On the accuracy of an application of the Richardson Extrapolation in the treatment of multi-dimensional advection equations

Z. Zlatev [1], Dimov [2], I. Faragó [3,4], K. Georgiev [2], Á. Havasi [3,4] and Tz. Ostromsky [2]

[1] Department of Environmental Science, Aarhus University, Roskilde, Denmark

[2] Institute of Information and Communication Technologies, Bulgarian Academy of Sciences, Sofia, Bulgaria

[3] Department of Applied Analysis and Computational Mathematics, Eötvös Loránd University, Budapest, Hungary

[4] MTA-ELTE Numerical Analysis and Large Networks Research Group, Budapest, Hungary

Abstract

A Crank-Nicolson type scheme, which is of order two with respect to all independent variables, is used in the numerical solution of multi-dimensional advection equations. Normally, the order of accuracy of any numerical scheme can be increased by one when the well-known Richardson Extrapolation is used. It is proved that in this particular case the order of accuracy of the combined numerical method consisting of the Crank-Nicolson scheme and the Richardson Extrapolation is not three but four.

Key words: Partial differential equations, Multi-dimensional advection, Crank-Nicolson numerical scheme, Richardson Extrapolation, Order of accuracy

1. Some introductory remarks about the Richardson Extrapolation

The main ideas related to the application of the Richardson Extrapolation will be sketched in this section (some more details can be found in [1], [3]-[6]).

Consider a time-dependent differential equation $\mathbf{F(t,c)=0}$ where $\mathbf{t} \in [\mathbf{a,b}]$ is the time variable (in the next sections it will be assumed that also some other independent variables, which will be denoted by $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_Q}$, are involved in $\mathbf{F(t,c)=0}$) and $\mathbf{c}$ is the unknown function. It is assumed here that $\mathbf{F}$ contains some derivatives of the unknown function with respect to the independent variables.

Richardson extrapolation is sometimes successfully used either to control the accuracy achieved in the solution of $\mathbf{F(t,c)=0}$ by the selected numerical method or in an attempt to improve the accuracy of the calculated approximate solution. This procedure can be introduced in the following way. Assume that $\mathbf{t_n \in [a,b]}$ is a given time-point and that $\mathbf{c(t_n)}$ is the exact solution of $\mathbf{F(t,c)=0}$ at this point. Assume also that two approximations of $\mathbf{c(t_n)}$ have been obtained when a numerical method of order $\mathbf{p}$ is applied by using two stepsizes $\mathbf{k}$ and $\mathbf{0.5k}$ (more precisely, starting from a point $\mathbf{t = t_{n-1} = t_n - k \in [a,b]}$ the two approximations are calculated by using first one large time-step and after that two small time-steps). Denoting the two approximations with $\mathbf{z_n}$ and $\mathbf{w_n}$ respectively, we can write:

$$\mathbf{(1)} \qquad \mathbf{c(t_n) = z_n + k^p\, K + O(k^{p+1})}$$

and

$$\mathbf{(2)} \qquad \mathbf{c(t_n) = w_n + (0.5k)^p\, K + O(k^{p+1})}\ ,$$

where $\mathbf{K}$ is some quantity depending on the selected numerical method. Eliminating the terms containing $\mathbf{K}$ in (1) and (2) gives:

$$\mathbf{(3)} \qquad \mathbf{c(t_n) = \frac{2^p\, w_n - z_n}{2^p - 1} + O(k^{p+1})}\ .$$

Denote:

$$\mathbf{(4)} \qquad \mathbf{c_n = \frac{2^p\, w_n - z_n}{2^p - 1}}\ .$$

It is clear that the approximation $\mathbf{c_n}$, being of order $\mathbf{p+1}$, will be more accurate than both $\mathbf{z_n}$ and $\mathbf{w_n}$ when the stepsize $\mathbf{k}$ is sufficiently small. Thus, the Richardson extrapolation can be used in the efforts to improve the accuracy.

The Richardson extrapolation can also be used in an attempt to evaluate the leading term of the local error of the approximation $\mathbf{w_n}$. Subtract (1) from (2), neglect the terms $\mathbf{O(k^{p+1})}$ and solve for $\mathbf{K}$. The result is:

$$(5) \qquad \mathbf{K} = \frac{2^p \left[\mathbf{w_n} - \mathbf{z_n} \right]}{\mathbf{k}^p (2^p - 1)}.$$

Substitute $\mathbf{K}$ from (5) in (2):

$$(6) \qquad \mathbf{c}(\mathbf{t_n}) - \mathbf{w_n} = \frac{\mathbf{w_n} - \mathbf{z_n}}{2^p - 1} + \mathbf{O}(\mathbf{k}^{p+1}) \ ,$$

which means that the quantity:

$$(7) \qquad \mathbf{E_n} = \frac{\mathbf{w_n} - \mathbf{z_n}}{2^p - 1}$$

can be used as an evaluation of the local error of the approximation $\mathbf{w_n}$ when the stepsize $\mathbf{k}$ is sufficiently small. If the evaluation of the local error, computed by using (7) is not acceptable, then the quantity $\mathbf{E_n}$ can also be used to determine a new stepsize $\mathbf{k}$ which will hopefully give an acceptable error. Assume that the requirement for the accuracy imposed by the user is $\mathbf{TOL}$. Then the new, hopefully better, stepsize $\mathbf{k_{new}}$ can be calculated by

$$(8) \qquad \mathbf{k_{new}} = \gamma \frac{\mathbf{TOL}}{\mathbf{E_n}} \mathbf{k},$$

where $\gamma < 1$ is used as a precaution factor (see, for example, [7]).

Thus, the Richardson extrapolation can be applied in codes with automatic stepsize control.

It must be emphasized here that the Richardson extrapolation does not depend too much on the particular method used ([1], [10]). It can be utilized both when classical numerical algorithms are applied in the solution of differential equations and when some more advanced numerical methods, which are combination of splitting procedures and classical numerical algorithms, are devised and used ([2], [11]). Two issues are important: (a) the large time-step and the two small time-steps must be handled by the same numerical method and (b) the order $\mathbf{p}$ of the selected method should be known.

The above remarks are made for the case where the time-variable $\mathbf{t}$ is the only independent variable. However the situation does not change too much when one or more spatial variables are added. The case where only one spatial variable appears is treated in [8] and [9]. The case where the unknown function $\mathbf{c}$ depends on several spatial variables will be handled in the remaining part of this paper.

2. Multi-dimensional advection equations

Consider the multi-dimensional advection equation:

$$(9)\quad \frac{\partial c}{\partial t} = -\sum_{q=1}^{Q} u_q \frac{\partial c}{\partial x_q}, \quad x_q \in \left[a_q, b_q\right] \quad \text{for} \quad q = 1, 2, \ldots, Q, \quad \text{with} \quad Q \geq 1, \quad t \in [a, b] .$$

It is assumed that the coefficients $u_q = u_q(t, x_1, x_2 \ldots, x_Q), \quad q = 1, 2, \ldots, Q,$ before the spatial partial derivatives in the right-hand-side of (9) are some given functions of the independent variables.

Let $\mathbf{D}$ be the domain in which the independent variables involved in (9) vary and assume that:

$$(10)\quad (t, x_1, x_2, \ldots, x_Q) \in D \quad \Rightarrow \quad t \in [a, b] \quad \wedge \quad x_q \in \left[a_q, b_q\right] \quad \text{for} \quad q = 1, 2, \ldots, Q .$$

By applying the definition proposed in (10), it is assumed here that the domain $\mathbf{D}$ is rather special, but this is done only for the sake of simplicity. In fact, many of the results will also be valid for some more complicated domains.

It will always be assumed that the unknown function $c = c(t, x_1, x_2 \ldots, x_Q)$ is continuously differentiable up to some order $2p$ with $p \geq 1$ in all points of the domain $\mathbf{D}$ and for all independent variables. Here p is the order of the numerical method which will be used in order to obtain some approximations of the unknown function at some mesh defined somehow in the domain (10). For some of the proofs it will also be necessary to assume that continuous derivatives up to order two of all functions u_q exist with respect to all independent variables.

The multi-dimensional advection equation (9) must always be considered together with appropriate initial and boundary conditions.

The following notation in connection with some given positive increments h_q will also be used in the paper:

(11) $\bar{x} = (x_1, x_2, \ldots, x_Q)$,

(12) $\bar{x}^{(+q)} = (x_1, x_2, \ldots, x_{q-1}, x_q + h_q, x_{q+1}, \ldots, x_Q), \quad q = 1, 2, \ldots, Q$,

(13) $\bar{x}^{(-q)} = (x_1, x_2, \ldots, x_{q-1}, x_q - h_q, x_{q+1}, \ldots, x_Q), \quad q = 1, 2, \ldots, Q$,

(14) $\bar{x}^{(+0.5q)} = (x_1, x_2, \ldots, x_{q-1}, x_q + 0.5h_q, x_{q+1}, \ldots, x_Q), \quad q = 1, 2, \ldots, Q$,

(15) $\bar{x}^{(-0.5q)} = (x_1, x_2, \ldots, x_{q-1}, x_q - 0.5h_q, x_{q+1}, \ldots, x_Q), \quad q = 1, 2, \ldots, Q$.

3. Expanding the unknown function in Taylor series

Theorem 1: Consider the multi-dimensional advection equation (9), assume that $(t, \bar{x}) \in D$ is an arbitrary but fixed point and introduce the increments $k > 0$ and $h_q > 0$ such that $t + k \in [a, b]$, $x_q - h_q \in [a_q, b_q]$ and $x_q + h_q \in [a_q, b_q]$ for all $q = 1, 2, \ldots, Q$. Assume furthermore that the unknown function $c = c(t, \bar{x})$ is continuously differentiable up to some order $2p$ with regard to all independent variables. Then there exists an expansion in Taylor series of the unknown function $c = c(t, \bar{x})$ around the point $(t + 0.5k, \bar{x})$ which contains terms involving only even degrees of the increments k and h_q ($q = 1, 2, \ldots, Q$).

