Cubic Graphs Have Bounded Slope Parameter

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Graph Drawing 2008

The Slope Parameter of a Graph

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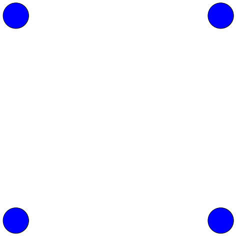
Example.



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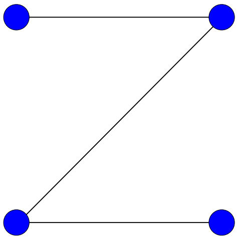

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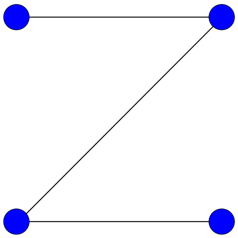
Example.

The slope parameter of P_4 is 2.



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Another Example.

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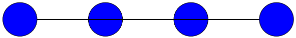

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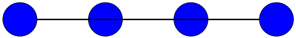

The slope parameter of K_4 is 1.

An easy observation

If G has a vertex of degree d whose neighbors are independent, then its slope parameter is at least d.

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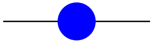

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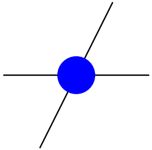

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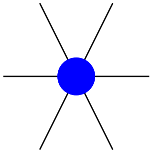

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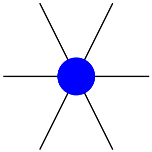

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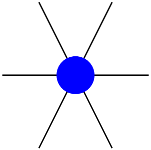

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Question: Can we bound slope parameter from above by a function of the maximum degree?

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Every cubic graph has slope parameter at most seven.

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Every subcubic graph has slope parameter at most five.

Moreover, these can be realized by a drawing such that no *three* vertices are collinear.

The case of maximum degree four remains open.

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A line is said to be of one of the five **basic directions** if its degree with the horizontal line is a multiple of $\pi/5$.

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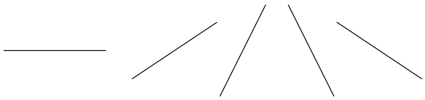

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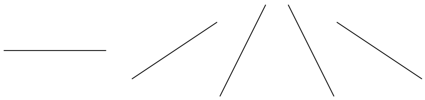

Theorem

Every subcubic graph has slope parameter at most five*. Moreover, this can be realized by a drawing such that no* three *vertices are collinear and each edge has one of the five basic slopes.*

Proof

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Then the omitted part is carefully reattached.

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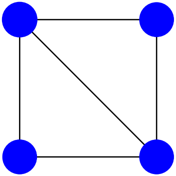

G'

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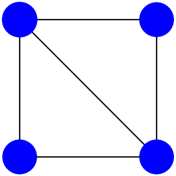
If the degree of $v \in V(G)$ is one, then $G' := G \setminus v$.

By induction, G' has a drawing using the five basic slopes.

And v can be added easily to this drawing, since its neighbor (eg., the upper-right vertex) can have only two edges in G'.



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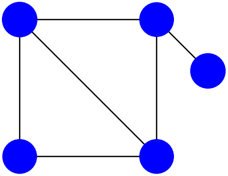

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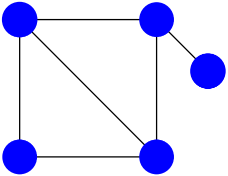

Note that if v is placed *close enough* to its neighbor, it cannot cause any troubles.

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Proposition

The degree of each vertex is at least two and G is two-connected.

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Note that there are no edges inside C because it is a shortest cycle passing through v.

Now let us delete C from the graph and using induction, take a drawing of $G' := G \setminus C$.

Imagine that G' is very small and put back C in a suitable way.

