Polychromatic Colorings of Arbitrary Rectangular Partitions

Dömötör Pálvölgyi

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Joint work with:

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What do Japanese and Hungarians have in common?

Polychromatic colorings

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Let G be a graph drawn in the plane without crossings.

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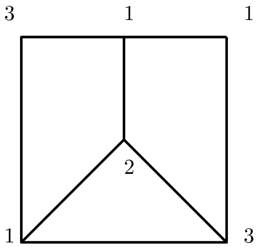

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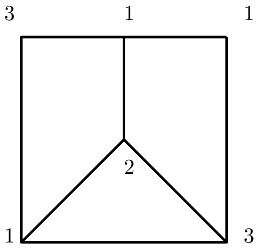

A polychromatic coloring is not necessarily a proper-coloring!

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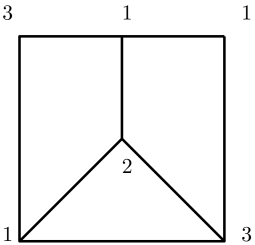

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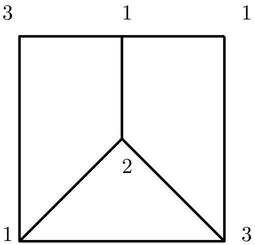

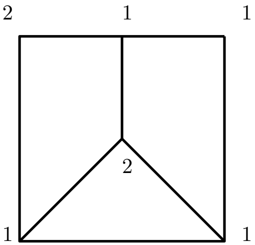

This graph's polychromatic number is 3, as it has a face that is a triangle.

General bounds

Remark

If g *is the size of the smallest face of* G *then* $\chi_f(G) \leq g$.

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$\chi_f(G) \geq \lfloor (3g-5)/4 \rfloor$.

This result is quite tight as there are graphs that do not have a polychromatic k-coloring with $k > \lfloor (3g+1)/4 \rfloor$.

Coloring with a small number of colors

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Theorem (Hoffmann and Kriegel)

Every 2-connected and bipartite plane graph admits a polychromatic 3-coloring.

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If the maximum degree of a plane graph is at most 3 then it admits a polychromatic 3-coloring unless it is K_4.

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Theorem (Horev, Katz and Krakovski)

Every 3-regular, 2-connected and bipartite plane graph admits a polychromatic 4-coloring.

Rectangular partitions

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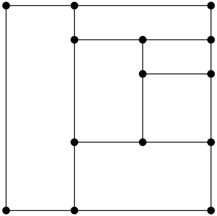

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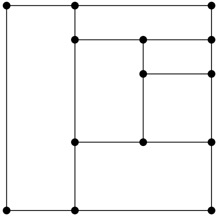

$\Rightarrow$ the maximum degree of G is at most 3

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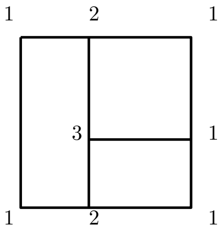

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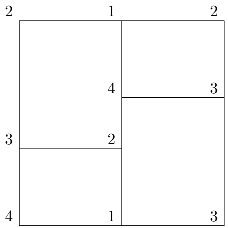
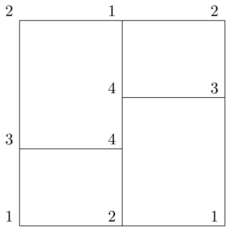


(a)



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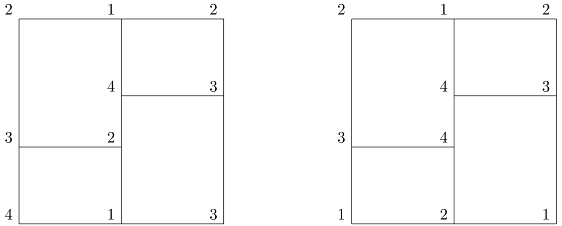

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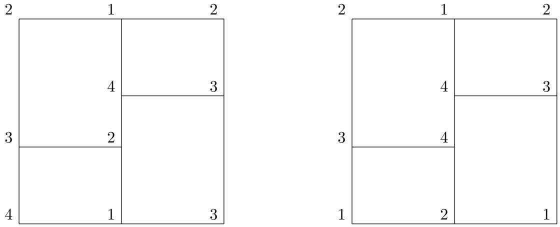

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(b) strong rectangle-respecting: 4 colors on the **corners** of every rectangle.