Proof: It is clear that the following two formulae hold when the assumptions of Theorem 1 are satisfied:

(16) $$c(t + k, \bar{x}) = c(t + \frac{k}{2}, \bar{x}) + \frac{k}{2} \frac{\partial c(t + 0.5k, \bar{x})}{\partial t} + \sum_{s=2}^{2p-1} \frac{k^s}{2^s s!} \frac{\partial^s c(t + 0.5k, \bar{x})}{\partial t^s} + O(k^{2p}),$$

$$(17)\qquad c(t,\bar{x}) = c(t+\frac{k}{2},\bar{x}) - \frac{k}{2}\frac{\partial c(t+0.5k,\bar{x})}{\partial t} + \sum_{s=2}^{2p-1}(-1)^s \frac{k^s}{2^s s!}\frac{\partial^s c(t+0.5k,\bar{x})}{\partial t^s} + O(k^{2p}).$$

Eliminate the quantity $c(t+0.5k,\bar{x})$ from (16) and (17), which can be achieved by subtracting (17) from (16). The result is:

$$(18)\qquad c(t+k,\bar{x}) - c(t,\bar{x}) = k\frac{\partial c(t+0.5k,\bar{x})}{\partial t} + 2\sum_{s=1}^{p-1}\frac{k^{2s+1}}{2^{2s+1}(2s+1)!}\frac{\partial^{2s+1} c(t+0.5k,\bar{x})}{\partial t^{2s+1}} + O(k^{2p}).$$

The last equality can be rewritten as

$$(19)\qquad \frac{\partial c(t+0.5k,\bar{x})}{\partial t} = \frac{c(t+k,\bar{x}) - c(t,\bar{x})}{k} - \sum_{s=1}^{p-1}\frac{k^{2s}}{2^{2s}(2s+1)!}\frac{\partial^{2s+1} c(t+0.5k,\bar{x})}{\partial t^{2s+1}} + O(k^{2p-1}).$$

Consider the following two relationships:

$$(20)\qquad \frac{\partial c(t+k,\bar{x})}{\partial x_q} = \frac{\partial c(t+0.5k,\bar{x})}{\partial x_q} + \sum_{s=1}^{2p-1}\frac{k^s}{2^s s!}\frac{\partial^{s+1} c(t+0.5k,\bar{x})}{\partial t^s \partial x_q} + O(k^{2p}),$$

$$(21)\qquad \frac{\partial c(t,\bar{x})}{\partial x_q} = \frac{\partial c(t+0.5k,\bar{x})}{\partial x_q} + \sum_{s=1}^{2p-1}(-1)^s\frac{k^s}{2^s s!}\frac{\partial^{s+1} c(t+0.5k,\bar{x})}{\partial t^s \partial x_q} + O(k^{2p}).$$

Add (20) to (21). The result is:

$$(22)\qquad \frac{\partial c(t+k,\bar{x})}{\partial x_q} + \frac{\partial c(t,\bar{x})}{\partial x_q} = 2\frac{\partial c(t+0.5k,\bar{x})}{\partial x_q} + 2\sum_{s=1}^{p-1}\frac{k^{2s}}{2^{2s}(2s)!}\frac{\partial^{2s+1} c(t+0.5k,\bar{x})}{\partial t^{2s} \partial x_q} + O(k^{2p}).$$

The last equality can be rewritten as:

$$(23)\qquad \frac{\partial c(t+0.5k,\bar{x})}{\partial x_q} = \frac{1}{2}\left[\frac{\partial c(t+k,\bar{x})}{\partial x_q} + \frac{\partial c(t,\bar{x})}{\partial x_q}\right] - \sum_{s=1}^{p-1}\frac{k^{2s}}{2^{2s}(2s)!}\frac{\partial^{2s+1} c(t+0.5k,\bar{x})}{\partial t^{2s} \partial x_q} + O(k^{2p}).$$

Now the following four relationships can be written:

$$(24)\qquad c(t+k,\bar{x}^{(+q)}) = c(t+k,\bar{x}) + h_q \frac{\partial c(t+k,\bar{x})}{\partial x_q} + \sum_{s=2}^{2p-1} \frac{h_q^s}{s!} \frac{\partial^s c(t+k,\bar{x})}{\partial x_q^s} + O(h_q^{2p}),$$

$$(25)\qquad c(t+k,\bar{x}^{(-q)}) = c(t+k,\bar{x}) - h_q \frac{\partial c(t+k,\bar{x})}{\partial x_q} + \sum_{s=2}^{2p-1} (-1)^s \frac{h_q^s}{s!} \frac{\partial^s c(t+k,\bar{x})}{\partial x_q^s} + O(h_q^{2p}),$$

$$(26)\qquad c(t,\bar{x}^{(+q)}) = c(t,\bar{x}) + h_q \frac{\partial c(t,\bar{x})}{\partial x_q} + \sum_{s=2}^{2p-1} \frac{h_q^s}{s!} \frac{\partial^s c(t,\bar{x})}{\partial x_q^s} + O(h_q^{2p}),$$

$$(27)\qquad c(t,\bar{x}^{(-q)}) = c(t,\bar{x}) - h_q \frac{\partial c(t,\bar{x})}{\partial x_q} + \sum_{s=2}^{2p-1} (-1)^s \frac{h_q^s}{s!} \frac{\partial^s c(t,\bar{x})}{\partial x_q^s} + O(h_q^{2p}).$$

Subtract (25) from (24) to obtain:

$$(28)\qquad \frac{\partial c(t+k,\bar{x})}{\partial x_q} = \frac{c(t+k,\bar{x}^{(+q)}) - c(t+k,\bar{x}^{(-q)})}{2h_q}$$

$$- \sum_{s=1}^{p-1} \frac{h_q^{2s}}{(2s+1)!} \frac{\partial^{2s+1} c(t+k,\bar{x})}{\partial x_q^{2s+1}} + O(h_q^{2p-1}).$$

Similarly, the following relationship can be obtained by subtracting (27) from (26):

$$(29)\qquad \frac{\partial c(t,\bar{x})}{\partial x_q} = \frac{c(t,\bar{x}^{(+q)}) - c(t,\bar{x}^{(-q)})}{2h_q} - \sum_{s=1}^{p-1} \frac{h_q^{2s}}{(2s+1)!} \frac{\partial^{2s+1} c(t,\bar{x})}{\partial x_q^{2s+1}} + O(h_q^{2p-1}).$$

Assume that $(t+0.5k,\bar{x}) = (t+0.5k, x_1, x_2, \ldots, x_Q)$ is some arbitrary but fixed point. Then use the abbreviation $u_q(t+0.5k,\bar{x}) = u_q(t+0.5k, x_1, x_2, \ldots, x_Q)$ in order to obtain, from (9), the following formula:

$$\textbf{(30)} \qquad \frac{\partial c(t+0.5k,\bar{x})}{\partial t} = -\sum_{q=1}^{Q} u_q(t+0.5k,\bar{x}) \frac{\partial c(t+0.5k,\bar{x})}{\partial x_q} .$$

Use (19) and (23) in (30) to obtain:

$$\textbf{(31)} \qquad \frac{c(t+k,\bar{x}) - c(t,\bar{x})}{k} = -\sum_{q=1}^{Q} u_q(t+0.5k,\bar{x}) \left\{ \frac{1}{2} \left[\frac{\partial c(t+k,\bar{x})}{\partial x_q} + \frac{\partial c(t,\bar{x})}{\partial x_q} \right] \right\}$$

$$+ \sum_{s=1}^{p-1} \frac{k^{2s}}{2^{2s}(2s+1)!} \frac{\partial^{2s+1} c(t+0.5k,\bar{x})}{\partial t^{2s+1}}$$

$$+ \sum_{q=1}^{Q} u_q(t+0.5k,\bar{x}) \sum_{s=1}^{p-1} \frac{k^{2s}}{2^{2s}(2s)!} \frac{\partial^{2s+1} c(t+0.5k,\bar{x})}{\partial t^{2s} \partial x_q} + O(k^{2p-1}).$$

The last equality can be rewritten as

$$\textbf{(32)} \qquad \frac{c(t+k,\bar{x}) - c(t,\bar{x})}{k} = -\sum_{q=1}^{Q} u_q(t+0.5k,\bar{x}) \left\{ \frac{1}{2} \left[\frac{\partial c(t+k,\bar{x})}{\partial x_q} + \frac{\partial c(t,\bar{x})}{\partial x_q} \right] \right\}$$

$$+ \sum_{s=1}^{p-1} \frac{k^{2s}}{2^{2s}(2s)!} \left\{ \frac{1}{2s+1} \frac{\partial^{2s+1} c(t+0.5k,\bar{x})}{\partial t^{2s+1}} + \sum_{q=1}^{Q} u_q(t+0.5k,\bar{x}) \frac{\partial^{2s+1} c(t+0.5k,\bar{x})}{\partial t^{2s} \partial x_q} \right\}$$

$$+ O(k^{2p-1}) .$$

Denote:

$$\textbf{(33)} \quad K_t^{(2s)} = \frac{1}{2^{2s}(2s)!} \left[\frac{1}{2s+1} \frac{\partial^{2s+1} c(t+0.5k,\bar{x})}{\partial t^{2s+1}} + \sum_{q=1}^{Q} u_q(t+0.5k,\bar{x}) \frac{\partial^{2s+1} c(t+0.5k,\bar{x})}{\partial t^{2s} \partial x_q} \right].$$