Putting Back $C = \{u_0, u_1, \ldots, u_4\}$

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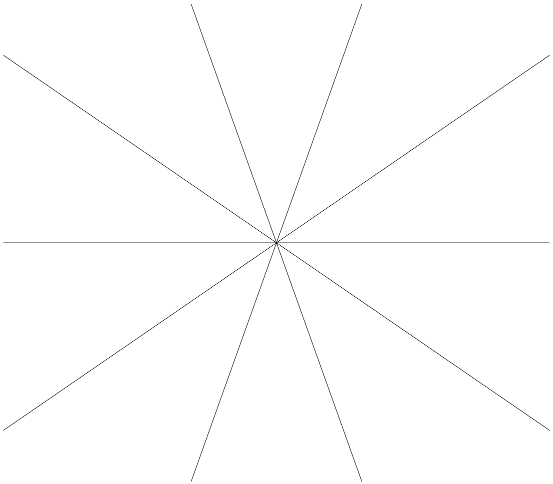
Take the five basic directions through origin.



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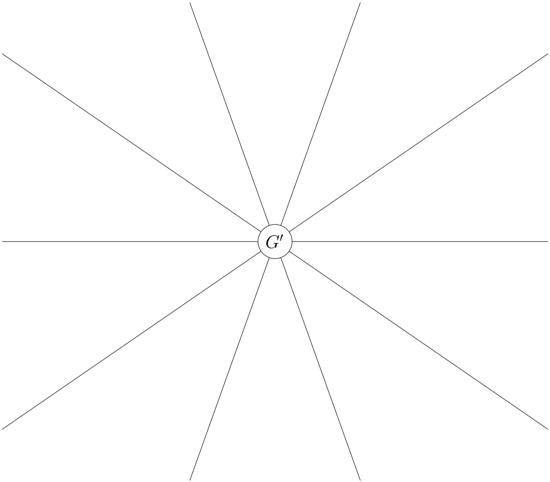
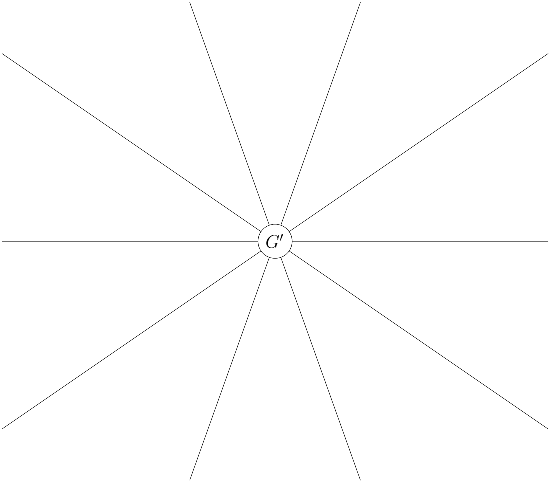
Put a small copy of G' into the middle.



Putting Back $C = \{u_0, u_1, \ldots, u_4\}$

Find a place for u_1.



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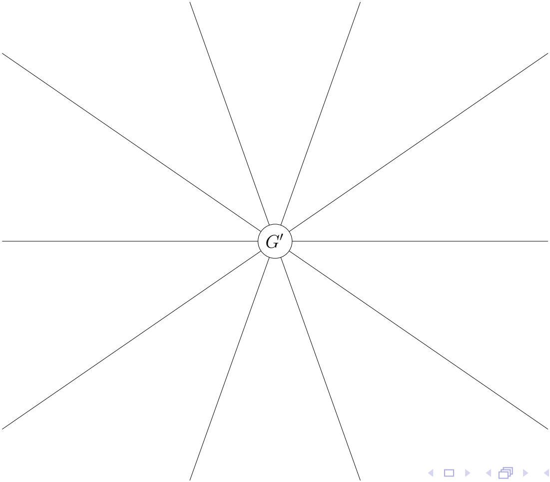
The neighbor of u_1 from G' can have at most two neighbors in G'.



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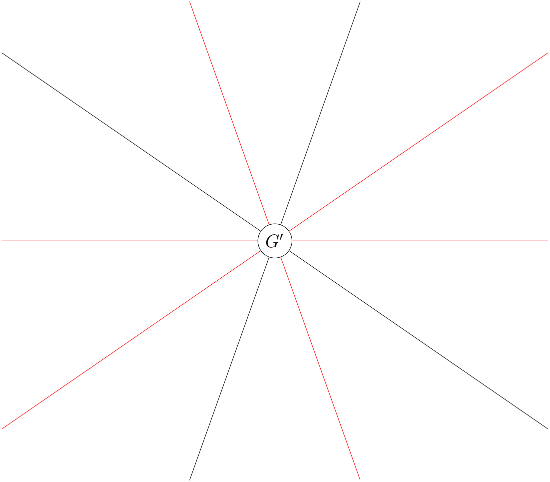
Therefore it has three *free* directions.



