Kutatási és szakdolgozati témák

Alább található az általam meghírdetett témák leírása. A leírások angolul vannak. A szakdolgozatot nem kell angolul írni, sőt eltekintve néhány esettől ezt nem is javaslom. Az ok amiért mégis angolul vannak a leírások az az, hogy aki ezeket nem érti annak semmiképpen nem érdemes hozzám jönni szakdolgozni vagy kutatni, mert a szakirodalom is angolul lesz.

Mielőtt úgy döntenél, hogy te mindenképp egyik vagy másik témából szeretnél szakdolgozni vagy kutatni, kérlek keress rá a témára az interneten, hogy ne legyen csalódás belőle. Megadtam néhány tájékoztató jellegű adatot, hogy mennyire tartom nehéznek az egyes témákat. Semmiképp ne válassz olyan témát szakdolgozatnak aminél nagyobb szám van, mint a legrosszabb jegyed az utolsó félévben, sőt inkább legyen a legrosszabb jegyed 1-gyel nagyobb már ha van erre egyáltalán lehetőség.

Egy évben legfeljebb két diák szakdolgozati témavezetését vállalom. Nincs limitem kutató diákra viszont csak nagyon kevés embernek javaslom. Ha egy témából valaki szakdolgozatot ír, akkor azt a témát néhány évig nem lehet választani, ez meg lesz jelölve a témánál. Az irodalomnál néhány link csak egyetemi számítógépről érhető el.

A **következő oldalon található táblázatban** van feltüntetve, hogy kinek mit ajánlok. A Research a szokásos TDK, az ÚNKP az elsőéveseknek szól akiktől csak az irodalom feldolgozását várják el, új eredményt nem. Természetesen a középiskolásoknál sem elvárás az új eredmény, itt is csak néhány cikk vagy könyvrészlet megértése a cél.

Ha valamely témánál nem szerepel kutatási lehetőség, az nem azt jelenti, hogy nem kutatják, csak azt, hogy én nem vagyok elég kompetens a témavezetéshez. Ha valaki mégis ilyen témában szeretne kutatni, akkor megpróbálom egy nálam kompetensebb emberrel összekapcsolni.

Témák

	Title	Bsc thesis	Msc Thesis	Research	ÚNKP	High school
1	Walks on graphs	✓			✓	
2	Strongly regular graphs	✓			✓	✓
3	Laplacian eigenvalues	✓			✓	
4	Pseudorandom graphs	✓	✓	✓	✓	
5	Chromatic polynomial	✓	✓	✓	✓	✓
6	Matching polynomial	✓	✓	✓	✓	✓
7	Tutte polynomial	✓	✓		✓	✓
8	Eulerian orientations	✓	✓		✓	✓
9	Moment sequences	✓	✓	✓	✓	
10	Extremal regular graphs	✓	✓	✓	✓	
11	Hyperplane arrangements	✓	✓		✓	✓
12	Ehrhart polynomial	✓	✓		✓	✓
13	Symmetric structures	✓	✓			
14	Ramanujan graphs	✓	✓	✓		
15	Nonabelian Fourier-analysis		✓			
16	Sphere packings		✓			

1. Spectral graph theory: walks on graphs

Difficulty level: Bsc 2/5

Required knowledge: linear algebra

Number of walks on graphs can be efficiently computed via the adjacency matrix of a graph. In return walks give various properties of the eigenvalues of the adjacency matrix.

The goal of the thesis would be to collect some of these results.

Readings

C. Godsil: Algebraic combinatorics, Chapter 2.

2. Spectral graph theory: strongly regular graphs

Difficulty level: Bsc 3/5

Required knowledge: linear algebra

Strongly regular graphs are defined with some simple properties about the number of common neighbors of various vertices and are described by 4 parameters: (n, d, a, b), where n is the number of vertices, the graph is d-regular, and if two vertices are adjacent then they have a common neighbors and if they are not adjacent then they have b common neighbors.

This simple definition gives rise to a particularly interesting graph class. The small strongly regular graphs are very symmetric. There are some results which (n, d, a, b) admits the existence of a strongly regular graph, mostly these results are deduced by linear algebraic tools.
The goal of the thesis would be to collect some of these results together with various nice constructions of strongly regular graphs.

Not available till 2022.

