

Drawing cubic graphs with the four basic slopes

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Graph Drawing '11 - Eindhoven

Definition

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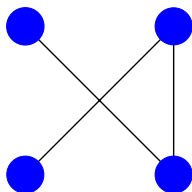
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Example



A straight-line drawing of P_3 .

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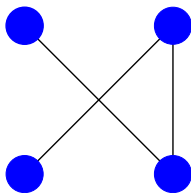
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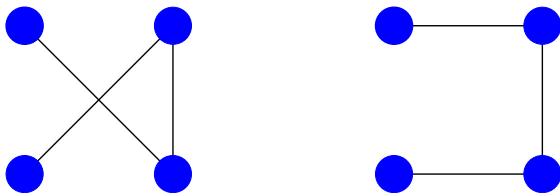
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The slope number of P_3 is one.

If G has a vertex of degree d , then its slope number is at least $\lceil d/2 \rceil$.



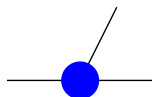
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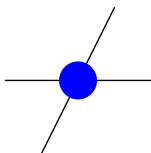
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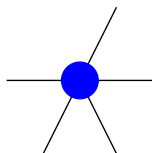
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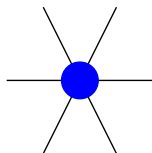
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Question: Bounding slope number from above by a function of the maximum degree?

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Graphs with maximum degree 5 can have arbitrarily large slope number.

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Theorem (Mukkamala, Szegedy)

The slope number of connected cubic graphs is at most 4.

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Our goal is to give an optimal answer for cubic graphs.

Reminder

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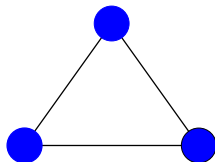
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The slope number of K_3 is three.

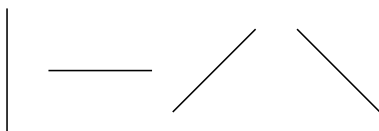
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A line is said to be of one of the four **basic directions** if it is parallel to one of the axes or to one of the lines $y = x$ and $y = -x$.

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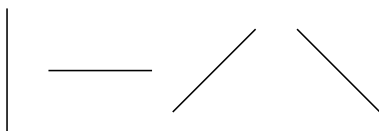
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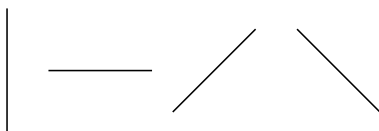
Theorem (M., Pálvölgyi)

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Note that all previous theorems about cubic graphs used unnatural slopes.

Subcubic Lemma

Theorem (Keszegh, Pach, P., Tóth)

Let G be a connected graph whose every vertex has degree at most three and has at least one vertex with degree less than three. Then G has a straight-line drawing with the four basic directions.

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- ▶ *The subgraph after cycle removal may not be connected.*
- ▶ *The neighbor of degree three vertex of the removed cycle need not have degree three.*

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Let G be a graph whose every component is subcubic and not a cycle. Denote by v_1, \dots, v_m the vertices of degree at most two ($m \geq 1$).

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- ▶ The x -coordinates of all the vertices are a linear combination with rational coefficients of x_1, \dots, x_n .

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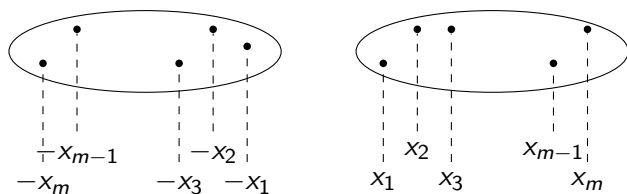


Figure: The x -coordinates of the degree 2 vertices is suitably chosen and one component is rotated and translated to make the M -cut vertical.

