Vectors in a Box

Kevin Buchin, Jiří Matoušek, Robin A. Moser, Dömötör Pálvölgyi

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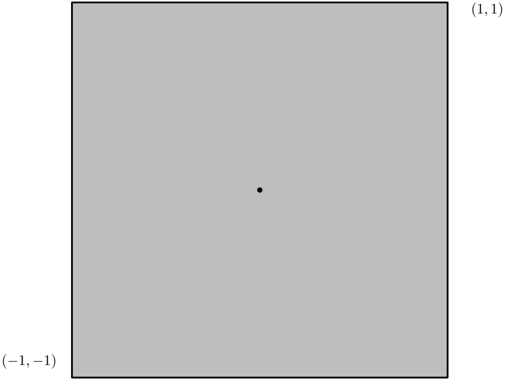

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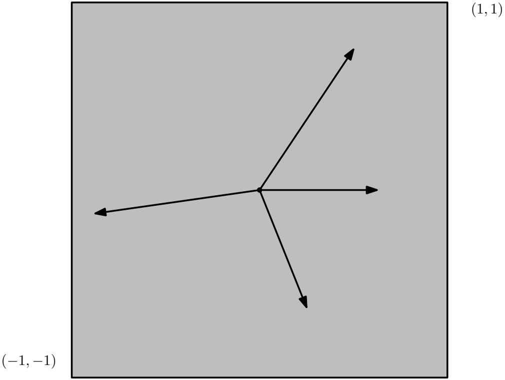

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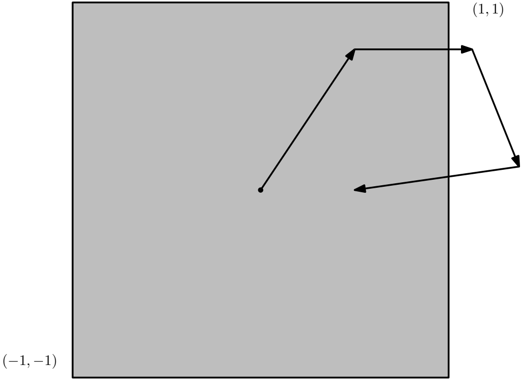

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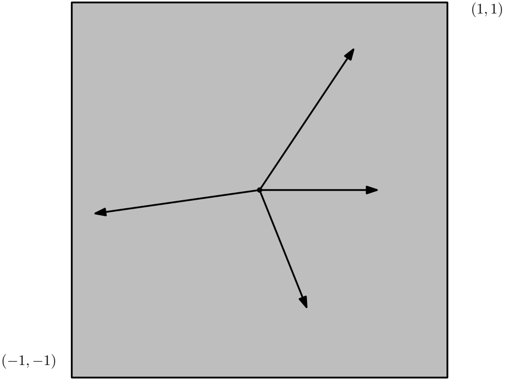

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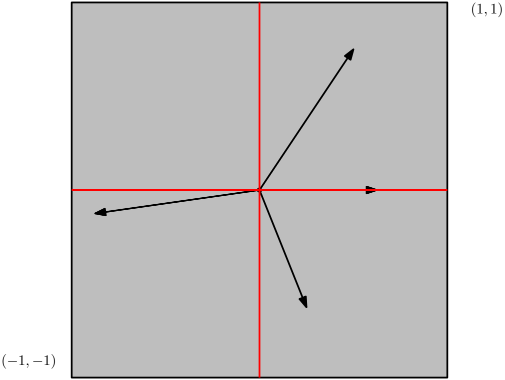

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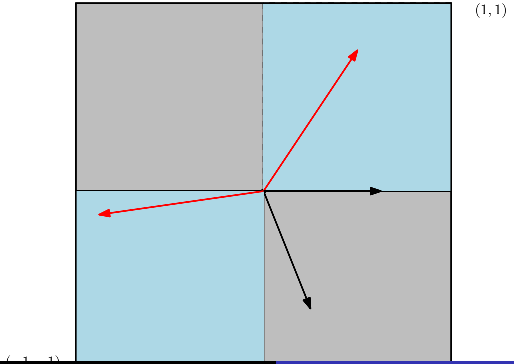