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Theorem (Dimitrov, Horev and Krakovski)

Every rectangular partition admits a strong rectangle-respecting 4*-coloring.*

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- Follows from a theorem of Guenin on edge-colorings of plane graphs (formerly a conjecture of Seymour).
- Gives a polychromatic 4-coloring, thus it implies the theorem of Dinitz, Katz and Krakovski.

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There is a general rectangular partition which does not admit a strong rectangle-respecting 4-coloring.

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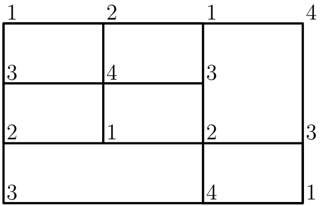

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Is there a a general rectangular partition which does not admit a polychromatic 4-coloring?

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Answer (GKLPPP)

Yes.

Proof

Observation

There are only two different polychromatic 4*-coloring of a* 3×3 *grid (up to rotation and permutation of the colors).*

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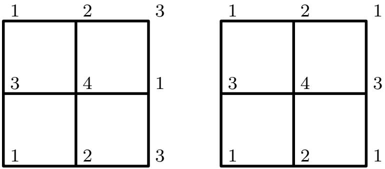

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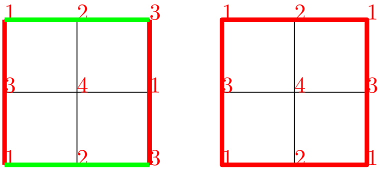

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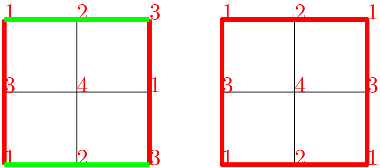

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A grid has the same colors on its opposing sides and has a red side.

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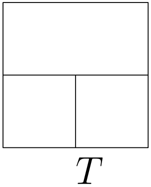
A polychromatic 4-coloring of a T*-shape assigns 3 distinct colors to the left side or the right side of the* T*.*



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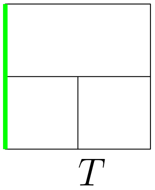

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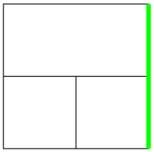

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Every T has a green side.

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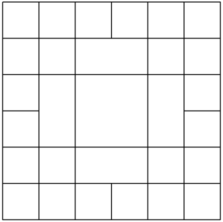
A polychromatic 4-coloring of the below graph assigns 3 distinct colors to 3 consecutive vertices on each side of the outer face.



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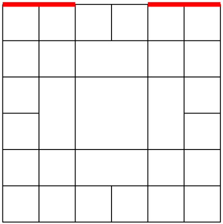

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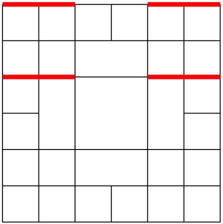

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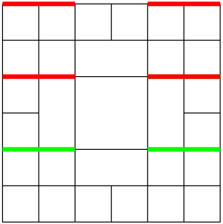

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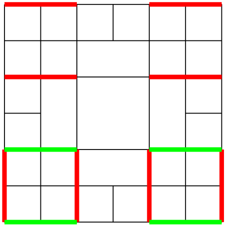

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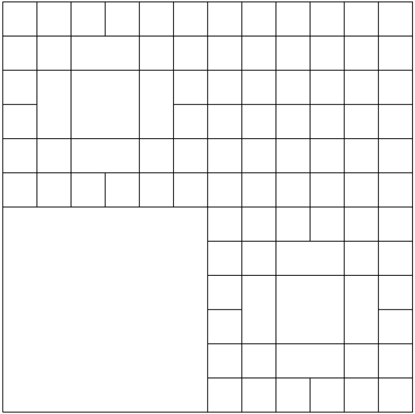

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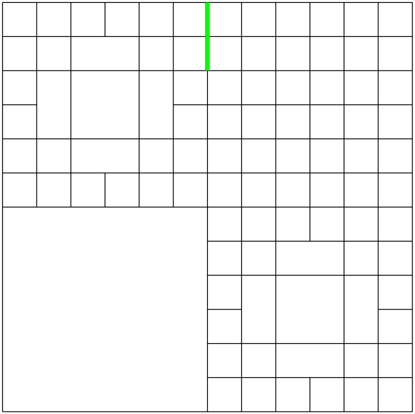

