Numerical stability for nonlinear evolution equations

Petra Csomós\textsuperscript{a}, István Faragó\textsuperscript{a,b}, Imre Fekete\textsuperscript{a,b}

\textsuperscript{a}MTA-ELTE Numerical Analysis and Large Networks Research Group, H-1117 Budapest, Pázmány P. s. 1/C, Hungary
\textsuperscript{b}Department of Applied Analysis and Computational Mathematics, Eötvös Loránd University, H-1117 Budapest, Pázmány P. s. 1/C, Hungary

Abstract

The paper deals with discretisation methods for nonlinear operator equations written as abstract nonlinear evolution equations. Brezis and Pazy showed that the solution of such problems is given by nonlinear semigroups whose theory was founded by Crandall and Liggett. By using the approximation theorem of Brezis and Pazy, we show the $N$-stability of the abstract nonlinear discrete problem for the implicit Euler method. Motivated by the rational approximation methods for linear semigroups, we propose a more general time discretisation method and prove its $N$-stability as well.

Keywords: nonlinear stability, nonlinear semigroups, nonlinear rational approximations

2000 MSC: 47H20, 65J08, 65M12

1. Introduction

The abstract framework of investigating nonlinear operator equations was first introduced by Stetter in [9]. Sanz-Serna and Palencia studied general linear operator equations in [8] when it turned out that this kind of abstract framework is feasible for the stability analysis of linear evolution equations as well. They considered general one-step finite difference schemes as time discretisations, and as a special case of their results they showed the Lax–Richtmyer stability introduced in [7]. Our aim is to set this abstract framework for nonlinear problems originated from nonlinear evolution equations. We will apply nonlinear operator semigroup theory established by Crandall and Liggett in [2], Brezis and Pazy in [1] and Goldstein in [5]. Their results on nonlinear operator semigroups can be viewed as numerical approximations by implicit Euler method. In the present paper we propose a more general class of discretisations, that is, one-step methods originated from rational approximations.

Based on Fekete and Faragó [3] we introduce the abstract setting and define the natural stability concept, the $N$-stability for nonlinear operator equations. In Section 2 we collect the basic results in the theory of nonlinear semigroups. Section 3 is devoted to the derivation of the space and time discretisation methods needed later on, especially, the implicit Euler method and the discretisations having the same form as the rational approximation schemes for linear operators.
Section 4 contains the proof of the $N$-stability for a special class of nonlinear operators. In Section 5 we show the correspondence with the linear stability theory presented in Sanz-Serna and Palencia [8].

1.1. Setting the problem

Let $(X, \| \cdot \|_X)$ and $(Y, \| \cdot \|_Y)$ be normed spaces and $F : D(F) \subset X \to Y$ be a (possibly unbounded and nonlinear) operator. We investigate the problem

\[ F(u) = 0 \quad \text{for} \quad u \in D(F). \tag{1.1} \]

If a certain discretisation is applied to solve equation (1.1), one defines an index set $\mathbb{I} \subset \mathbb{N}^p$ for $p \in \mathbb{N}$, the normed spaces $(X_n, \| \cdot \|_{X_n}), (Y_n, \| \cdot \|_{Y_n})$, and the operator $F_n : D(F_n) \subset X_n \to Y_n$ and considers the problem

\[ F_n(u_n) = 0 \quad \text{for} \quad u_n \in D(F_n) \quad \text{and} \quad n \in \mathbb{I}. \tag{1.2} \]

Throughout the paper we assume that there exist unique solutions $u^*$ and $u^*_n$ to both problems (1.1) and (1.2), respectively. To be able to compare the solutions $u^*$ and $u^*_n$, the mappings $\varphi_n : X \to X_n$ and $\psi_n : Y \to Y_n$ need to be defined for all $n \in \mathbb{I}$. For the analysis of the discretised problem (1.2) the following notions play an important role.