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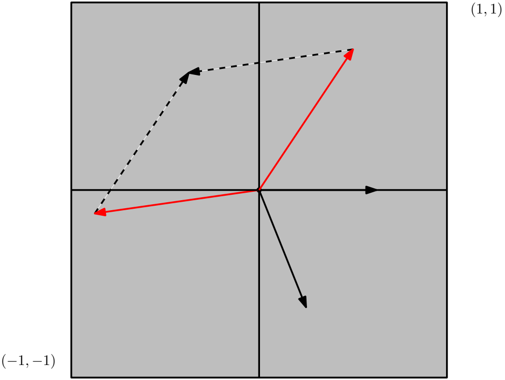

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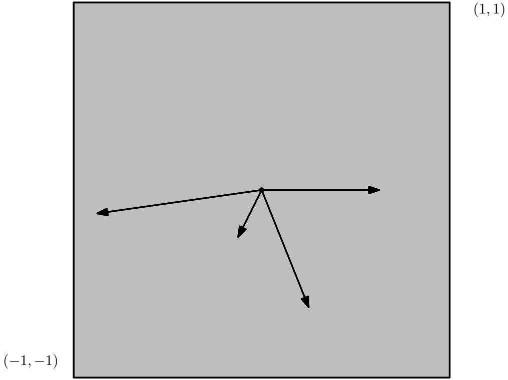

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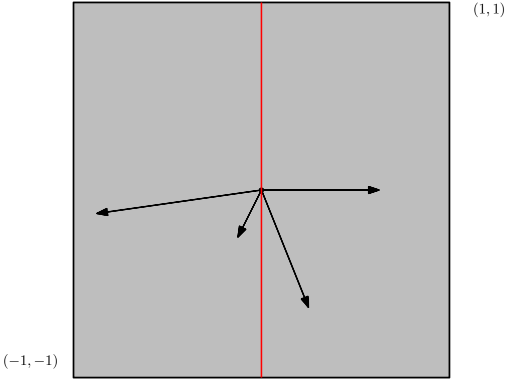

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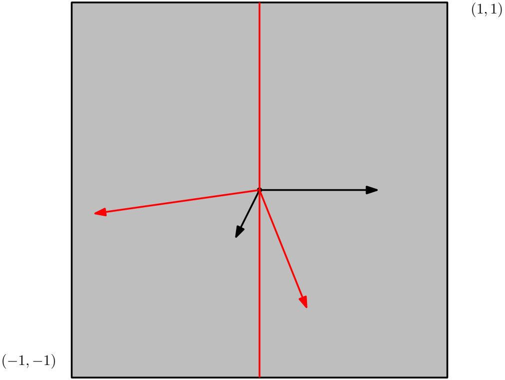

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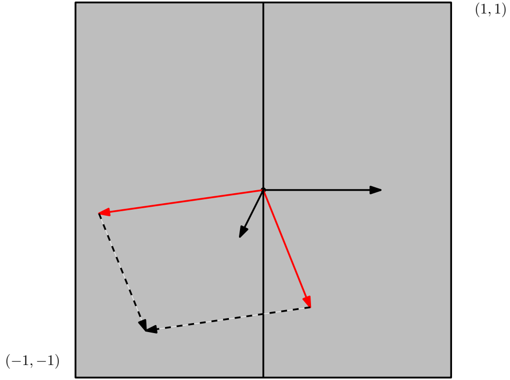

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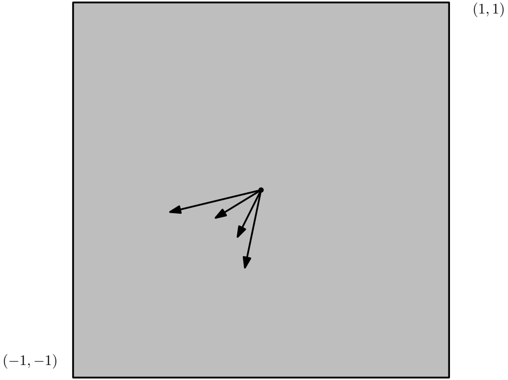

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Our goal is to determine the order of magnitude of $\tau(d)$.

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Our result is an asymptotics for $\tau(d)$.

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Let B be a symmetric convex body in $\mathbb{R}^d$, and let $\mathbf{v}_i \subset B$ be a finite set (or multiset) of vectors satisfying $\sum \mathbf{v}_i = 0$. Then there is a permutation of the indices, π such that for all $k = 1, 2, \ldots, n$, we have $\sum_{i=1}^{k} \mathbf{v}_{\pi(i)} \in d \cdot B$.

