

On weakly intersecting pairs of sets

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Abstract

Let \mathcal{F} be a family of pairs of sets. We call \mathcal{F} a (k, l) -set system if for $(A, B) \in \mathcal{F}$ we have that $|A| = k$, $|B| = l$, and $A \cap B = \emptyset$. Furthermore, \mathcal{F} is weakly intersecting if for any $(A_i, B_i), (A_j, B_j) \in \mathcal{F}$ with $i \neq j$ we have that $A_i \cap B_j$ and $A_j \cap B_i$ are not both empty. We investigate the maximum possible size of weakly intersecting (k, l) -set systems. We improve on the lower bound of Tuza [10] and compute the exact value for $k = l = 2$.

1 Introduction

Let \mathcal{F} be a family of pairs of sets. We call \mathcal{F} a (k, l) -set system if for $(A, B) \in \mathcal{F}$ we have that $|A| = k$, $|B| = l$, $A \cap B = \emptyset$. The following classical result on intersecting pairs of sets is due to Bollobás [5]. Let \mathcal{F} be a (k, l) -set system with the intersecting property, i.e., for $(A_i, B_i), (A_j, B_j) \in \mathcal{F}$ with $i \neq j$ we have that $A_i \cap B_j$ and $A_j \cap B_i$ are both non-empty. The maximum possible size of such a set system is $\binom{k+l}{k}$ and this is sharp. Interestingly, the same upper bound holds even if for $(A_i, B_i), (A_j, B_j) \in \mathcal{F}$ we only require that $A_i \cap B_j \neq \emptyset$ for $i < j$, as was shown in [6]. Several further generalizations were investigated in [1, 7, 8, 9]. In this paper we consider the following variant of the problem, introduced by Tuza [10] (for more details on the history we refer to the surveys [11, 12].)

Definition 1.1. *Let \mathcal{F} be a (k, l) -set system. \mathcal{F} is weakly intersecting if for $(A_i, B_i), (A_j, B_j) \in \mathcal{F}$ with $i \neq j$ we have that $A_i \cap B_j$ and $A_j \cap B_i$ are not both empty.*

The goal of this paper is to investigate the maximum possible size of such a system, which we denote by $g(k, l)$. In Section 2 we give a construction based on lattice paths and prove that $\liminf_{k+l \rightarrow \infty} g(k, l) / \binom{k+l}{k} \geq 2 - o(1)$.¹ This improves on the lower bound in [10]. In Section 3 we recall the known upper bounds and in Section 4 we present some computational results for small values of k and l , in particular, we show that $g(2, 2) = 10$. We conclude by posing several interesting open problems.

¹Whenever we use $o(1)$, we mean that the number tends to 0 as $k + l$ tends to infinity.

2 Lower bounds

We start from the following two claims made by Tuza [10].

Claim 2.1. $g(k, 1) \geq 2k + 1$.

Proof. Let $B_i = \{i\}$ and $A_i = \{i + j \pmod{2k + 1} \mid 1 \leq j \leq k\}$ for $i = 0, \dots, 2k$. □

Claim 2.2. $g(k, l) \geq g(k - 1, l) + g(k, l - 1)$.

Proof. Suppose we have a construction $\{(A_i, B_i)\}$ of cardinality $g(k - 1, l)$ and another construction $\{(A'_j, B'_j)\}$ of cardinality $g(k, l - 1)$. Let x be an element not contained in any of these sets. Then, $\{(A_i \cup \{x\}, B_i)\} \cup \{(A'_j, B'_j \cup \{x\})\}$ gives a construction for a (k, l) -set system of cardinality $g(k - 1, l) + g(k, l - 1)$. □

From these claims one can easily obtain the following corollary.

Corollary 2.3. $g(k, l) \geq 2 \binom{k+l}{k} - \binom{k+l-2}{k-1} = \left(2 - \frac{kl}{(k+l)(k+l-1)}\right) \binom{k+l}{k}$.