Then (32) can be rewritten as:

$$\textbf{(34)}\qquad \frac{c(t+k,\bar{x})-c(t,\bar{x})}{k} = -\sum_{q=1}^{Q} u_q(t+0.5k,\bar{x}) \left\{ \frac{1}{2}\left[\frac{\partial c(t+k,\bar{x})}{\partial x_q} + \frac{\partial c(t,\bar{x})}{\partial x_q} \right] \right\}$$

$$+\sum_{s=1}^{p-1} k^{2s} K_t^{(2s)} + O(k^{2p-1}).$$

Use (28) and (29) in the expression in the square bracket of (34) to obtain

$$\textbf{(35)}\qquad \frac{c(t+k,\bar{x})-c(t,\bar{x})}{k} = -\sum_{q=1}^{Q} u_q(t+0.5k,\bar{x}) \frac{c(t+k,\bar{x}^{(+q)})-c(t+k,\bar{x}^{(-q)})}{4h_q}$$

$$-\sum_{q=1}^{Q} u_q(t+0.5k,\bar{x}) \frac{c(t,\bar{x}^{(+q)})-c(t,\bar{x}^{(-q)})}{4h_q}$$

$$+\frac{1}{2}\sum_{q=1}^{Q} u_q(t+0.5k,\bar{x}) \left\{ \sum_{s=1}^{p-1} \frac{h_q^{2s}}{(2s+1)!} \left[\frac{\partial^{2s+1} c(t+k,\bar{x})}{\partial x_q^{2s+1}} + \frac{\partial^{2s+1} c(t,\bar{x})}{\partial x_q^{2s+1}} \right] \right\}$$

$$+\sum_{s=1}^{p-1} k^{2s} K_t^{(2s)} + O(k^{2p-1}) + O(h_q^{2p-1}).$$

The last equality can be rewritten in the following form:

$$(36)\qquad \frac{c(t+k,\bar{x})-c(t,\bar{x})}{k} = -\sum_{q=1}^{Q} u_q(t+0.5k,\bar{x})\,\frac{c(t+k,\bar{x}^{(+q)})-c(t+k,\bar{x}^{(-q)})}{4h_q}$$

$$-\sum_{q=1}^{Q} u_q(t+0.5k,\bar{x})\,\frac{c(t,\bar{x}^{(+q)})-c(t,\bar{x}^{(-q)})}{4h_q}$$

$$+\sum_{s=1}^{p-1}\left\{\sum_{q=1}^{Q}\frac{1}{2}\,\frac{h_q^{2s}}{(2s+1)!}u_q(t+0.5k,\bar{x})\left[\frac{\partial^{2s+1}c(t+k,\bar{x})}{\partial x_q^{2s+1}}+\frac{\partial^{2s+1}c(t,\bar{x})}{\partial x_q^{2s+1}}\right]\right\}$$

$$+\sum_{s=1}^{p-1}k^{2s}\,K_t^{(2s)}+O(k^{2p-1})+O(h_q^{2p-1}).$$

Denote:

$$(37)\qquad K_q^{(2s)}=\frac{1}{2}\,\frac{1}{(2s+1)!}\sum_{q=1}^{Q}u_q(t+0.5k,\bar{x})\left[\frac{\partial^{2s+1}c(t+k,\bar{x})}{\partial x_q^{2s+1}}+\frac{\partial^{2s+1}c(t,\bar{x})}{\partial x_q^{2s+1}}\right].$$

Substitute this value of $K_q^{(2s)}$ in (36). The result is:

$$(38)\qquad \frac{c(t+k,\bar{x})-c(t,\bar{x})}{k} = -\sum_{q=1}^{Q} u_q(t+0.5k,\bar{x})\,\frac{c(t+k,\bar{x}^{(+q)})-c(t+k,\bar{x}^{(-q)})}{4h_q}$$

$$-\sum_{q=1}^{Q} u_q(t+0.5k,\bar{x})\,\frac{c(t,\bar{x}^{(+q)})-c(t,\bar{x}^{(-q)})}{4h_q}$$

$$+\sum_{s=1}^{p-1}\left[k^{2s}K_t^{(2s)}+\sum_{q=1}^{Q}h_q^{2s}K_q^{(2s)}\right]+O(k^{2p-1})+O(h_q^{2p-1}).$$

The last equality can also be rewritten as:

$$(39)\qquad \frac{c(t+k,\bar{x})-c(t,\bar{x})}{k} = -\sum_{q=1}^{Q} u_q(t+0.5k,\bar{x})\ \frac{c(t+k,\bar{x}^{(+q)})-c(t+k,\bar{x}^{(-q)})}{4h_q}$$

$$-\sum_{q=1}^{Q} u_q(t+0.5k,\bar{x})\ \frac{c(t,\bar{x}^{(+q)})-c(t,\bar{x}^{(-q)})}{4h_q}$$

$$+\sum_{s=1}^{p-1} k^{2s}K^{(2s)} + O(k^{2p-1}),$$

where

$$(40)\qquad K^{(2s)} = K_t^{(2s)} + \sum_{q=1}^{Q} \frac{h_q^{2s}}{k^{2s}} K_q^{(2s)} .$$

Assume that all ratios h_q/k, $q=1,2,\ldots,Q$, remain constants when $k \to 0$ (which can easily be achieved for example by reducing all h_q by a factor of two when k is reduced by a factor of two). Then, the last two equalities, (39) and (40), show that the assertion of Theorem 1 holds.

Corollary 1 (the one-dimensional case): If $Q=1$ then (9) can be written as

$$(41)\qquad \frac{\partial c}{\partial t} = -u_1 \frac{\partial c}{\partial x_1}, \qquad x_1 \in [a_1, b_1], \qquad t \in [a,b],$$

and (39) reduces to the following equality:

$$(42) \qquad \frac{c(t+k,x_1)-c(t,x_1)}{k} = -u_1(t+0.5k,x_1)\frac{c(t+k,x_1+h_1)-c(t+k,x_1-h_1)}{4h_1}$$

$$-u_1(t+0.5k,x_1)\frac{c(t,x_1+h_1)-c(t,x_1-h_1)}{4h_1}$$

$$+k^2K^{(2)}+k^4K^{(4)}+\ldots+k^{2p-2}K^{(2p-2)}+O(k^{2p-1})+O(h_1^{2p-1}),$$

where the values of $K^{(2s)}$, $s = 1, 2, \ldots, 2p - 2$ for the one-dimensional case can be obtained in an obvious way from (33), (37) and (40).

Corollary 2 (the two-dimensional case): If $Q = 2$ then (9) can be written as

$$(43) \qquad \frac{\partial c}{\partial t} = -u_1\frac{\partial c}{\partial x_1} - u_2\frac{\partial c}{\partial x_2}, \qquad x_1 \in [a_1, b_1], \quad x_2 \in [a_2, b_2], \quad t \in [a,b],$$

and (39) reduces to the following equality:

$$(44) \quad \frac{c(t+k,x_1,x_2)-c(t,x_1,x_2)}{k} = -u_1(t+0.5k,x_1,x_2)\frac{c(t+k,x_1+h_1,x_2)-c(t+k,x_1-h_1,x_2)}{4h_1}$$

$$-u_1(t+0.5k,x_1,x_2)\frac{c(t,x_1+h_1,x_2)-c(t,x_1-h_1,x_2)}{4h_1}$$

$$-u_2(t+0.5k,x_1,x_2)\frac{c(t+k,x_1,x_2+h_2)-c(t+k,x_1,x_2-h_2)}{4h_2}$$

$$-u_2(t+0.5k,x_1,x_2)\frac{c(t,x_1,x_2+h_2)-c(t,x_1,x_2-h_2)}{4h_2}$$

$$+k^2K^{(2)}+k^4K^{(4)}+\ldots+k^{2p-2}K^{(2p-2)}+O(k^{2p-1})+O(h_1^{2p-1})+O(h_2^{2p-1}),$$

where the values of $\mathbf{K}^{(2s)}$, $\mathbf{s = 1, 2, \ldots, 2p - 2}$ for the two-dimensional case can be obtained in an obvious way from (33), (37) and (40).

Corollary 3 (the three-dimensional case): If $\mathbf{Q = 3}$ then (9) can be written as

$$\textbf{(45)} \quad \frac{\partial \mathbf{c}}{\partial \mathbf{t}} = -\mathbf{u_1} \frac{\partial \mathbf{c}}{\partial \mathbf{x}} - \mathbf{u_2} \frac{\partial \mathbf{c}}{\partial \mathbf{y}} - \mathbf{u_3} \frac{\partial \mathbf{c}}{\partial \mathbf{z}}, \quad \mathbf{x} \in \left[\mathbf{a_1}, \mathbf{b_1} \right], \quad \mathbf{y} \in \left[\mathbf{a_2}, \mathbf{b_2} \right], \quad \mathbf{z} \in \left[\mathbf{a_3}, \mathbf{b_3} \right], \quad \mathbf{t} \in [\mathbf{a}, \mathbf{b}],$$

and (39) reduces to the following equality:

$$\textbf{(46)} \qquad \frac{c(t+k,x_1,x_2,x_3)-c(t,x_1,x_2,x_3)}{k}$$

$$= -u_1(t+0.5k,x_1,x_2,x_3)\frac{c(t+k,x_1+h_1,x_2,x_3)-c(t+k,x_1-h_1,x_2,x_3)}{4h_1}$$

$$-u_1(t+0.5k,x_1,x_2,x_3)\frac{c(t,x_1+h_1,x_2,x_3)-c(t,x_1-h_1,x_2,x_3)}{4h_1}$$

$$-u_2(t+0.5k,x_1,x_2,x_3)\frac{c(t+k,x_1,x_2+h_2,x_3)-c(t+k,x_1,x_2-h_2,x_3)}{4h_2}$$

$$-u_2(t+0.5k,x_1,x_2,x_3)\frac{c(t,x_1,x_2+h_2,x_3)-c(t,x_1,x_2-h_2,x_3)}{4h_2}$$

$$-u_3(t+0.5k,x_1,x_2,x_3)\frac{c(t+k,x_1,x_2,x_3+h_3)-c(t+k,x_1,x_2,x_3-h_3)}{4h_3}$$

$$-u_3(t+0.5k,x_1,x_2,x_3)\frac{c(t,x_1,x_2,x_3+h_3)-c(t,x_1,x_2,x_3-h_3)}{4h_3}$$

$$+k^2K^{(2)}+k^4K^{(4)}+\ldots+k^{2p-2}K^{(2p-2)}++O(k^{2p-1})$$

$$+O(h_1^{2p-1})+O(h_2^{2p-1})+O(h_3^{2p-1}) \;,$$

where the values of $K^{(2s)}$, $s = 1, 2, \ldots, 2p - 2$ for the three-dimensional case can be obtained in an obvious way from (33), (37) and (40).

