_{k-3}^2 - n_{k-4}^2, \dots, n_2^2 - n_1^2, n_1^2)$$

are integral. We also know that

$$T_1 \sim (n_k^2 - n_{k-1}^2)T_2 = T_{k-1}((n_k^2 - n_{k-1}^2) + (n_{k-1}^2 - n_{k-2}^2), n_{k-2}^2 - n_{k-3}^2, \dots, n_2^2 - n_1^2, n_1^2)$$

is integral by the second part of Lemma 2.3 and by the induction. Hence by Corollary 3.4 and the first part of Lemma 2.3 we have that

$$T_2 \sim (n_k^2 - n_{k-1}^2)T_1 = T_k(n_k^2 - n_{k-1}^2, n_{k-1}^2 - n_{k-2}^2, n_{k-2}^2 - n_{k-3}^2, \dots, n_2^2 - n_1^2, n_1^2)$$

is also integral. □

Remark 3.6. One can prove Theorem 2.9 of [2] in the very same way.

4. Open problems

Question 1.: Are there integral trees for arbitrary odd diamater?

Note that for a fixed value, say 9, one may wish to distinguish two questions:

Question 1.1. Is there an integral tree of diameter 9?

Question 1.2. Is there an infinite class of integral trees of diameter 9?

The constructed trees also provide some questions.

Question 2.1: Are the constructed trees are determined by their spectrum among trees? (I think the answer is yes.)

Question 2.2: Are the constructed trees are determined by their spectrum among graphs? (I think the answer is no for the most of the trees. For instance, $T(3, 1)$ has the same spectrum as the 6-cycle plus an isolated vertex.)

The constructed trees are so large that makes me conjecture the following.

Conjecture 3.1 For a given set S of positive integers, the constructed tree is the largest among the trees whose different positive eigenvalues are the elements of S.

In fact, one can conjecture the same for almost-integral trees (trees having eigenvalues only square roots of integers).

Conjecture 3.2 For a given set S of positive integers, the constructed tree is the largest among the the trees whose different positive eigenvalues are square roots of the elements of S.

Two weak heuristics supply these conjectures: from the set of eigenvalues one can bound the diameter and the largest degree (and so the number of vertices). For the constucted trees the diameters are as large as possible, while the largest degrees are not far from the best possible.

Since we require for the constructed trees that $r_1 \geq 2$ there is a little gap between the two largest eigenvalues. This motivates the following question.

Question 4. Is there a tree whose eigenvalues are square roots of integers and the two largest eigenvalues are $\sqrt{n-1}$ and $\sqrt{n}$ for some integer n?

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References

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Eötvös Loránd University, Department of Computer Science, H-1117 Budapest, Pázmány Péter sétány 1/C, Hungary

E-mail address: csiki@cs.elte.hu