Readings

C. Godsil: Algebraic combinatorics, chapters 2,5,10,11 and 12.

3. Spectral graph theory: Laplacian eigenvalues and spanning trees

Difficulty level: Bsc 3/5

Required knowledge: linear algebra

The Laplacian-matrix L of a graph G on vertices is defined as follows: $L_{ii} = d_i$ for vertex i, and $L_{ij} = -a_{ij}$, where a_{ij} is the number of edges between v_i and v_j. The Laplacian-matrix is a positive definite matrix, it is best known about its connection with the number of spanning trees, but also plays important roles in electric networks and even in graph drawing. The goal of the thesis would be to collect some of these results.

4. Spectral graph theory: pseudorandom graphs

Difficulty level: Bsc 3/5, Msc 2/5

Required knowledge: linear algebra

A graph G is (n, d, λ)-pseudorandom if it has n vertices, d–regular and $\max(\lambda_2, -\lambda_n) \leq \lambda$, where the eigenvalues of the adjacency matrix are $d = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

Many statements about random graphs have a natural analogue about pseudorandom graphs with sufficiently small λ. The goal of the thesis would be to collect some of these results together with some applications in computer science.

Readings

S. Hoory, N. Linial, A. Wigderson: Expander graphs and their applications

5. Graph polynomials: chromatic polynomial

Difficulty level: Bsc 2-4/5, Msc 3/5 or 5/5

The chromatic polynomial $\mathrm{ch}(G, q)$ counts the number of proper colorings with q colors. It turns out that it is indeed a polynomial.

The goal of the thesis would be to collect the basic properties of the chromatic polynomial. The difficulty varies what kind of theorems one includes, for instance, the proof of Sokal's theorem on the absolute value of the largest zero is non-trivial. If one plans to include June Huh's result on the log-concavity of the coefficients of the polynomial then the difficulty level is 5 even for an Msc student (and I don't suggest it for a Bsc student).

Readings

R. C. Read: An introduction to chromatic polynomials, Journal of Combinatorial Theory **4**, 52-71 (1968)

A. D. Sokal: Bounds on the complex zeros of (di) chromatic polynomials and Potts-model partition functions, Combinatorics, Probability and Computing (2001) 10, 41–77.

P. Csikvári: Proof of Sokal's theorem, preprint

J. Huh: Milnor numbers of projective hypersurfaces and the chromatic polynomial of graphs, Journal of the American Mathematical Society **25** (2012), Pages 907–927

6. Graph polynomials: matching and independence polynomial

Difficulty level: Bsc 3/5, Msc 2/5.

The matching polynomial $\mu(G,x)$ of a graph G on n vertices is defined as follows:

$$\mu(G,x) = \sum_{k=0}^{n} (-1)^k m_k(G) x^{n-2k},$$

where $m_k(G)$ denotes the number of matchings of size k. It turns out that this polynomial has only real zeros. There is a similarly defined graph polynomial called the independence polynomial:

$$I(G,x) = \sum_{k=0}^{n} (-1)^k i_k(G) x^k,$$

where $i_k(G)$ denotes the number of independent sets of size k. The two polynomials are related to each other in many ways.

The goal of the thesis would be to collect the basic properties of the matching and independence polynomials.

Readings

C. Godsil and I. Gutman: On the theory of the matching polynomial, J. Graph Theory, **5** (1981)

C. Godsil: Algebraic combinatorics, chapters 1 and 6

O. Heilmann and E. Lieb: Theory of monomer-dimer systems, Commun. math. Phys. 25, 190–232 (1972)

V. E. Levit and E. Mandrescu: The independence polynomial of a graph– a survey

F. Bencs: Christoffel-Darboux type identities for independence polynomial

7. Graph polynomials: Tutte polynomial

Difficulty level: Bsc 3/5, Msc 2/5.

Tutte polynomial is a 2-variable polynomial associated to graphs. Among many others it encodes the chromatic polynomial, flow polynomial, number of forests and spanning trees.

The goal of the thesis would be to collect the basic properties of the Tutte polynomial.

Readings

D. Welsh: The Tutte polynomial

8. Counting Eulerian orientations

Difficulty level: Bsc 3/5, Msc 2/5.

It is well-known that if all degrees of a graph G are even, then it has an Eulerian orientation: the in-degree and out-degree are the same at each vertex. A. Schrijver gave a lower bound for the number of Eulerian orientations in terms of the degree sequence. L. Vergnas gave some refinement of Schrijver's result. M. Mihail and P. Winkler gave an efficient randomized algorithm for approximating the of Eulerian orientations in an arbitrary Eulerian graph. E. Lieb gave an asymptotical formula for the Eulerian orientations of large grid graphs.