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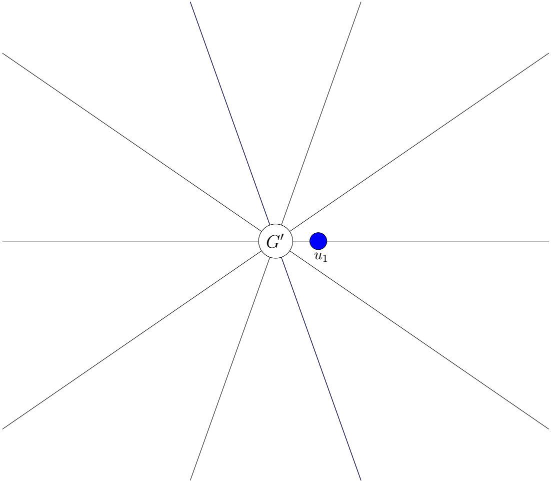
Place u_1 on one of them.



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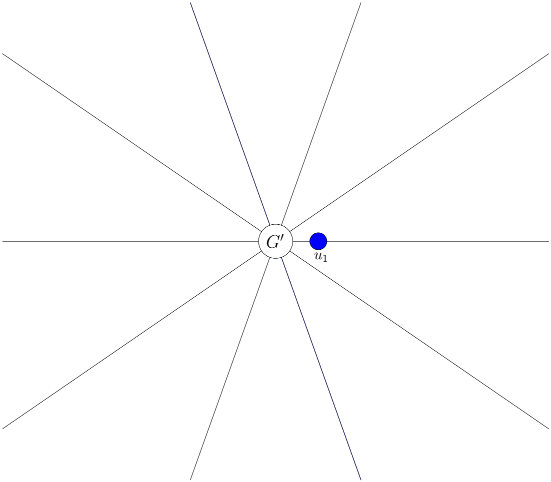
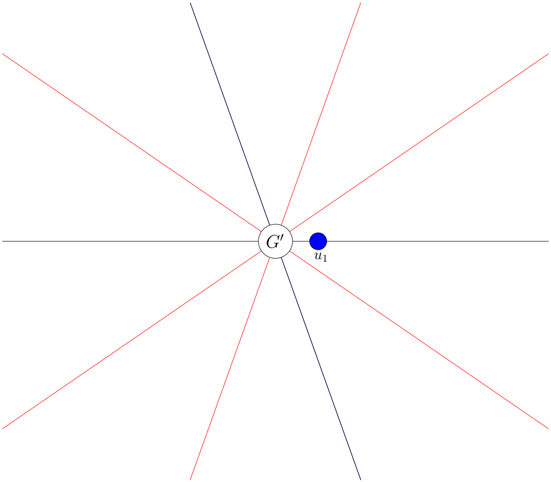
Find a place for u_2.



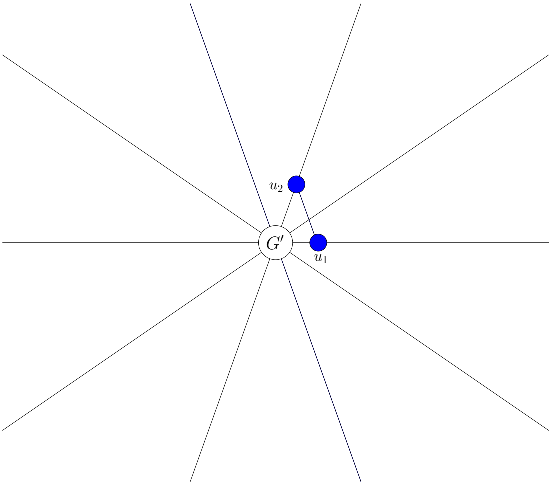
Putting Back $C = \{u_0, u_1, \ldots, u_4\}$

The neighbor of u_2 has three free directions.