Proof. We prove by induction. Let $\tilde{g}(k, l) = 2 \binom{k+l}{k} - \binom{k+l-2}{k-1}$. From Claim 2.1 we have that $g(k, l) \geq \tilde{g}(k, l)$ for all $(k, 1)$ and $(1, l)$ which settles the base case. The inductive step follows from Claim 2.2 and the fact that $\tilde{g}(k, l) = \tilde{g}(k - 1, l) + \tilde{g}(k, l - 1)$. □

We would like to point out that from this we get that $\lim_{k \rightarrow \infty} \tilde{g}(k, k) / \binom{2k}{k} = \frac{7}{4}$ and not 2 as was claimed in [10]. In what follows, we give an explicit construction that proves that the above ratio for the lower bound is at least $2 - o(1)$ for all k and l as $k + l$ tends to infinity.

2.1 A construction using lattice paths

Now we describe the construction of a (k, l) -set system \mathcal{F} for arbitrary k and l using lattice paths. Let $m = 2k + 2l - 1$. We will construct $\mathcal{F} = \{(A, B) : |A| = k, |B| = l, A, B \subset \{0, \dots, m - 1\}\}$. Denote by $\mathcal{L}(k, l)$ the set of lattice paths on a $k \times l$ grid from $(0, 0)$ to (k, l) , where each move is to the right or up, and the path is strictly below the diagonal except at the two endpoints. We will identify a path $\pi \in \mathcal{L}(k, l)$ with its binary representation $\pi \in \{0, 1\}^{k+l}$ using 0 and 1 for steps to the right and up, resp. For each path $\pi \in \mathcal{L}(k, l)$ we create the following m set pairs $(A_{\pi, i}, B_{\pi, i})$ where $i = 0, \dots, m - 1$. Let $A_{\pi, i} = \{i + t \pmod{m} \mid 1 \leq t \leq k + l, \pi(t) = 0\}$ and $B_{\pi, i} = \{i + t \pmod{m}, \mid 1 \leq t \leq k + l, \pi(t) = 1\}$.

We have to show that the obtained \mathcal{F} indeed satisfies Definition 1.1. Clearly, it is a (k, l) -set system by construction. To show the weakly intersecting property, first observe that $A_{\pi, i} \cap B_{\sigma, i} \neq \emptyset$ for any $\pi \neq \sigma$ as the paths are different and hence $\exists t : \pi(t) = 0, \sigma(t) = 1$. If $i \neq j$ then $A_{\pi, i} \cap B_{\sigma, j} \neq \emptyset$ or $A_{\sigma, j} \cap B_{\pi, i} \neq \emptyset$ follows from the fact that any proper suffix of a lattice path in $\mathcal{L}(k, l)$ cannot equal the prefix of another (or even the same) lattice path, and hence $\exists t : \pi(t) \neq \sigma(t + i - j)$. One way to see this is to consider the ratio of the right and up steps: in the prefix this ratio has to be greater than k/l whereas in the suffix it has to be strictly less than that.

Remark 2.4. We note that for $\gcd(k, l) = 1$ this (k, l) -set system is maximal in the sense that we cannot add another pair of sets to it without violating the weakly intersecting property.

Unfortunately, there is no simple closed formula for $|\mathcal{L}(k, l)|$ for general (k, l) . However, for relative primes k and l we have the following theorem.

Theorem 2.5 (Bizley [3]). $|\mathcal{L}(k, l)| = \binom{k+l}{k}/(k+l)$ if $\gcd(k, l) = 1$.

From Theorem 2.5 and the construction described above we obtain a better lower bound than in Corollary 2.3.

Theorem 2.6. $g(k, l) \geq (2k + 2l - 1)|\mathcal{L}(k, l)| = (2k + 2l - 1)\binom{k+l}{k}/(k+l)$ if $\gcd(k, l) = 1$.

Corollary 2.7. $g(k, k-1) \geq (2 - \frac{1}{2k-1}) \cdot \binom{2k-1}{k}$.

Corollary 2.8. $g(k, k) \geq (2 - \frac{1}{2k-1}) \cdot \binom{2k}{k}$.

Proof. Claim 2.2 and Corollary 2.7 gives $g(k, k) \geq 2g(k, k-1) = (2 - \frac{1}{2k-1}) \cdot \binom{2k}{k}$. \square

Applying this trick multiple times we obtain our main theorem, a stronger lower bound for all (k, l) .

Theorem 2.9. $g(k, l) \geq (2 - o(1))\binom{k+l}{k}$.

