Decomposability of polygon coverings

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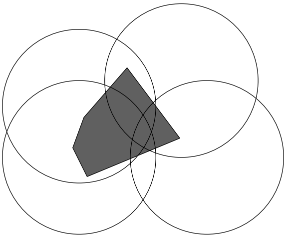

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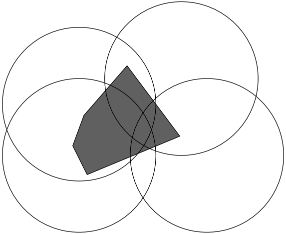

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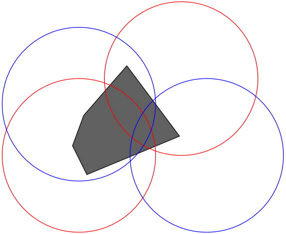

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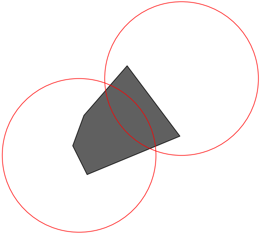

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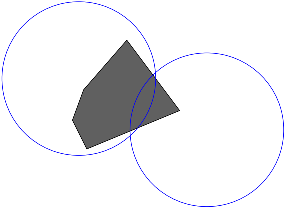

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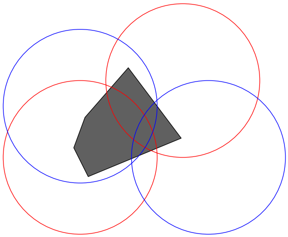

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Deciding plane-cover-decomposability seems harder.

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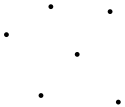

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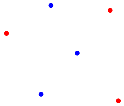

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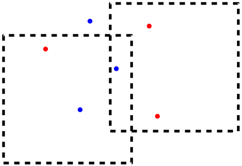

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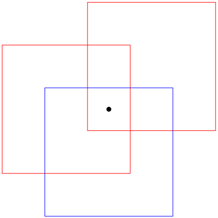

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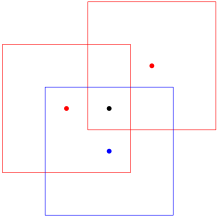

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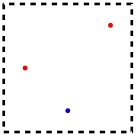

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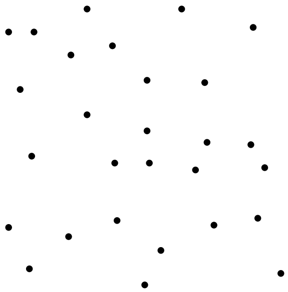

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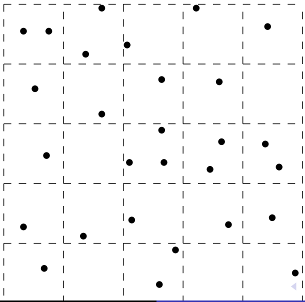

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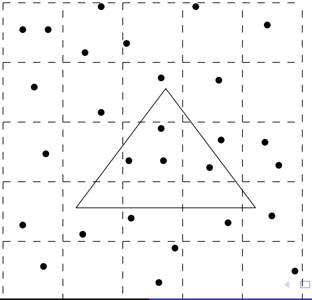

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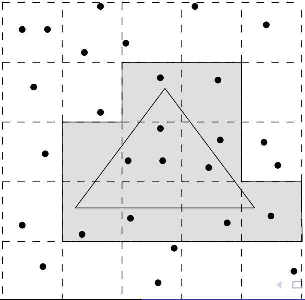

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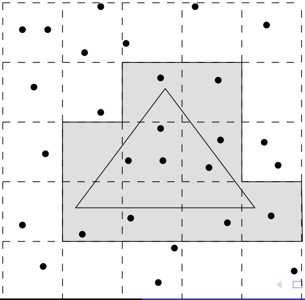