4. Designing a second-order numerical method

Consider the grids:

$$\textbf{(47)} \qquad G_t = \left\{ t_n ,\ n = 0,1,\ldots,N_t \ \middle|\ t_0 = a,\ t_n = t_{n-1} + k,\ n = 1,2,\ \ldots,N_t,\ k = \frac{b-a}{N_t},\ t_{N_t} = b \right\}$$

and (for $q = 1,\ 2,\ \ldots,\ Q$ and $h_q = (b_q - a_q)/N_q$)

$$\textbf{(48)} \quad G_x^{(q)} = \left\{ x_q^{i_q},\ i_q = 0,1,\ldots,N_q \,|\, x_q^0 = a_q,\ x_q^{i_q} = x_q^{i_q - 1} + h_q,\ i = 1,2,\ldots,\ N_q,\ \ x_q^{N_q} = b_q \right\}$$

Consider

$$\textbf{(49)} \qquad \tilde{x} = \left(x_1^{i_1}, x_2^{i_2}, \ldots, x_Q^{i_Q} \right),$$

$$\textbf{(50)} \qquad \tilde{x}^{(+q)} = \left(x_1^{i_1}, x_2^{i_2}, \ldots, x_{q-1}^{i_{q-1}}, x_q^{i_q} + h_q, x_{q+1}^{i_{q+1}}, \ldots, x_Q^{i_Q} \right),$$

$$\textbf{(51)} \qquad \tilde{x}^{(-q)} = \left(x_1^{i_1}, x_2^{i_2}, \ldots, x_{q-1}^{i_{q-1}}, x_q^{i_q} - h_q, x_{q+1}^{i_{q+1}}, \ldots, x_Q^{i_Q} \right),$$

where $x_q^{i_q} \in G_x^{(q)}$ **for** $q = 1,\ 2,\ \ldots,\ Q$.

In this notation the following numerical method can be defined:

$$\textbf{(52)} \quad \frac{\tilde{c}(t_{n+1},\tilde{x}) - \tilde{c}(t_n,\tilde{x})}{k}$$

$$= -\sum_{q=1}^{Q} u_q(t_n + 0.5k, \tilde{x}) \ \frac{\tilde{c}(t_{n+1},\tilde{x}^{(+q)}) - \tilde{c}(t_{n+1},\tilde{x}^{(-q)}) + \tilde{c}(t_n,\tilde{x}^{(+q)}) - \tilde{c}(t_n,\tilde{x}^{(-q)})}{4h_q}$$

It is clear that (52) can be obtained from (39) by neglecting the terms in the last line and assuming that an arbitrary inner point of the grids defined by (47) and (48) is considered.

The quantities $\tilde{c}(t_n, \tilde{x})$ can be considered as approximations of the exact values of the unknown function $c(t_n, \tilde{x})$ at the grid-points defined by (47) and (48). It is clear that the method defined by (52) is of order two with respect to all independent variables.

Assume that the values of $\tilde{c}(t_n, \tilde{x})$ have been calculated for all grid-points of (52). Then the values $\tilde{c}(t_{n+1}, \tilde{x})$ of the unknown function at the next time-point $t_{n+1} = t_n + k$ can be obtained by solving a huge system of linear algebraic equations of dimension $\tilde{N}$ where $\tilde{N}$ is defined by

$$\text{(53)} \qquad \tilde{N} = \prod_{q=1}^{Q} \left(N_q - 1 \right)$$

5. Application of Richardson Extrapolation

Consider (52) with $\tilde{c}$ replaced by z when $t = t_{n+1}$

$$\text{(54)} \qquad \frac{z(t_n + k, \tilde{x}) - \tilde{c}(t_n, \tilde{x})}{k}$$

$$= - \sum_{q=1}^{Q} u_q(t_n + 0.5k, \tilde{x}) \frac{z(t_n + k, \tilde{x}^{(+q)}) - z(t_n + k, \tilde{x}^{(-q)}) + \tilde{c}(t_n, \tilde{x}^{(+q)}) - \tilde{c}(t_n, \tilde{x}^{(-q)})}{4h_q}$$

Suppose that $0.5k$ and $0.5h_q$ are considered instead of k and h_q $(q = 1, 2, \ldots, Q)$ respectively. Consider, as in formulae (14) and (15) but in the points of the grids (47) and (48), the two vectors $\tilde{x}^{(+0.5q)} = (x_1^{i_1}, x_2^{i_2}, \ldots x_{q-1}^{i_{q-1}}, x_q^{i_q} + 0.5h_q, x_{q+1}^{i_{q+1}}, \ldots, x_Q^{i_{N_q}})$ and $\tilde{x}^{(-0.5q)} = (x_1^{i_1}, x_2^{i_2}, \ldots x_{q-1}^{i_{q-1}}, x_q^{i_q} - 0.5h_q, x_{q+1}^{i_{q+1}}, \ldots, x_Q^{i_{N_q}})$ for $q = 1, 2, \ldots, Q$.

Write the following two formulae:

$$(55)\quad \frac{w(t_n+0.5k,\tilde{x})-\tilde{c}(t_n,\tilde{x})}{0.5k} = -\sum_{q=1}^{Q} u_q(t_n+0.25k,\tilde{x})\,\frac{w(t_n+0.5k,\tilde{x}^{(+0.5q)})-w(t_n+0.5k,\tilde{x}^{(-0.5q)})}{4(0.5h_q)}$$

$$-\sum_{q=1}^{Q} u_q(t_n+0.25k,\tilde{x})\,\frac{\tilde{c}(t_n,\tilde{x}^{(+0.5q)})-\tilde{c}(t_n,\tilde{x}^{(-0.5q)})}{4(0.5h_q)},$$

$$(56)\quad \frac{w(t_n+k,\tilde{x})-w(t_n+0.5k,\tilde{x})}{0.5k} = -\sum_{q=1}^{Q} u_q(t_n+0.75k,\tilde{x})\,\frac{w(t_n+k,\tilde{x}^{(+0.5q)})-w(t_n+k,\tilde{x}^{(-0.5q)})}{4(0.5h_q)}$$

$$-\sum_{q=1}^{Q} u_q(t_n+0.75k,\tilde{x})\,\frac{w(t_n+0.5k,\tilde{x}^{(+0.5q)})-w(t_n+0.5k,\tilde{x}^{(-0.5q)})}{4(0.5h_q)}.$$

Add (56) to (55) and multiply by 0.5 the obtained equation. The result is:

$$(57)\quad \frac{w(t_n+k,\tilde{x})-\tilde{c}(t_n,\tilde{x})}{k} = -\sum_{q=1}^{Q} u_q(t_n+0.75k,\tilde{x})\,\frac{w(t_n+k,\tilde{x}^{(+0.5q)})-w(t_n+k,\tilde{x}^{(-0.5q)})}{4h_q}$$

$$-\sum_{q=1}^{Q} u_q(t_n+0.75k,\tilde{x})\,\frac{w(t_n+0.5k,\tilde{x}^{(+0.5q)})-w(t_n+0.5k,\tilde{x}^{(-0.5q)})}{4h_q}$$

$$-\sum_{q=1}^{Q} u_q(t_n+0.25k,\tilde{x})\,\frac{w(t_n+0.5k,\tilde{x}^{(+0.5q)})-w(t_n+0.5k,\tilde{x}^{(-0.5q)})}{4h_q}$$

$$-\sum_{q=1}^{Q} u_q(t_n+0.25k,\tilde{x})\,\frac{\tilde{c}(t_n,\tilde{x}^{(+0.5q)})-\tilde{c}(t_n,\tilde{x}^{(-0.5q)})}{4h_q}.$$

Multiply (54) by $1/3$ and (57) by $4/3$. Subtract the modified equality (54) from the modified equality (57). Then the following equality will be obtained:

$$(58)\qquad \frac{4}{3}\frac{w(t_n+k,\tilde{x})-\tilde{c}(t_n,\tilde{x})}{k}-\frac{1}{3}\frac{z(t_n+k,\tilde{x})-\tilde{c}(t_n,\tilde{x})}{k}$$

$$=-\frac{4}{3}\sum_{q=1}^{Q}u_q(t_n+0.75k,\tilde{x})\,\frac{w(t_n+k,\tilde{x}^{(+0.5q)})-w(t_n+k,\tilde{x}^{(-0.5q)})}{4h_q}$$

$$-\frac{4}{3}\sum_{q=1}^{Q}u_q(t_n+0.75k,\tilde{x})\,\frac{w(t_n+0.5k,\tilde{x}^{(+0.5q)})-w(t_n+0.5k,\tilde{x}^{(-0.5q)})}{4h_q}$$

$$-\frac{4}{3}\sum_{q=1}^{Q}u_q(t_n+0.25k,\tilde{x})\,\frac{w(t_n+0.5k,\tilde{x}^{(+0.5q)})-w(t_n+0.5k,\tilde{x}^{(-0.5q)})}{4h_q}$$

$$-\frac{4}{3}\sum_{q=1}^{Q}u_q(t_n+0.25k,\tilde{x})\,\frac{\tilde{c}(t_n,\tilde{x}^{(+0.5q)})-\tilde{c}(t_n,\tilde{x}^{(-0.5q)})}{4h_q}$$

$$+\frac{1}{3}\sum_{q=1}^{Q}u_q(t_n+0.5k,\tilde{x})\,\frac{z(t_n+k,\tilde{x}^{(+q)})-z(t_n+k,\tilde{x}^{(-q)})+\tilde{c}(t_n,\tilde{x}^{(+q)})-\tilde{c}(t_n,\tilde{x}^{(-q)})}{4h_q}.$$