Putting Back $C = \{u_0, u_1, \ldots, u_4\}$

Place u_2 on one of them that differs from the line of u_1.



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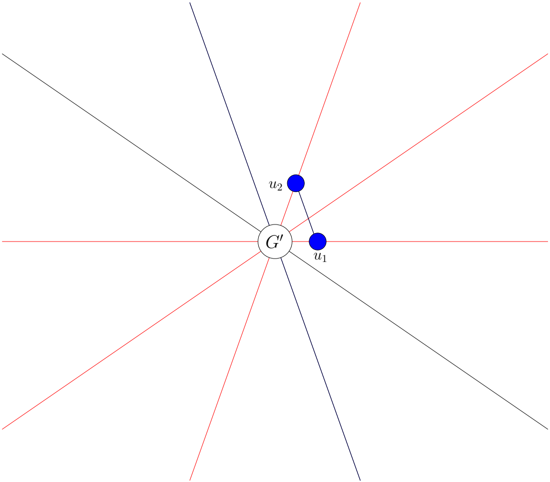
The neighbor of u_3 has three free directions.



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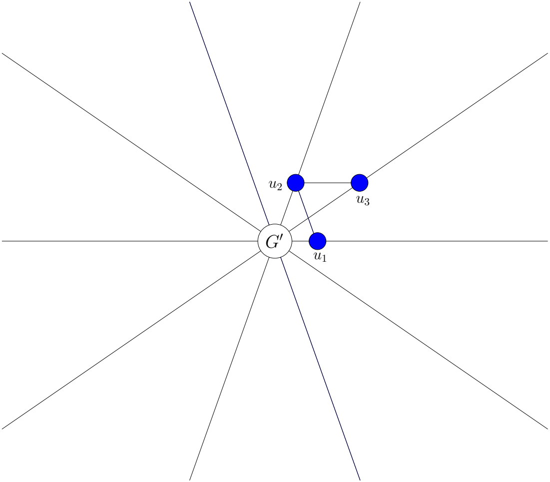
Place u_3 on one of them that differs from the line of u_2.



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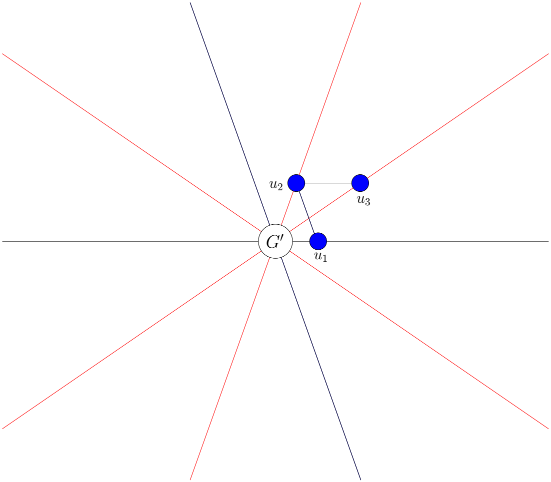
The neighbor of u_4 has three free directions.



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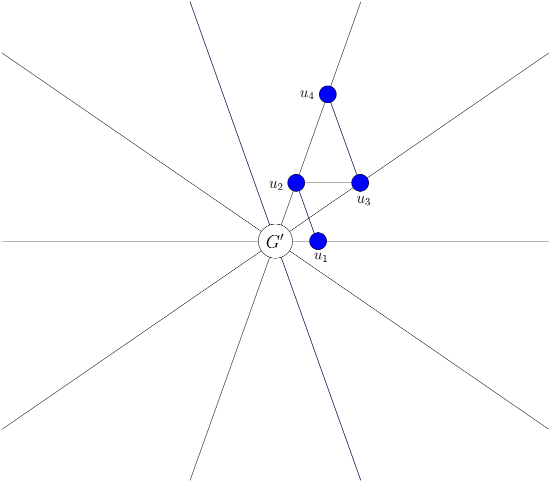
Place u_4 on one of them that differs from the line of u_3.