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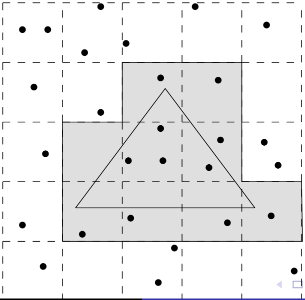

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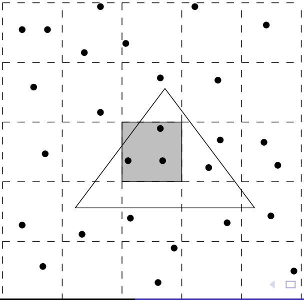

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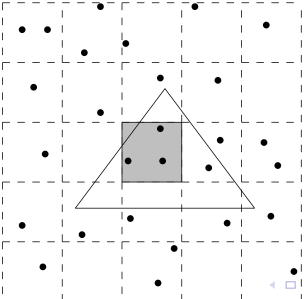
We can assume that all the points are in a small region.



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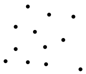

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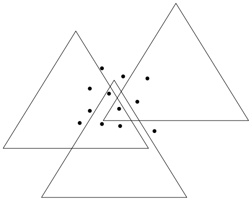

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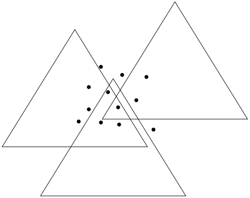

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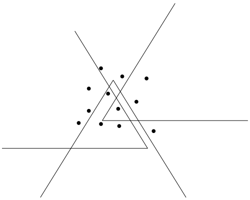

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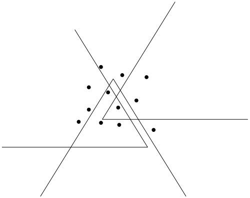

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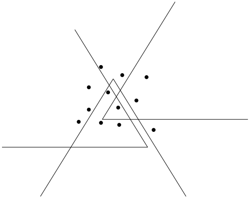
Ie. color points with two colors such that any translate of the given wedges contains both colors if it contains at least k points.



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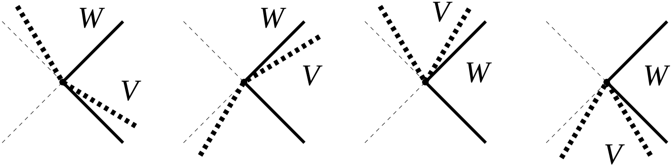

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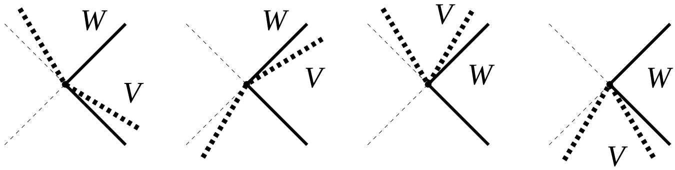
They form a special pair if their relative position is one of these.



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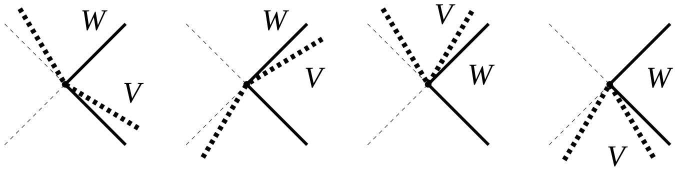
That is, the union of the wedges is in an open halfplane whose boundary contains the origin, but none of them contain the other.



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For any pair of special wedges V, W and for every k, l, there is a point set of cardinality $\binom{k+l}{k}$ such that for every coloring of P with red and blue either there is a translate of V containing k red and no blue points or there is a translate of W containing l blue and no red points.