Substitute the exact values of the unknown function instead of the approximations w, z and $\tilde{c}$ in (58):

$$(59)\qquad \frac{4}{3}\frac{c(t_n+k,\tilde{x})-c(t_n,\tilde{x})}{k}-\frac{1}{3}\frac{c(t_n+k,\tilde{x})-c(t_n,\tilde{x})}{k}$$

$$=-\frac{4}{3}\sum_{q=1}^{Q}u_q(t_n+0.75k,\tilde{x})\,\frac{c(t_n+k,\tilde{x}^{(+0.5q)})-c(t_n+k,\tilde{x}^{(-0.5q)})}{4\,h_q}$$

$$-\frac{4}{3}\sum_{q=1}^{Q}u_q(t_n+0.75k,\tilde{x})\,\frac{c(t_n+0.5k,\tilde{x}^{(+0.5q)})-c(t_n+0.5k,\tilde{x}^{(-0.5q)})}{4\,h_q}$$

$$-\frac{4}{3}\sum_{q=1}^{Q}u_q(t_n+0.25k,\tilde{x})\,\frac{c(t_n+0.5k,\tilde{x}^{(+0.5q)})-c(t_n+0.5k,\tilde{x}^{(-0.5q)})}{4\,h_q}$$

$$-\frac{4}{3}\sum_{q=1}^{Q}u_q(t_n+0.25k,\tilde{x})\,\frac{c(t_n,\tilde{x}^{(+0.5q)})-c(t_n,\tilde{x}^{(-0.5q)})}{4h_q}$$

$$+\frac{1}{3}\sum_{q=1}^{Q}u_q(t_n+0.5k,\tilde{x})\,\frac{c(t_n+k,\tilde{x}^{(+q)})-c(t_n+k,\tilde{x}^{(-q)})+c(t_n,\tilde{x}^{(+q)})-c(t_n,\tilde{x}^{(-q)})}{4h_q}$$

$$+\frac{4}{3}k^2\left(0.5\overline{K}^{(2)}+0.5\tilde{K}^{(2)}\right)-\frac{1}{3}k^2K^{(2)}+O(k^4)+\sum_{q=1}^{Q}O(h_q^4)$$

Equation (39) with $\mathbf{p=2}$ can be used, together with transformations similar to those made in Section 1 (some of the needed transformations will be performed in the remaining part of this section), in order to obtain the last terms in (59).

Subtract (58) from (59) and use the notation ε for the differences between the corresponding exact and approximate values of the unknown function. The result is:

$$(60) \qquad \frac{4}{3}\frac{\varepsilon(t_n+k,\tilde{x})-\varepsilon(t_n,\tilde{x})}{k}-\frac{1}{3}\frac{\varepsilon(t_n+k,\tilde{x})-\varepsilon(t_n,\tilde{x})}{k}$$

$$=-\frac{4}{3}\sum_{q=1}^{Q}u_q(t_n+0.75k,\tilde{x})\,\frac{\varepsilon(t_n+k,\tilde{x}^{(+0.5q)})-\varepsilon(t_n+k,\tilde{x}^{(-0.5)})}{4h_q}$$

$$-\frac{4}{3}\sum_{q=1}^{Q}u_q(t_n+0.75k,\tilde{x})\,\frac{\varepsilon(t_n+0.5k,\tilde{x}^{(+0.5q)})-\varepsilon(t_n+0.5k,\tilde{x}^{(+0.5)})}{4h_q}$$

$$-\frac{4}{3}\sum_{q=1}^{Q}u_q(t_n+0.25k,\tilde{x})\,\frac{\varepsilon(t_n+0.5k,\tilde{x}^{(+0.5)})-\varepsilon(t_n+0.5k,\tilde{x}^{(-0.5)})}{4h_q}$$

$$-\frac{4}{3}\sum_{q=1}^{Q}u_q(t_n+0.25k,\tilde{x})\,\frac{\varepsilon(t_n,\tilde{x}^{(+0.5q)})-\varepsilon(t_n,\tilde{x}^{(-0.5)})}{4\,h_q}$$

$$+\frac{1}{3}\sum_{q=1}^{Q}u_q(t_n+0.5k,\tilde{x})\,\frac{\varepsilon(t_n+k,\tilde{x}^{(+q)})-\varepsilon(t_n+k,\tilde{x}^{(-q)})+\varepsilon(t_n,\tilde{x}^{(+q)})-\varepsilon(t_n,\tilde{x}^{(-q)})}{4h_q}$$

$$+k^2\left[\frac{2}{3}\left(\bar{K}^{(2)}+\tilde{K}^{(2)}\right)-\frac{1}{3}K^{(2)}\right]+O(k^4)+\sum_{q=1}^{Q}O(h_q^4)$$

The interesting term in (60) is the expression in the square brackets:

$$\textbf{(61)} \qquad \hat{\mathbf{K}} = \frac{2}{3}\left(\bar{\mathbf{K}}^{(2)} + \tilde{\mathbf{K}}^{(2)} \right) - \frac{1}{3}\mathbf{K}^{(2)}$$

$$= \frac{2}{3}\left\{ \left[\bar{\mathbf{K}}_{\mathbf{t}}^{(2)} + \sum_{\mathbf{q}=1}^{\mathbf{Q}} \frac{\mathbf{h}_{\mathbf{q}}^2}{\mathbf{k}^2} \bar{\mathbf{K}}_{\mathbf{q}}^{(2)} \right] + \left[\tilde{\mathbf{K}}_{\mathbf{t}}^{(2)} + \sum_{\mathbf{q}=1}^{\mathbf{Q}} \frac{\mathbf{h}_{\mathbf{q}}^2}{\mathbf{k}^2} \tilde{\mathbf{K}}_{\mathbf{q}}^{(2)} \right] \right\}$$

$$- \frac{1}{3}\left[\mathbf{K}_{\mathbf{t}}^{(2)} + \sum_{\mathbf{q}=1}^{\mathbf{Q}} \frac{\mathbf{h}_{\mathbf{q}}^2}{\mathbf{k}^2} \mathbf{K}_{\mathbf{q}}^{(2)} \right]$$

$$= \frac{1}{3}\left[2\bar{\mathbf{K}}_{\mathbf{t}}^{(2)} + 2\tilde{\mathbf{K}}_{\mathbf{t}}^{(2)} - \mathbf{K}_{\mathbf{t}}^{(2)} \right] + \frac{1}{3} \sum_{\mathbf{q}=1}^{\mathbf{Q}} \frac{\mathbf{h}_{\mathbf{q}}^2}{\mathbf{k}^2} \left[2\bar{\mathbf{K}}_{\mathbf{q}}^{(2)} + 2\tilde{\mathbf{K}}_{\mathbf{q}}^{(2)} - \mathbf{K}_{\mathbf{q}}^{(2)} \right]$$

In the derivation of (61) it is assumed that the quantities $\bar{\mathbf{K}}^{(2s)}$ and $\tilde{\mathbf{K}}^{(2s)}$ have the same structure as the expression for $\mathbf{K}^{(2s)}$ in (40). It is immediately clear that this assumption is true (some additional information will be given in the remaining part of this section). Furthermore $\mathbf{s} = \mathbf{1}$ is used to obtain all three quantities involved in the last line of (61).

It is quite clear that if $\hat{\mathbf{K}} = \mathbf{O}(\mathbf{k}^2)$ then the Richardson Extrapolation will be a numerical method of order four. Consider the last row of (61). It obvious that it will be quite sufficient to prove that the terms of the expression in the first square brackets are of order $\mathbf{O}(\mathbf{k}^2)$ and that also the terms of the expression in the second square brackets of (61) are of order $\mathbf{O}(\mathbf{k}^2)$ for an arbitrary value of $\mathbf{q}$.

The following equality can easily be obtained from (33) by setting $\mathbf{s} = \mathbf{1}$ and replacing $\bar{\mathbf{x}}$ with $\tilde{\mathbf{x}}$:

$$\textbf{(62)} \qquad \mathbf{K}_{\mathbf{t}}^{(2)} = \frac{1}{2^2} \frac{1}{(2)!} \left[\frac{1}{3} \frac{\partial^3 \mathbf{c}(\mathbf{t}_{\mathbf{n}} + 0.5\mathbf{k}, \tilde{\mathbf{x}})}{\partial \mathbf{t}^3} + \sum_{\mathbf{q}=1}^{\mathbf{Q}} \mathbf{u}_{\mathbf{q}}(\mathbf{t}_{\mathbf{n}} + 0.5\mathbf{k}, \tilde{\mathbf{x}}) \frac{\partial^3 \mathbf{c}\left(\mathbf{t}_{\mathbf{n}} + 0.5\mathbf{k}, \tilde{\mathbf{x}}\right)}{\partial \mathbf{t}^2 \partial \mathbf{x}_{\mathbf{q}}} \right]$$

Similar expressions for the quantities $\bar{\mathbf{K}}_{\mathbf{t}}^{(2)}$ and $\tilde{\mathbf{K}}_{\mathbf{t}}^{(2)}$ as well as for the quantities $\bar{\mathbf{K}}_{\mathbf{q}}^{(2)}$ and $\tilde{\mathbf{K}}_{\mathbf{q}}^{(2)}$ are derived below by following closely the procedure applied in Section 1.