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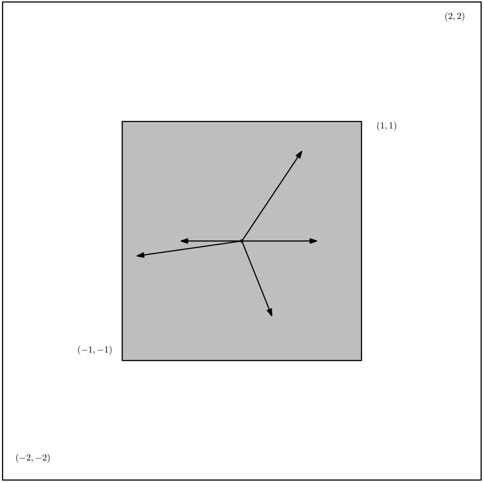

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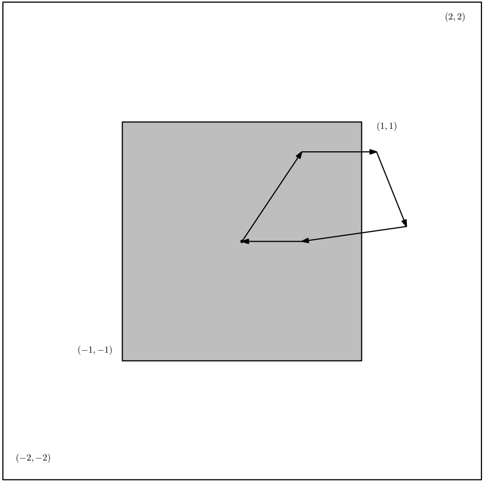

Add $\mathbf{w} := -\sum \mathbf{v}_i$ to our vectors to ensure they sum up to zero.

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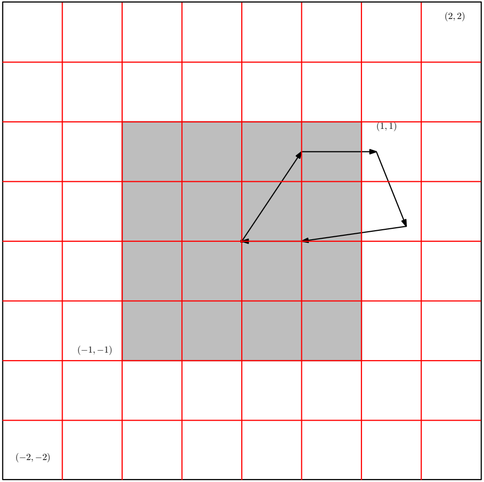

Dive $d\cdot$ box to small boxes.

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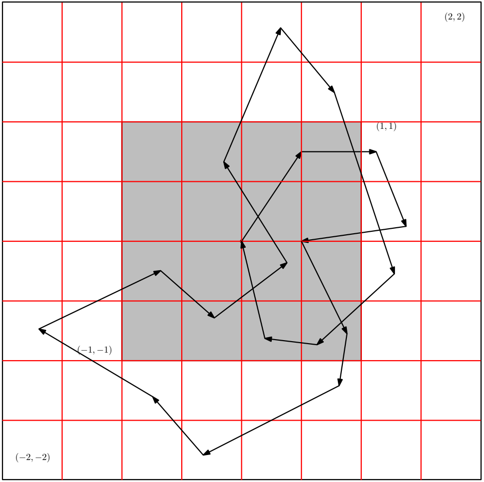

Take ordering such that $\mathbf{s}_k = \sum_{i=1}^{k} \mathbf{v}'_i \in d\cdot$ box.

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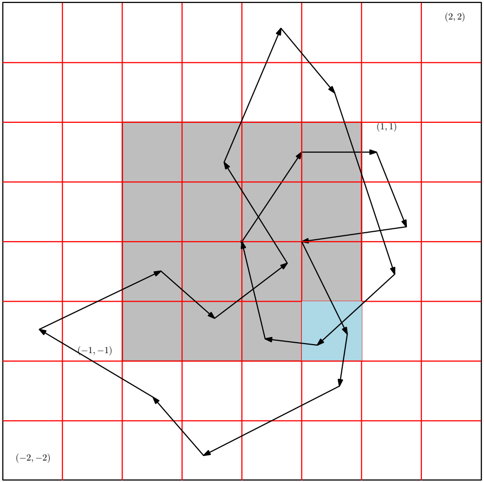

Two of the s_k must be close to each other.