The goal of the thesis would be to collect these results together with other results.

Readings

A. Schrijver: Bounds on the number of Eulerian orientations

L. M. Vergnas: An upper bound for the number of Eulerian orientations of a regular graph

E. Lieb: Residual entropy of square ice

M. Mihail and P. Winkler: On the number of Eularian orientations of a graph

9. Combinatorial sequences as moment sequences of probability distributions

Difficulty level: Bsc 3/5, Msc 3/5

Required courses and skills: Probability theory and some programming skill

It turns out that many notable combinatorial sequences, like the Catalan-numbers or the central binomial coefficients, are moment sequences of certain probability distributions. In other words, for the sequence $(c_n)_{n=1}^{\infty}$ there is a probability measure ρ such that

$$c_n = \int_{\mathbb{R}} x^n \, d\rho(x).$$

There are some conditions which describe when a sequence can be a moment sequence of a probability measure and how we can get back the density function (if exists) of the probability distribution from the moment sequence under certain conditions.

The goal of the thesis would be to collect these theorems and search for such sequences in the Encyclopedia of Integer Sequences and to determine the corresponding density function in case of some interesting sequences. Some programming skill can be helpful for automatic search in the database.

Readings

T. S. Chihara: An introduction to orthogonal polynomials, chapters 1 and 2.

10. Extremal regular graphs

Difficulty level: Bsc 3-5/5 and Msc 2-4/5

Let G be a graph, and let $I(G)$ denote the number of independent sets. Alon conjectured and Kahn and Zhao proved that for any d–regular graph G we have

$$I(G)^{1/v(G)} \leq I(K_{d,d})^{1/v(K_{d,d})},$$

where $v(G)$ is the number of vertices of G, and $K_{d,d}$ is the complete bipartite graph on $d+d$ vertices.

There are many similar results in the literature. It turns out that in many cases $K_{d,d}$, the complete graph K_{d+1}, or somewhat surprisingly the infinite d–regular tree is the extremal graph among d–regular graphs. An example for the last case is a theorem of Schrijver: let $\mathrm{pm}(G)$ be the number of perfect matchings then if G is a d–regular bipartite graph on $2n$ vertices then

$$\mathrm{pm}(G) \geq \left(\frac{(d-1)^{d-1}}{d^{d-2}}\right)^n.$$

The mysterious constant comes from the infinite d–regular tree. Similar theorem holds true for the number of spanning trees of d–regular graphs.

The difficulty level of this topic heavily depends on the results included in the thesis.

Readings

Y. Zhao: Extremal regular graphs: independent sets and graph homomorphisms, Journal The American Mathematical Monthly **124**, 2017

D. Galvin and P. Tetali: On weighted graph homomorphisms

E. Davies, M. Jenssen, W. Perkins and B. Roberts: Independent sets, matchings, and occupancy fractions

11. Hyperplane arrangements

Difficulty level: Bsc 3/5 and Msc 3/5

A hyperplane arrangement is simply a set of hyperplanes in some $\mathbb{R}^d$. There are many natural questions: how many regions do the hyperplanes cut the space into? How many of these regions are bounded? It turns out that many of these questions can be easily answered using the characteristic polynomial associated to the hyperplane arrangement. This is an algebraically defined invariant.

The goal of the thesis would be to collect various results on hyperplane arrangements.

Readings

R. Stanley: An introduction to hyperplane arrangements

12. Ehrhart polynomial

Difficulty level: Bsc 3/5 and Msc 3/5

Let P be a polytope in d dimension with vertices having only integer coordinates. For an integer k let $L(P,k) = |kP \cap \mathbb{Z}^d|$, that is, the number of lattice points in the k-the dilate of the polytope. It turns out that $L(P,x)$ is a polynomial of x called the Ehrhart polynomial.

Another related polynomial can be defined as follows. Consider the generating function

$$\mathrm{Ehr}_P(z) = \sum_{k=0}^{\infty} L(P,k) z^k.$$

It turns out that this can be written as

$$\mathrm{Ehr}_P(z) = \frac{\sum_{j=0}^{d} h_j^*(P) z^j}{(1-z)^{d+1}},$$

where $h_j^*(P)$ are non-negative.