Consider the point $(\mathbf{t_n} + \mathbf{0.25k}, \tilde{\mathbf{x}})$ and the following two relationships:

$$\textbf{(63)} \qquad c(t_n + 0.5k, \tilde{x}) = c(t_n + 0.25k, \tilde{x}) + \frac{k}{4}\frac{\partial c(t_n + 0.25k, \tilde{x})}{\partial t}$$

$$+ \sum_{s=2}^{2p-1} \frac{k^s}{4^s s!} \frac{\partial^s c(t_n + 0.25k, \tilde{x})}{\partial t^s} + O(k^{2p}),$$

$$\textbf{(64)} \qquad c(t_n, \tilde{x}) = c(t_n + 0.25k, \tilde{x}) - \frac{k}{4}\frac{\partial c(t_n + 0.25k, \tilde{x})}{\partial t}$$

$$+ \sum_{s=2}^{2p-1} (-1)^s \frac{k^s}{4^s s!} \frac{\partial^s c(t_n + 0.25k, \tilde{x})}{\partial t^s} + O(k^{2p}).$$

Eliminate the quantity $\mathbf{c(t_n} + \mathbf{0.25k}, \tilde{\mathbf{x}})$ from (63) and (64), which can be achieved by subtracting (64) from (63). The result is:

$$\textbf{(65)} \qquad c(t_n + 0.5k, \tilde{x}) - c(t_n, \bar{x}) = \frac{k}{2}\frac{\partial c(t_n + 0.25k, \tilde{x})}{\partial t}$$

$$+ 2 \sum_{s=1}^{p-1} \frac{k^{2s+1}}{4^{2s+1}(2s+1)!} \frac{\partial^{2s+1} c(t_n + 0.25k, \tilde{x})}{\partial t^{2s+1}} + O(k^{2p}).$$

The last equality can be rewritten as

$$\textbf{(66)} \qquad \frac{\partial c(t_n + 0.25k, \tilde{x})}{\partial t} = \frac{c(t_n + 0.5k, \tilde{x}) - c(t_n, \tilde{x})}{0.5k}$$

$$- \sum_{s=1}^{p-1} \frac{k^{2s}}{4^{2s}(2s+1)!} \frac{\partial^{2s+1} c(t_n + 0.25k, \tilde{x})}{\partial t^{2s+1}} + O(k^{2p-1}).$$

It can be seen that (66) can be obtained from (19) by replacing $\mathbf{t+k}$ with $\mathbf{t_n+0.5k}$, $\mathbf{t+0.5k}$ with $\mathbf{t_n+0.25k}$, $\bar{\mathbf{x}}$ with $\tilde{\mathbf{x}}$ and 2^{2s} with 4^{2s} which is quite understandable.

Consider the following two relationships:

$$\textbf{(67)} \qquad \frac{\partial c(t_n+0.5k,\tilde{x})}{\partial x_q} = \frac{\partial c(t_n+0.25k,\tilde{x})}{\partial x_q} + \sum_{s=1}^{2p-1} \frac{k^s}{4^s s!} \frac{\partial^{s+1} c(t_n+0.25k,\tilde{x})}{\partial t^s \partial x_q} + O(k^{2p}) ,$$

$$\textbf{(68)} \qquad \frac{\partial c(t_n,\tilde{x})}{\partial x_q} = \frac{\partial c(t_n+0.25k,\tilde{x})}{\partial x_q} + \sum_{s=1}^{2p-1} (-1)^s \frac{k^s}{4^s s!} \frac{\partial^{s+1} c(t_n+0.25k,\tilde{x})}{\partial t^s \partial x_q} + O(k^{2p}).$$

Add (67) to (68). The result is:

$$\textbf{(69)} \qquad \frac{\partial c(t_n+0.5k,\tilde{x})}{\partial x_q} + \frac{\partial c(t_n,\tilde{x})}{\partial x_q} = 2\,\frac{\partial c(t_n+0.25k,\tilde{x})}{\partial x_q}$$

$$+\,2\sum_{s=1}^{p-1} \frac{k^{2s}}{4^{2s}(2s)!} \frac{\partial^{2s+1} c(t_n+0.25k,\tilde{x})}{\partial t^{2s} \partial x_q} + O(k^{2p}).$$

The last equality can be rewritten as:

$$\textbf{(70)} \qquad \frac{\partial c(t_n+0.25k,\tilde{x})}{\partial x_q} = \frac{1}{2}\left[\frac{\partial c(t_n+0.5k,\tilde{x})}{\partial x_q} + \frac{\partial c(t_n,\tilde{x})}{\partial x_q}\right]$$

$$-\sum_{s=1}^{p-1} \frac{k^{2s}}{4^{2s}(2s)!} \frac{\partial^{2s+1} c(t_n+0.25k,\tilde{x})}{\partial t^{2s} \partial x_q} + O(k^{2p}).$$

Note that (70) can be obtained from (23) in a similar way as (66) can be obtained from (19), i.e., by replacing $\mathbf{t+k}$ with $\mathbf{t_n + 0.5k}$, $\mathbf{t+0.5k}$ with $\mathbf{t_n + 0.25k}$, $\bar{\mathbf{x}}$ with $\tilde{\mathbf{x}}$ and $\mathbf{2^{2s}}$ with $\mathbf{4^{2s}}$ which is again quite understandable.

Now the following four relationships can be written:

$$\textbf{(71)} \qquad c(t_n + 0.5k, \tilde{x}^{(+0.5q)}) = c(t_n + 0.5k, \tilde{x}) + \frac{h_q}{2} \frac{\partial c(t_n + 0.5k, \tilde{x})}{\partial x_q} + \sum_{s=2}^{2p-1} \frac{h_q^s}{2^s s!} \frac{\partial^s c(t_n + 0.5k, \tilde{x})}{\partial x_q^s} + O(h_q^{2p}),$$

$$\textbf{(72)} \qquad c(t_n + 0.5k, \tilde{x}^{(-0.5q)}) = c(t_n + 0.5k, \tilde{x}) - \frac{h_q}{2} \frac{\partial c(t_n + 0.5k, \tilde{x})}{\partial x_q} + \sum_{s=2}^{2p-1} (-1)^s \frac{h_q^s}{2^s s!} \frac{\partial^s c(t_n + 0.5k, \tilde{x})}{\partial x_q^s} + O(h_q^{2p}),$$

$$\textbf{(73)} \qquad c(t_n, \tilde{x}^{(+0.5q)}) = c(t_n, \bar{x}) + \frac{h_q}{2} \frac{\partial c(t_n, \tilde{x})}{\partial x_q} + \sum_{s=2}^{2p-1} \frac{h_q^s}{2^s s!} \frac{\partial^s c(t_n, \tilde{x})}{\partial x_q^s} + O(h_q^{2p}),$$

$$\textbf{(74)} \qquad c(t_n, \tilde{x}^{(-0.5q)}) = c(t_n, \bar{x}) - \frac{h_q}{2} \frac{\partial c(t_n, \tilde{x})}{\partial x_q} + \sum_{s=2}^{2p-1} (-1)^s \frac{h_q^s}{2^s s!} \frac{\partial^s c(t_n, \tilde{x})}{\partial x_q^s} + O(h_q^{2p}).$$

Subtract (72) from (71) to obtain:

$$(75) \qquad \frac{\partial c(t_n + 0.5k, \tilde{x})}{\partial x_q} = \frac{c(t_n + 0.5k, \tilde{x}^{(+0.5q)}) - c(t_n + 0.5k, \tilde{x}^{(-0.5q)})}{2(0.5h_q)}$$

$$- \sum_{s=1}^{p-1} \frac{h_q^{2s}}{2^{2s}(2s+1)!} \frac{\partial^{2s+1} c(t_n + 0.5k, \tilde{x})}{\partial x_q^{2s+1}} + O(h_q^{2p-1}).$$

Similarly, the following relationship can be obtained by subtracting (74) from (73):

$$(76) \qquad \frac{\partial c(t_n, \tilde{x})}{\partial x_q} = \frac{c(t_n, \tilde{x}^{(+0.5q)}) - c(t_n, \tilde{x}^{(-0.5q)})}{2(0.5h_q)}$$

$$- \sum_{s=1}^{p-1} \frac{h_q^{2s}}{2^{2s}(2s+1)!} \frac{\partial^{2s+1} c(t_n, \tilde{x})}{\partial x_q^{2s+1}} + O(h_q^{2p-1}).$$

It can easily be seen how (75) and (76) can be obtained from (28) and (29) respectively (it is necessary to replace $t+k$ with $t+0.5k$, $(+q)$ with $(+0.5q)$, $(-q)$ with $(-0.5q)$, h_q with $0.5h_q$ $\bar{x}$ with $\tilde{x}$ and, finally, 1 with 2^{2s} in the denominators of the sums).