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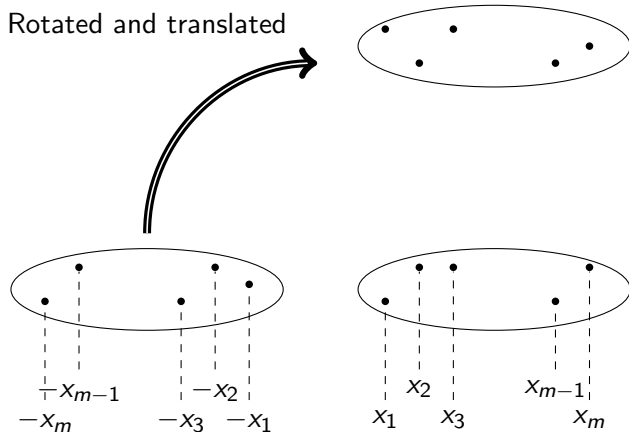


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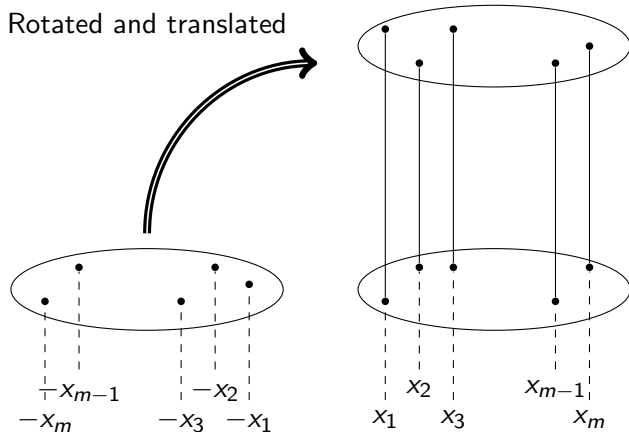


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Corollary

Cubic graphs that cannot be drawn with the four basic slopes must be three connected.

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Remark

Intuition: For a connected cubic graph G on n vertices, if there is a component C , not a cycle, on s vertices with $s \ll n$ such that the degree of all vertices in C is at least 2, then G has a M -cut.

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Lemma

Every connected cubic graph on n vertices contains a cycle of length at most $2\lceil \log(\frac{n}{3} + 1) \rceil$.

Finding a short cycle

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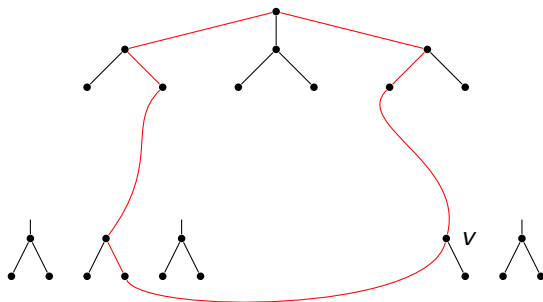


Figure: Finding a cycle in the BFS tree using that the left child of v already occurred.

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Lemma (Intuition in numbers)

Every connected cubic graph on $n > 2s - 2$ vertices with a supercycle with s vertices contains a suitable M -cut of size at most $s - 2$.

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The only graphs that requires to be checked are Petersen and Tietze's graph.

Proofs

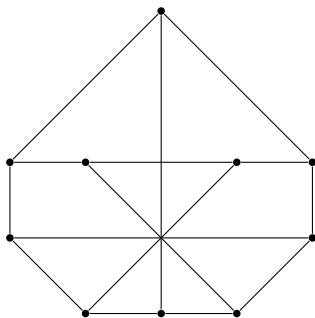


Figure: The Petersen graph drawn with the four basic slopes.

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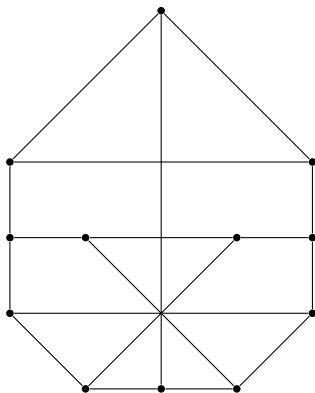


Figure: The Tietze's graph drawn with the four basic slopes.

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- ▶ None of the slopes in S can occur three times in the drawing of K_4 .
- ▶ Two slopes, say s_1 and s_2 , occur twice.
- ▶ These slopes must form a parallelogram that can be transformed to a square with an affine transformation.



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What about higher maximum degree and other graph classes?

Thank you for your attention!