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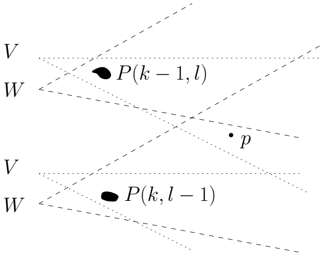

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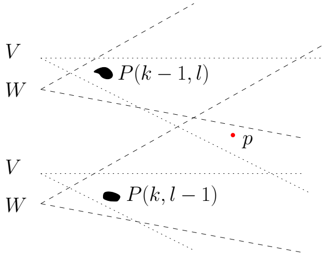

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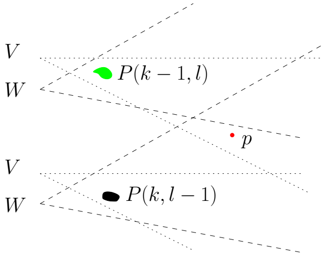

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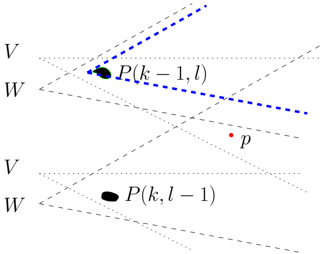

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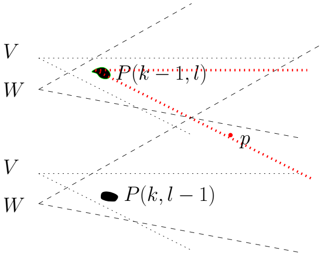

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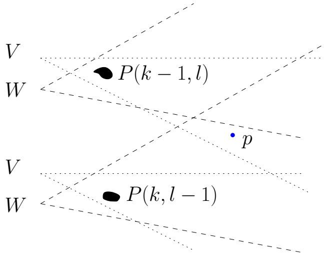

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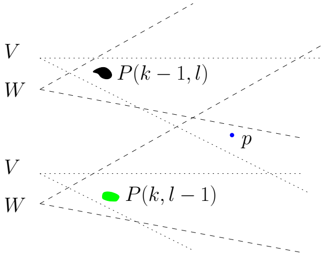

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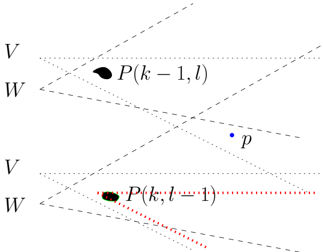

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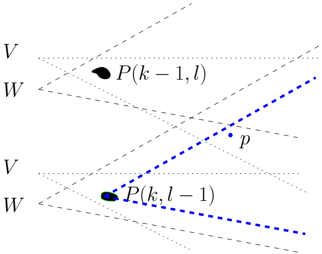

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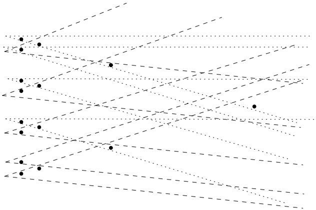

Proof.

Induction. □

Theorem

If a concave polygon has no parallel sides, then it has a special pair of wedges.

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If a concave polygon has no parallel sides, then it has a special pair of wedges. Thus it is not cover-decomposable.

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Theorem

For a system of wedges we can color the points with two colors such that any translate of the given wedges contains both colors if it contains at least k points IFF none of the pairs are special.

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Theorem

A polygon is cover-decomposable IFF none of the pair of its angles are special.

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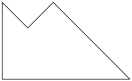

Figure: Cover-decomposable but not union of translates of a convex set.

Theorem

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Theorem

For a system of wedges we can color the points with two colors such that any translate of the given wedges contains both colors if it contains at least k points IFF none of the pairs are special.

Theorem

A polygon is cover-decomposable IFF none of the pair of its angles are special.

Corollary

Convex polygons are cover-decomposable.

Theorem

If a concave polygon has no parallel sides, then it has a special pair of wedges. Thus it is not cover-decomposable.

Theorem

For a system of wedges we can color the points with two colors such that any translate of the given wedges contains both colors if it contains at least k points IFF none of the pairs are special.

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A polygon is cover-decomposable IFF none of the pair of its angles are special.

Corollary

Convex polygons are cover-decomposable.

Theorem

Polyhedrons are not cover-decomposable.

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Theorem

For a system of wedges we can color the points with two colors such that any translate of the given wedges contains both colors if it contains at least k points IFF none of the pairs are special.