Readings

Matthias Beck and Sinai Robins: Computing the continuous discretely

13. Symmetric structures in 24 dimensions

Difficulty level: Bsc 4/5, Msc 3/5

Suggested course: Symmetric structures

There are various symmetric objects of different fields of mathematics which turn out to be intimately related to each other: Golay-codes (coding theory), Witt-designs (designs), Mathieu-groups (group theory) and Leech-lattice (geometry). Practically, these objects are the manifestations of the very same symmetricity phenomenon. Even the derived symmetric structures are very exotic like the Higman-Sims strongly regular graph or the system of 276 equiangular lines in 23 dimensions.

The goal of the thesis would be to describe the various objects and their relationships to each other.

14. Ramanujan graphs

Difficulty level: Bsc 5/5, Msc 4/5

Required knowledge: some spectral graph theory and some number theory

Ramanujan-graphs are d–regular graphs for which $\max(\lambda_2, -\lambda_n) \leq 2\sqrt{d-1}$. This means that Ramanujan-graphs are the best possible expanders since for any constant $c < 2\sqrt{d-1}$ there are only finitely many graphs for which $\max(\lambda_2, -\lambda_n) \leq c$ according to the Alon-Boppana theorem.

We know that there are infinitely many Ramanujan-graphs for $d = p^\alpha + 1$, and for any d there are infinitely many bipartite Ramanujan-graphs (note that for bipartite graphs one has to modify the definition only requiring $\lambda_2 \leq 2\sqrt{d-1}$).

The goal of the thesis would be to collect some fundamental theorems about Ramanujan-graphs (Alon-Boppana theorem, expander mixing lemma) and some constructions (Phillips-Sarnak-Lubotzky and Marcus-Spielman-Srivastava).

Readings

A. Nilli: On the second eigenvalue of a graph
(Alon-Boppana first proof)

A. Nilli: Tight estimates for eigenvalues of regular graphs
(Alon-Boppana second proof)

S. Cioaba: On the extreme eigenvalues of regular graphs, Journal of Combinatorial Theory, Series B 96 (2006) 367 – 373
(Alon-Boppana third proof)

M. Ram Murty, Ramanujan graphs, J. Ramanujan Math. Soc. 18, No.1 (2003) 1–20

A. Lubotzky, R. Phillips and P. Sarnak: Ramanujan graphs, Combinatorica, **8** (3) (1988), 261–277

A. Marcus, D. A. Spielmann, N. Srivastava: Interlacing families I: Bipartite Ramanujan graphs of all degrees

15. Nonabelian Fourier-analysis

Difficulty level: Msc 5/5

Required courses: Representation theory of finite groups and Exponential sums in number theory

Let G be a finite group. A set A is called product-free if $ab \neq c$ for any $a, b, c \in A$. It is easy to show that in any abelian group the size of the largest product-free set is at least $\frac{2}{7}|G|$. On the other hand, Gowers showed that in $G = SL_2(q)$, the size of the largest product-free set is at most $O(|G|^{8/9})$. He proved it via the machinery of nonabelian Fourier-analysis.

The goal of the thesis would be to explore various applications of the nonabelian Fourier-analysis.

16. Sphere packings in 8 and 24 dimensions

Difficulty level: Msc 5/5

Required courses: Complex function theory and Special functions

The sphere packing problem in d dimensions seeks for the densest possible arrangement of unit spheres in $\mathbb{R}^d$ in such a way that the interiors of the spheres are disjoint. This problem has been solved in the following cases: $d = 1$ (trivial), $d = 2$ (not very very hard, but somewhat tricky), $d = 3$ (the proof is humanly non-checkable, but it is good), and very recently for $d = 8$ and $d = 24$. The best arrangements in the last two cases are to place the centers of the spheres to the points of the E8 lattice and the Leech-lattice, respectively.

The goal of thesis would be to understand the proofs of the last two cases. There is an extra research problem here: the corresponding auxiliary function used in the proofs of $d = 8$ and $d = 24$ is not known for the case $d = 2$ in spite of the fact that it should exist.

Readings

M. Viazovska: The sphere packing problem in dimension 8, Annals of Mathematics, 991–1015

H. Cohn, A. Kumar, S. D. Miller, D. Radchenko, M. Viazovska: The sphere packing problem in dimension 24, Annals of Mathematics, 1017–1033