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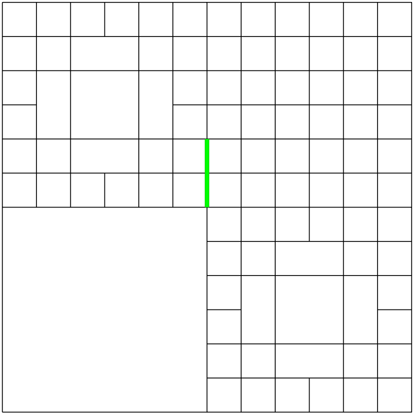

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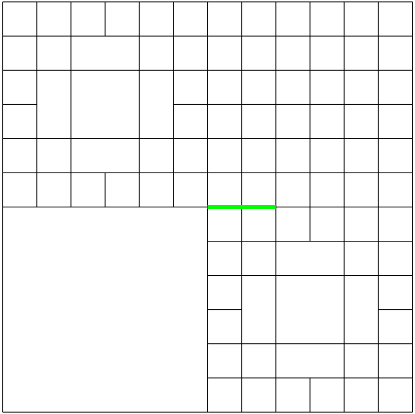

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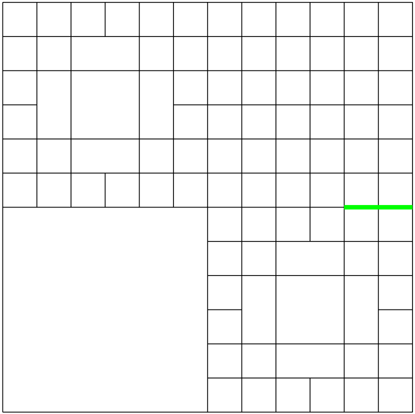

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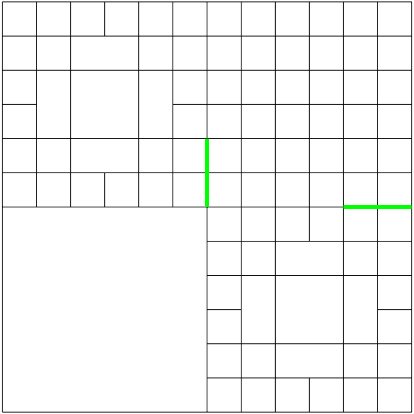

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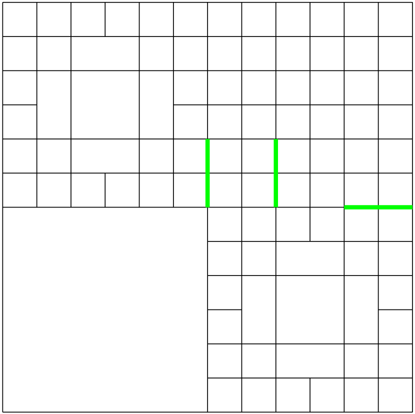

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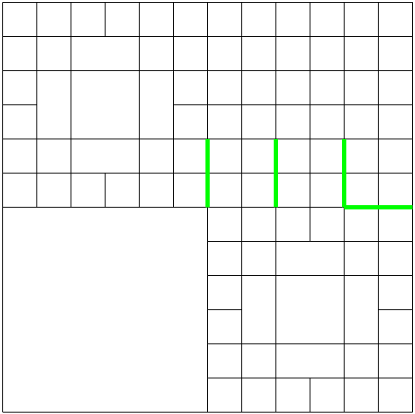

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The best that we know of has 46 rectangles and 65 vertices.

Polychromatic k-colorings of rectangular partitions

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Every general rectangular partition admits a "polychromatic" 5-coloring, i.e. 4 of 5 different colors on each boundary.

Strong rectangle-respecting k-colorings

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Strong rectangle-respecting k-colorings

Proposition (GKLPPP)

Every general rectangular partition admits a strong rectangle-respecting 2-coloring i.e. 2 colors on the corners.

Open for $k = 3$.

Proposition (GKLPPP)

Every general rectangular partition admits a "strong rectangle-respecting 6-coloring" i.e. 4 of 6 different colors on the corners.

Open for $k = 5$.

Thank you for your attention.