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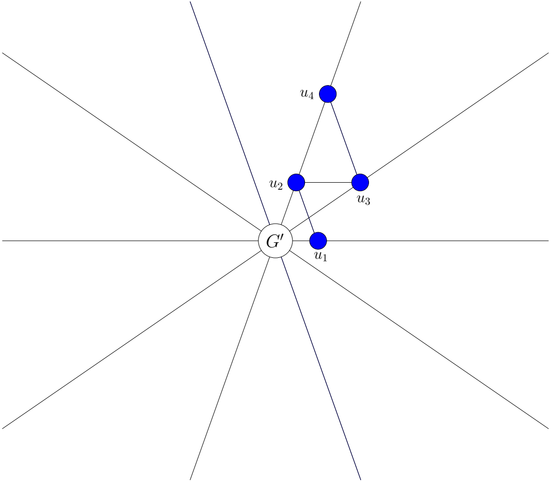
Find a place for u_0.



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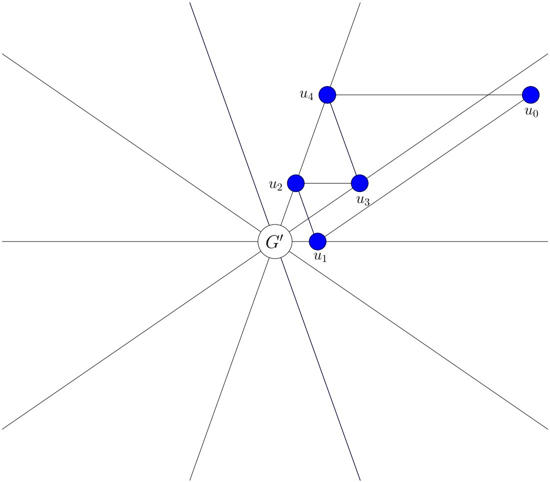
Tricky but can be done if its line does not neighbor the line of u_1.



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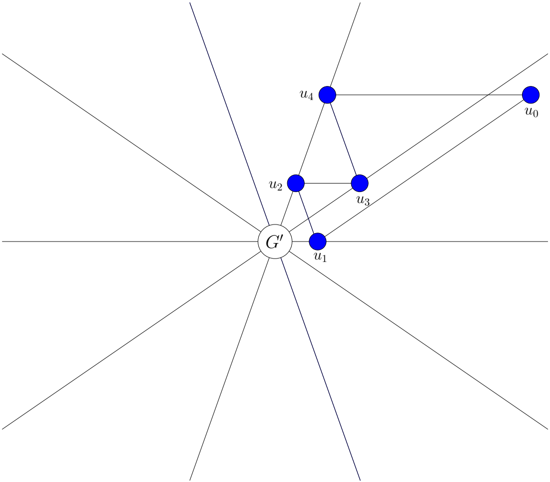
This can be achived by using the freedom that we had earlier.



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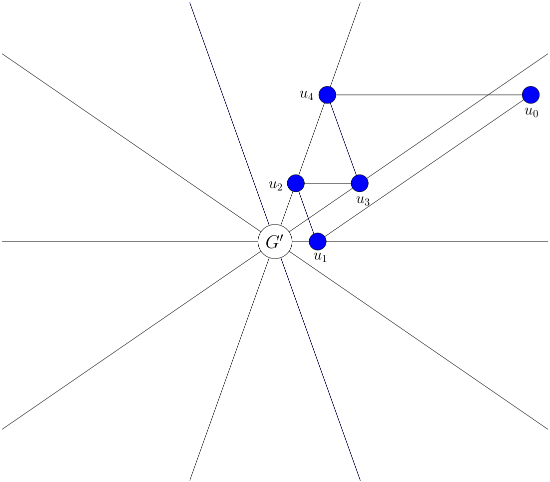
This finishes the proof of the subcubic theorem.



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Every cubic graph has slope parameter at most seven.

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Proof for connected graphs.

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Delete any vertex.

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Draw rest of graph using theorem for subcubic graphs.

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Draw rest of graph using theorem for subcubic graphs.

Put back deleted vertex. Carefully.

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Proof for graphs with more components.

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In each component take a shortest cycle and one vertex form it.

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In each component take a shortest cycle and one vertex form it.

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Then delete this cycle, draw rest of graph with basic directions.

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Put back cycle, as usual, except the degree two vertex.

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Proof of the last step is not that trivial and quite technical.

Thank you for your attention!