Let $(t_n + 0.25k, \tilde{x}) = (t_n + 0.25k, x_1^{i_1}, x_2^{i_2}, \ldots x_{q-1}^{i_{q-1}}, x_q^{i_q} - 0.5h_q, x_{q+1}^{i_{q+1}}, \ldots, x_Q^{i_{N_q}})$ be some arbitrary but fixed point. Then use the abbreviation $u_q(t + 0.25k, \tilde{x}) = u_q(t + 0.25k, x_1^{i_1}, x_2^{i_2}, \ldots x_{q-1}^{i_{q-1}}, x_q^{i_q} - 0.5h_q, x_{q+1}^{i_{q+1}}, \ldots, x_Q^{i_{N_q}})$ in order to obtain, from (9), the following formula:

$$(77) \qquad \frac{\partial c(t_n + 0.25k, \tilde{x})}{\partial t} = - \sum_{q=1}^{Q} u_q(t_n + 0.25k, \tilde{x}) \frac{\partial c(t_n + 0.25k, \tilde{x})}{\partial x_q}.$$

Use (66) and (70) in (77) to obtain:

$$(78) \quad \frac{c(t_n + 0.5k, \tilde{x}) - c(t_n, \tilde{x})}{0.5k}$$

$$= - \sum_{q=1}^{Q} u_q(t_n + 0.25k, \tilde{x}) \left\{ \frac{1}{2} \left[\frac{\partial c(t_n + 0.5k, \tilde{x})}{\partial x_q} + \frac{\partial c(t, \tilde{x})}{\partial x_q} \right] \right\}$$

$$+ \sum_{s=1}^{p-1} \frac{k^{2s}}{4^{2s}(2s+1)!} \frac{\partial^{2s+1} c(t_n + 0.25k, \tilde{x})}{\partial t^{2s+1}}$$

$$+ \sum_{q=1}^{Q} u_q(t_n + 0.25k, \tilde{x}) \sum_{s=1}^{p-1} \frac{k^{2s}}{4^{2s}(2s)!} \frac{\partial^{2s+1} c(t_n + 0.25k, \tilde{x})}{\partial t^{2s} \partial x_q} + O(k^{2p-1}).$$

The last equality can be rewritten as

$$(79) \quad \frac{c(t_n + 0.5k, \tilde{x}) - c(t_n, \tilde{x})}{0.5k}$$

$$= - \sum_{q=1}^{Q} u_q(t_n + 0.25k, \tilde{x}) \left\{ \frac{1}{2} \left[\frac{\partial c(t_n + 0.5k, \tilde{x})}{\partial x_q} + \frac{\partial c(t_n, \tilde{x})}{\partial x_q} \right] \right\}$$

$$+ \sum_{s=1}^{p-1} \frac{k^{2s}}{4^{2s}(2s)!} \left\{ \frac{1}{2s+1} \frac{\partial^{2s+1} c(t_n + 0.25k, \tilde{x})}{\partial t^{2s+1}} + \sum_{q=1}^{Q} u_q(t_n + 0.25k, \tilde{x}) \frac{\partial^{2s+1} c(t_n + 0.25k, \tilde{x})}{\partial t^{2s} \partial x_q} \right\}$$

$$+ O(k^{2p-1}) .$$

Denote:

$$(80) \quad \bar{K}_t^{(2s)} = \frac{1}{4^{2s}(2s)!} \left[\frac{1}{2s+1} \frac{\partial^{2s+1} c(t_n + 0.25k, \tilde{x})}{\partial t^{2s+1}} + \sum_{q=1}^{Q} u_q(t_n + 0.25k, \tilde{x}) \frac{\partial^{2s+1} c(t_n + 0.25k, \tilde{x})}{\partial t^{2s} \partial x_q} \right].$$

Note that (80) can be obtained from (33) by performing similar replacements as those made in connection with formulae (66) and (70).

Then (79) can be rewritten as:

$$\textbf{(81)} \qquad \frac{c(t_n+0.5k,\tilde{x})-c(t_n,\tilde{x})}{0.5k}$$

$$=-\sum_{q=1}^{Q} u_q(t_n+0.25k,\tilde{x})\left\{\frac{1}{2}\left[\frac{\partial c(t_n+0.5k,\tilde{x})}{\partial x_q}+\frac{\partial c(t_n,\tilde{x})}{\partial x_q}\right]\right\}$$

$$+\sum_{s=1}^{p-1} k^{2s} K_t^{(2s)} + O(k^{2p-1}).$$

Use (75) and (76) in the expression in the square bracket from (81) to obtain

$$\textbf{(82)} \qquad \frac{c(t_n+0.5k,\tilde{x})-c(t_n,\tilde{x})}{0.5k}$$

$$=-\sum_{q=1}^{Q} u_q(t_n+0.25k,\tilde{x})\,\frac{c(t_n+0.5k,\tilde{x}^{(+0.5q)})-c(t_n+0.5k,\tilde{x}^{(-0.5q)})}{4(0.5h_q)}$$

$$-\sum_{q=1}^{Q} u_q(t_n+0.25k,\tilde{x})\,\frac{c(t_n,\tilde{x}^{(+0.5q)})-c(t_n,\tilde{x}^{(-0.5q)})}{4(0.5h_q)}$$

$$+\frac{1}{2}\sum_{q=1}^{Q} u_q(t_n+0.25k,\tilde{x})\left\{\sum_{s=1}^{p-1}\frac{h_q^{2s}}{2^{2s}(2s+1)!}\left[\frac{\partial^{2s+1}c(t_n+0.5k,\tilde{x})}{\partial x_q^{2s+1}}+\frac{\partial^{2s+1}c(t_n,\tilde{x})}{\partial x_q^{2s+1}}\right]\right\}$$

$$+\sum_{s=1}^{p-1} k^{2s} K_t^{(2s)} + O(k^{2p-1}) + O(h_q^{2p-1}).$$

The last equality can be rewritten in the following form:

(83) $$\frac{c(t_n+0.5k,\tilde{x})-c(t_n,\tilde{x})}{0.5k}$$

$$=-\sum_{q=1}^{Q} u_q(t_n+0.25k,\tilde{x})\,\frac{c(t_n+0.5k,\tilde{x}^{(+0.5q)})-c(t_n+0.5k,\tilde{x}^{(-0.5q)})}{4(0.5h_q)}$$

$$-\sum_{q=1}^{Q} u_q(t_n+0.25k,\tilde{x})\,\frac{c(t_n,\tilde{x}^{(+0.5q)})-c(t_n,\tilde{x}^{(-0.5q)})}{4(0.5h_q)}$$

$$+\sum_{s=1}^{p-1}\left\{\frac{h_q^{2s}}{2^{2s+1}(2s+1)!}\sum_{q=1}^{Q} u_q(t_n+0.25k,\tilde{x})\left[\frac{\partial^{2s+1}c(t_n+0.5k,\tilde{x})}{\partial x_q^{2s+1}}+\frac{\partial^{2s+1}c(t_n,\tilde{x})}{\partial x_q^{2s+1}}\right]\right\}$$

$$+\sum_{s=1}^{p-1} k^{2s}\,\overline{K}_t^{(2s)}+O(k^{2p-1})+O(h_q^{2p-1}).$$

Denote:

Substitute this value of $\overline{K}_q^{(2s)}$ in (83). The result is:

(85) $$\frac{c(t_n+0.5k,\tilde{x})-c(t_n,\tilde{x})}{0.5k}$$

$$=-\sum_{q=1}^{Q} u_q(t_n+0.25k,\tilde{x})\,\frac{c(t_n+0.5k,\tilde{x}^{(+0.5q)})-c(t_n+0.5k,\tilde{x}^{(-0.5q)})}{4(0.5h_q)}$$

$$-\sum_{q=1}^{Q} u_q(t_n+0.25k,\tilde{x})\,\frac{c(t_n,\tilde{x}^{(+0.5q)})-c(t_n,\tilde{x}^{(-0.5q)})}{4(0.5h_q)}$$

$$+\sum_{s=1}^{p-1}\left(k^{2s}\,\overline{K}_t^{(2s)}+h_q^{2s}\,\overline{K}_q^{(2s)}\right)+O(k^{2p-1})+O(h_q^{2p-1}).$$

Performing similar transformations around the point $t_n+0.75k$ the following two equalities can be obtained:

(86) $\tilde{K}_t^{(2s)}$

$$= \frac{1}{4^{2s}(2s)!}\left[\frac{1}{2s+1}\frac{\partial^{2s+1} c(t_n+0.75k,\tilde{x})}{\partial t^{2s+1}} + \sum_{q=1}^{Q} u_q(t_n+0.75k,\tilde{x})\frac{\partial^{2s+1} c(t_n+0.75k,\tilde{x})}{\partial t^{2s}\partial x_q}\right],$$

(87) $\tilde{K}_q^{(2s)}$

$$= \frac{1}{2^{2s+1}}\frac{1}{(2s+1)!}\sum_{q=1}^{Q} u_q(t_n+0.75k,\tilde{x})\left[\frac{\partial^{2s+1} c(t_n+k,\tilde{x})}{\partial x_q^{2s+1}} + \frac{\partial^{2s+1} c(t_n+0.5k,\tilde{x})}{\partial x_q^{2s+1}}\right].$$

Consider now the expression $2\bar{K}_t^{(2)} + 2\tilde{K}_t^{(2)} - K_t^{(2)}$ in (61).

(88) $2\bar{K}_t^{(2)} + 2\tilde{K}_t^{(2)} - K_t^{(2)}$

$$= \frac{1}{24}\left[0.5\frac{\partial^3 c(t_n+0.25k,\tilde{x})}{\partial t^3} + 0.5\frac{\partial^3 c(t_n+0.75k,\tilde{x})}{\partial t^3} - \frac{\partial^3 c(t_n+0.5k,\tilde{x})}{\partial t^3}\right]$$

$$+ \frac{1}{16}\sum_{q=1}^{Q} u_q(t_n+0.25k,\tilde{x})\frac{\partial^3 c(t_n+0.25k,\tilde{x})}{\partial t^2\partial x_q}$$

$$+ \frac{1}{16}\sum_{q=1}^{Q} u_q(t_n+0.75k,\tilde{x})\frac{\partial^3 c(t_n+0.75k,\tilde{x})}{\partial t^2\partial x_q}$$

$$- \frac{1}{8}\sum_{q=1}^{Q} u_q(t_n+0.5k,\tilde{x})\frac{\partial^3 c(t_n+0.5k,\tilde{x})}{\partial t^2\partial x_q}.$$

Since

(89) $$\frac{\partial^3 c(t_n+0.25k,\tilde{x})}{\partial t^3} + \frac{\partial^3 c(t_n+0.75k,\tilde{x})}{\partial t^3} = 2\frac{\partial^3 c(t_n+0.5k,\tilde{x})}{\partial t^3} + O(k^2),$$

it is clear that the expression in the square brackets in (88) is of order $O(k^2)$.