Proof. The idea is to use Claim 2.2 repeatedly until we decrease $k+l$ to a prime p for which we automatically have $\gcd(p-q, q) = 1$ for $0 < q < p$ and then invoke Theorem 2.6.

Baker et al. proved in [4] that for $x > x_0$ the interval $[x - x^{0.525}, x]$ contains a prime. We will use this result for $\mathbf{x} := \mathbf{k} + \mathbf{1}$. If either k or l is smaller than $x^{0.525}$ then by Corollary 2.3 we have $g(k, l)/\binom{k+l}{k} \geq 2 - \frac{x \cdot x^{0.525}}{x(x-1)} \geq 2 - o(1)$. Otherwise, we will use the result that there is a prime p such that $0 \leq x - p \leq x^{0.525} \leq k, l$. By applying Claim 2.2 several times and then Theorem 2.6 we obtain

$$\begin{aligned} g(k, l) &\geq \sum_{i=0}^{x-p} \binom{x-p}{i} g(k - (x-p) + i, l - i) = \sum_{i=0}^{k+l-p} \binom{k+l-p}{i} g(p - (l-i), l-i) \geq \\ &\geq \sum_{i=0}^{k+l-p} \binom{k+l-p}{i} (2p-1) \binom{p}{l-i} / p = \frac{2p-1}{p} \binom{k+l}{l} \end{aligned}$$

and since $(2p-1)/p \geq 2 - 1/(x - x^{0.525}) = 2 - o(1)$, we have proven the theorem. \square

3 Upper bounds

For the sake of completeness we also mention the upper bounds which appeared in [10].

Claim 3.1. $g(k, 1) = 2k + 1$.

For general k and l we do not have matching upper bounds. In fact, the following theorem (also appeared as an exercise in [2]) is the best known.

Theorem 3.2 (Tuza, [10]). $g(k, l) < \frac{(k+l)^{k+l}}{k^k l^l}$.

4 Proving bounds using computer programs

Next we present a straightforward method for computing $g(k, l)$ for fixed k and l with the help of a computer program. If we know the size (or an upper bound on the size) of the ground set we can find a maximum size weakly intersecting (k, l) -set system as follows. First, generate all possible (k, l) -set pairs. Then create a graph, whose vertices represent set pairs and edges are drawn between vertices if the corresponding set pairs weakly intersect. Finally, find a maximum clique in this graph (this corresponds to a maximal set system).

Let us denote the size of the smallest ground set on which it is possible to realize a weakly intersecting (k, l) -set system with N set-pairs by $n(k, l, N)$. We are only interested in the case where $N > \binom{k+l}{l}$, consequently $n(k, l, N) > k + l$.

Claim 4.1. $\binom{n(k, l, N)}{2} \leq kl \cdot N$.

Proof. Suppose we have a construction of size N on a set of size n and $\binom{n}{2} > kl \cdot N$. Define a graph on the ground set in the following way. Connect x and y if there is an i such that A_i contains one of them and B_i the other one. By our assumption, there must be a pair of points that are not connected to each other. Contracting these points gives a smaller ground set, where still any two set-pairs intersect. It might happen that x and y are both in some A_j (or B_j), in this case just add to the shrunken set an arbitrary element that is not in $A_j \cup B_j$, as $n > k + l$ we can choose such an element. \square

Theorem 4.2. $g(2, 2) = 10$.

Proof. By Corollary 2.8 we have that $g(2, 2) \geq 10$. So, we have to prove $g(2, 2) \leq 10$. We prove by contradiction. Suppose that there is a weakly intersecting $(2, 2)$ -set system with $N = 11$ set-pairs. Then Claim 4.1 implies that this has a realization on a ground set with at most 9 elements. We performed an exhaustive search on the ground set of size 9 with our computer program, verifying that the maximum size of a $(2, 2)$ -set system is 10. \square

Unfortunately, the running time of this brute force search grows superexponentially and we could not compute further values. We found a $(3, 2)$ -set system of size 19, hence $g(3, 2) \geq 19$ (which also implies $g(3, 3) \geq 38$), thus our lower bound is not always optimal.

We would like to conclude with several interesting questions.

Open problems.

- Is $g(k, k) = 2g(k - 1, k)$ for all $k \geq 2$?
- Is $g(k, l) < 2\binom{k+l}{k}$?
- Is $g(k, k) = o(2^{2k})$?

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