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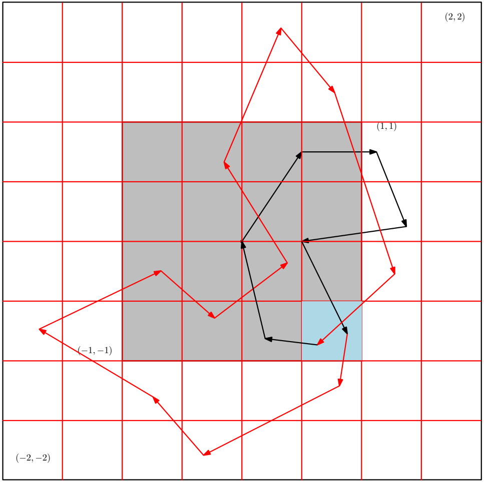

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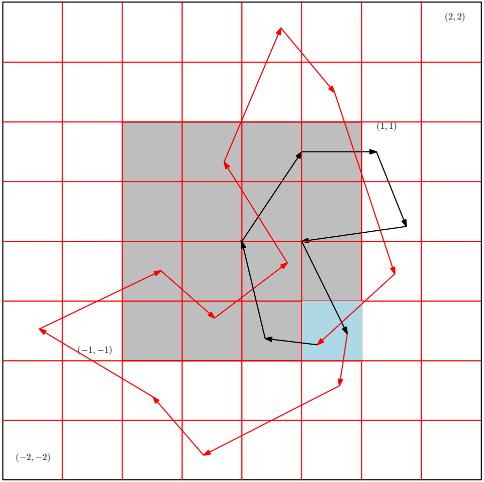

In fact, consider only half without $\mathbf{w}$ and take every second endpoint.

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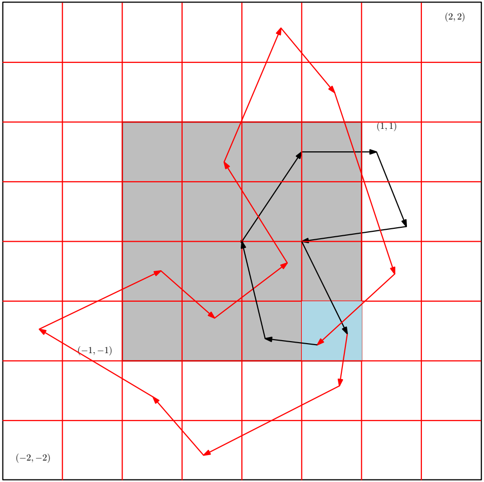

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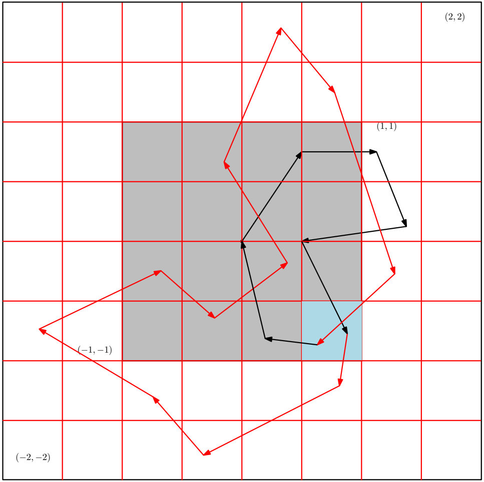

So we can even get into $\delta\cdot$ box if we have enough vectors.

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We use this C to construct a large number of vectors in the box such that there sum is in the box but for all I such that $1 < |I| < t$ we have $\sum_{i \in I} v_i \notin$ box.

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There are several ways to modify proof to work for boundary too.

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If $\sum_{i \in I} \mathbf{w}_i \in$ box, then $|I|$ is odd.

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$\mathbf{w}_i'' = \mathbf{w}_i$ with replacing $+1$'s by $(1-\varepsilon)$'s.
$\mathbf{v}_i = \mathbf{w}_i' \circ \mathbf{w}_i''$.
So $\mathbf{w}_i = (+1, +1, -1)$ becomes $\mathbf{v}_i = (+1, +1, -(1-\varepsilon), 1-\varepsilon, 1-\varepsilon, -1)$.
If $\sum_{i \in I} \mathbf{w}_i \in$ box, then $|I|$ is odd.
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Being a little more careful, each $\mathbf{v}_i$ can be a ± 1 vectors.

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For Euclidean ball and box, not even $o(d)$ is known.

Thank you for your attention!