Consider now the following relations:

$$(90) \qquad u_q(t_n + 0.75k, \tilde{x}) = u_q(t_n + 0.5k, \tilde{x}) + 0.25k \frac{\partial u_q(t_n + 0.5k, \tilde{x})}{\partial t} + O(k^2),$$

$$(91) \qquad u_q(t_n + 0.25k, \tilde{x}) = u_q(t_n + 0.5k, \tilde{x}) - 0.25k \frac{\partial u_q(t_n + 0.5k, \tilde{x})}{\partial t} + O(k^2),$$

$$(92) \qquad \frac{\partial^3 c(t_n + 0.75k, \tilde{x})}{\partial t^2 \partial x_q} = \frac{\partial^3 c(t_n + 0.5k, \tilde{x})}{\partial t^2 \partial x_q} + 0.25k \frac{\partial^4 c(t_n + 0.5k, \tilde{x})}{\partial t^3 \partial x_q} + O(k^2),$$

$$(93) \qquad \frac{\partial^3 c(t_n + 0.25k, \tilde{x})}{\partial t^2 \partial x_q} = \frac{\partial^3 c(t_n + 0.5k, \tilde{x})}{\partial t^2 \partial x_q} - 0.25k \frac{\partial^4 c(t_n + 0.5k, \tilde{x})}{\partial t^3 \partial x_q} + O(k^2).$$

Denote

$$(94) \qquad A = u_q(t_n + 0.5k, \tilde{x}), \qquad B = 0.25 \frac{\partial u_q(t_n + 0.5k, \tilde{x})}{\partial t},$$

$$(95) \qquad C = \frac{\partial^3 c(t_n + 0.5k, \tilde{x})}{\partial t^2 \partial x_q} \qquad D = 0.25 \frac{\partial^4 c(t_n + 0.5k, \tilde{x})}{\partial t^3 \partial x_q}.$$

Then for an arbitrary value of q it is possible to obtain from the last three terms of (88) the following relationship by omitting always the terms containing multiplier k^2:

$$(96) \qquad \frac{1}{16}(A - kB)(C - kD) + \frac{1}{16}(A + kB)(C + kD) - \frac{1}{8}AC$$

$$= \frac{1}{16}(AC - kAD - kBC) + \frac{1}{16}(AC + kAD + kBC) - \frac{1}{8}AC = 0 .$$

This shows that the sum of the last three terms of (88) is also of order $\mathbf{k}^2$.

Consider now the expression $2\bar{\mathbf{K}}_q^{(2)} + 2\tilde{\mathbf{K}}_q^{(2)} - \mathbf{K}_q^{(2)}$ in the last line of (61). The following three equalities can be obtained by using (37) with $\mathbf{s = 1}$:

$$\textbf{(97)} \qquad K_q^{(2)} = \frac{1}{12} \sum_{q=1}^{Q} u_q(t_n + 0.5k, \tilde{x}) \left[\frac{\partial^3 c(t_n + k, \tilde{x})}{\partial x_q^3} + \frac{\partial^3 c(t_n, \tilde{x})}{\partial x_q^3} \right]$$

$$= \frac{1}{6} \sum_{q=1}^{Q} u_q(t_n + 0.5k, \tilde{x}) \left[\frac{\partial^3 c(t_n + 0.5k, \tilde{x})}{\partial x_q^3} + O(k^2) \right],$$

$$\textbf{(98)} \qquad 2\bar{K}_q^{(2)} = \frac{1}{24} \sum_{q=1}^{Q} u_q(t_n + 0.25k, \tilde{x}) \left[\frac{\partial^3 c(t_n + 0.5k, \tilde{x})}{\partial x_q^3} + \frac{\partial^3 c(t_n, \tilde{x})}{\partial x_q^3} \right]$$

$$= \frac{1}{12} \sum_{q=1}^{Q} u_q(t_n + 0.25k, \tilde{x}) \left[\frac{\partial^3 c(t_n + 0.25k, \tilde{x})}{\partial x_q^3} + O(k^2) \right],$$

$$\textbf{(99)} \qquad 2\tilde{K}_q^{(2)} = \frac{1}{24} \sum_{q=1}^{Q} u_q(t_n + 0.75k, \tilde{x}) \left[\frac{\partial^3 c(t_n + k, \tilde{x})}{\partial x_q^3} + \frac{\partial^3 c(t_n, 0.5\tilde{x})}{\partial x_q^3} \right]$$

$$= \frac{1}{12} \sum_{q=1}^{Q} u_q(t_n + 0.25k, \tilde{x}) \left[\frac{\partial^3 c(t_n + 0.75k, \tilde{x})}{\partial x_q^3} + O(k^2) \right].$$

The following two relationships are used in the derivation of (97):

(100) $$\frac{\partial^3 c(t_n + k, \tilde{x})}{\partial x_q^3} = \frac{\partial^3 c(t_n + 0.5k, \tilde{x})}{\partial x_q^3} + 0.5k \frac{\partial^4 c(t_n + 0.5k, \tilde{x})}{\partial x_q^4} + O(k^2) ,$$

(101) $$\frac{\partial^3 c(t_n, \tilde{x})}{\partial x_q^3} = \frac{\partial^3 c(t_n + 0.5k, \tilde{x})}{\partial x_q^3} - 0.5k \frac{\partial^4 c(t_n + 0.5k, \tilde{x})}{\partial x_q^4} + O(k^2) .$$

Now (100) and (101) should be added to obtain:

(102) $$\frac{\partial^3 c(t_n + k, \tilde{x})}{\partial x_q^3} + \frac{\partial^3 c(t_n, \tilde{x})}{\partial x_q^3} = 2 \frac{\partial^3 c(t_n + 0.5k, \tilde{x})}{\partial x_q^3} + O(k^2) ,$$

Equality (102) shows how (97) can be obtained. Similar operations are to be used to obtain (98) and (99).

By using (97), (98) and (99) and by omitting terms containing $O(k^2)$ it is possible to transform the expression $2\overline{K}_q^{(2)} + 2\tilde{K}_q^{(2)} - K_q^{(2)}$ in the following way:

(103) $$2\overline{K}_q^{(2s)} + 2\tilde{K}_q^{(2s)} - K_q^{(2s)} = \frac{1}{12} \sum_{q=1}^{Q} u_q(t_n + 0.25k, \tilde{x}) \left[\frac{\partial^3 c(t_n + 0.25k, \tilde{x})}{\partial x_q^3} \right]$$

$$+ \frac{1}{12} \sum_{q=1}^{Q} u_q(t_n + 0.75k, \tilde{x}) \left[\frac{\partial^3 c(t_n + 0.75k, \tilde{x})}{\partial x_q^3} \right]$$

$$- \frac{1}{6} \sum_{q=1}^{Q} u_q(t_n + 0.5k, \tilde{x}) \left[\frac{\partial^3 c(t_n + 0.5k, \tilde{x})}{\partial x_q^3} \right]$$

It is now necessary to apply (90) and (91) together with the following two equalities:

$$\text{(104)} \qquad \frac{\partial^3 c(t_n + 0.75k, \tilde{x})}{\partial x_q^3} = \frac{\partial^3 c(t_n + 0.5k, \tilde{x})}{\partial x_q^3} + 0.25k \frac{\partial^4 c(t_n + 0.5k, \tilde{x})}{\partial t \partial x_q^3} + O(k^2),$$

$$\text{(105)} \qquad \frac{\partial^3 c(t_n + 0.25k, \tilde{x})}{\partial x_q^3} = \frac{\partial^3 c(t_n + 0.5k, \tilde{x})}{\partial x_q^3} - 0.25k \frac{\partial^4 c(t_n + 0.5k, \tilde{x})}{\partial t \partial x_q^3} + O(k^2),$$

Denote:

$$\text{(106)} \qquad E = \frac{\partial^3 c(t_n + 0.25k, \tilde{x})}{\partial x_q^3} \qquad F = 0.25 \frac{\partial^4 c(t_n + 0.5k, \tilde{\tilde{x}})}{\partial t \partial x_q^3} .$$

Then for an arbitrary value of q it is possible to obtain the following relationship from (103) by omitting always the terms containing multiplier k^2:

$$\text{(107)} \qquad \frac{1}{12}(A - kB)(E - kF) + \frac{1}{12}(A + kB)(E + kF) - \frac{1}{6} AE$$

$$= \frac{1}{12}(AE - kAF - kBE) + \frac{1}{12}(AE + kAF + kBE) - \frac{1}{6} AE = 0 .$$

Therefore, the expression in (103) is of order k^2.

This means that the expression $\hat{K}$ in (61) is of order $O(k^2)$ and it can be concluded from (52) that if the multi-dimensional advection is treated by the second-order numerical method defined by (52) and combined with the Richardson Extrapolation, then it results in a numerical method of order four (not of order three as should be expected). In this way the following theorem has been proved:

Theorem 2: Consider the multi-dimensional advection equation (9). Assume that the coefficients u_q before the spatial derivatives in (9) are continuously differentiable up to order two with respect to all independent variables and continuous derivatives of the unknown function c up to order four exist, again with respect to all variables. Then the combination of the numerical method (52) and the Richardson Extrapolation is of order four.

6. Conclusions

The result proved in this paper is a generalization of the result proved in [8], where the one-dimensional advection is handled.

It is interesting whether the result proved in Theorem 2 for equation (9) can be extended for the more general equation:

$$(108) \quad \frac{\partial c}{\partial t} = -\sum_{q=1}^{Q} \frac{\partial (u_q c)}{\partial x_q}, \quad x_q \in [a_q, b_q] \quad \text{for} \quad q = 1, 2, \ldots, Q, \quad \text{with} \quad Q \geq 1, \quad t \in [a, b] .$$

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