Theorem

A polygon is cover-decomposable IFF none of the pair of its angles are special.

Corollary

Convex polygons are cover-decomposable.

Theorem

Polyhedrons are not cover-decomposable.
Not even space-cover-decomposable.

Different cover-decomposability notions

Different cover-decomposability notions

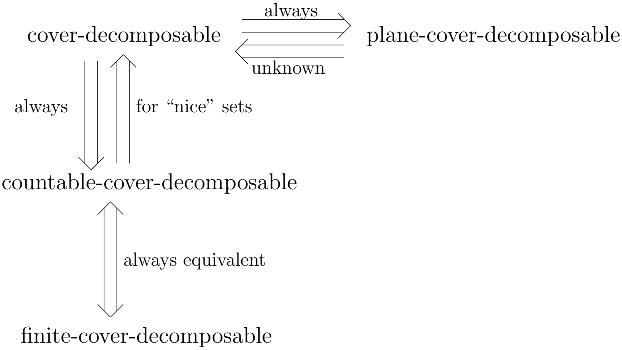


Figure: Connections between variants of cover-decomposability.

Different cover-decomposability notions

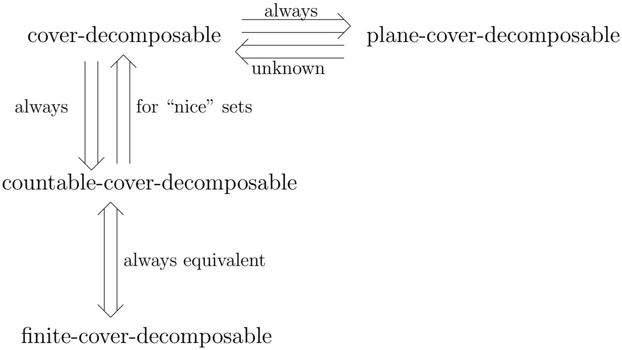


Figure: Connections between variants of cover-decomposability.

Corollary

It does not matter if the set is open or closed.

Extending the counter example to the whole plane

Extending the counter example to the whole plane

Theorem

If a concave polygon C has two special wedges that have a common touching line, then it is not plane-cover-decomposable.

Extending the counter example to the whole plane

Theorem

If a concave polygon C has two special wedges that have a common touching line, then it is not plane-cover-decomposable.

Proof.

The counter example can be extended. □

Extending the counter example to the whole plane

Theorem

If a concave polygon C has two special wedges that have a common touching line, then it is not plane-cover-decomposable.

Proof.

The counter example can be extended. □

Corollary

If a concave pentagon is not cover-decomposable, then it is not plane-cover-decomposable.

Extending the counter example to the whole plane

Theorem

If a concave polygon C has two special wedges that have a common touching line, then it is not plane-cover-decomposable.

Proof.

The counter example can be extended. □

Corollary

If a concave pentagon is not cover-decomposable, then it is not plane-cover-decomposable.

This statement does not hold for hexagons.

Extending the counter example to the whole plane

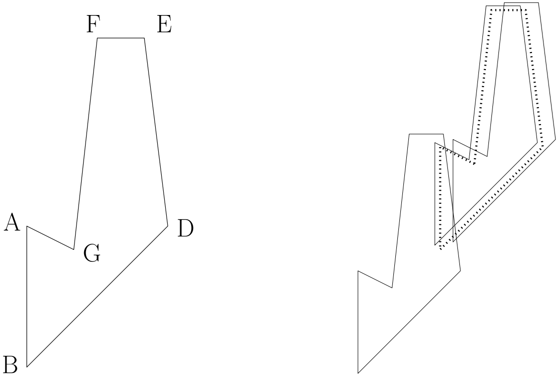
Theorem

There is a hexagon that is not cover-decomposable but it is plane-cover-decomposable.

Extending the counter example to the whole plane

Theorem

There is a hexagon that is not cover-decomposable but it is plane-cover-decomposable.



Thank you for your attention!