**Definition 1.1.**

(a) The discretisation scheme (1.2) is called convergent if

\[ \lim_{n \to \infty} \| \varphi_n(u^*) - u^*_n \|_{X_n} = 0. \]

(b) The scheme is called consistent on the element $v \in D(F)$ if $\varphi_n(v) \in D(F_n)$, $n \in \mathbb{I}$ and

\[ \lim_{n \to \infty} \| F_n(\varphi_n(v)) - \varphi_n(F(v)) \|_{Y_n} = 0. \]

(c) We call the scheme $N$-stable if there exists a constant $C > 0$ such that the estimate

\[ \| v_n - z_n \|_{X_n} \leq C \| F_n(v_n) - F_n(z_n) \|_{Y_n} \tag{1.3} \]

holds for all $v_n, z_n \in D(F_n)$ and the stability constant $C$ is independent of $n$.

We remark that the limit is understood simultaneously in all indices of $\mathbb{I}$. We note that convergence follows from $N$-stability if the scheme is assumed to be consistent on the exact solution $u^*$ and we further assume that

\[ \lim_{n \to \infty} \| \psi_n(0) - 0 \|_{Y_n} = 0. \]

In this case, namely, we have

\[ \| \varphi_n(u^*) - u^*_n \|_{X_n} \leq C \| F_n(\varphi_n(u^*)) - F_n(u^*_n) \|_{Y_n} \]

\[ \leq C \| F_n(\varphi_n(u^*)) - \varphi_n(F(u^*)) \|_{Y_n} + C \| \varphi_n(F(u^*)) - F_n(u^*_n) \|_{Y_n}, \]

where the first term converges to zero as $n$ goes to infinity due to consistency and the second term converges to zero because we have $F(u^*) = 0$ in $Y$ and $F_n(u^*_n) = 0$ in $Y_n$, $n \in \mathbb{I}$. This result shows the role of both stability and consistency for obtaining convergence.
2. Nonlinear semigroups

In this section we summarise the results about the nonlinear theory we will need. Our main reference is the textbook by Ito and Kappel [4]. Let \( (\mathcal{X}, \| \cdot \|_{\mathcal{X}}) \) denote a Banach space. From now on we identify the operator \( A : D(A) \subset \mathcal{X} \to \mathcal{X} \) with its graph in \( \mathcal{X} \times \mathcal{X} \).

**Definition 2.1** (Prop. 1.8, [4]). For \( \omega \in \mathbb{R} \), an operator \( A \) on \( \mathcal{X} \), i.e., \( A \subset \mathcal{X} \times \mathcal{X} \), is called \( \omega \)-dissipative if for all \( \tau \in (0, \frac{1}{\omega}) \) and \( f, g \in D(A) \) we have

\[
\| (I - \tau A)(f) - (I - \tau A)(g) \|_{\mathcal{X}} \geq (1 - \tau \omega) \| f - g \|_{\mathcal{X}}. \tag{2.4}
\]

For \( \omega = 0 \) the operator \( A \) is called dissipative. We note that for \( \omega = 0 \), we have \( \tau \in (0, \infty) \).

**Remark 2.2** (Prop. 1.9, [4]). Let \( A \) be an \( \omega \)-dissipative operator on \( \mathcal{X} \). Then, for any \( \tau \in \left(0, \frac{1}{\omega}\right) \), the operator \( (I - \tau A)^{-1} \) is single-valued and for any \( \tau \in \left(0, \frac{1}{\omega}\right) \) and \( f, g \in \text{ran} (I - \tau A) \), we have

\[
\| (I - \tau A)^{-1}(f) - (I - \tau A)^{-1}(g) \|_{\mathcal{X}} \leq \frac{1}{1 - \tau \omega} \| f - g \|_{\mathcal{X}}. \]

**Definition 2.3** (Def. 5.1, [4]). Let \( \mathcal{X}_0 \) be a subset of \( \mathcal{X} \), \( \omega \in \mathbb{R} \) and \((S(t))_{t \geq 0}\) be a family of (nonlinear) operators \( \mathcal{X}_0 \to \mathcal{X}_0 \). The family \((S(t))_{t \geq 0}\) is called a strongly continuous semigroup of type \( \omega \) on \( \mathcal{X}_0 \) if it possesses the following properties.

(i) \( S(0)(f) = f \) for all \( f \in \mathcal{X}_0 \).
(ii) \( S(t + s)(f) = S(t)S(s)(f) \) for all \( t, s \geq 0 \) and \( f \in \mathcal{X}_0 \).
(iii) For any \( f \in \mathcal{X}_0 \) the function \((0, \infty) \ni t \to S(t)(f) \in \mathcal{X}_0 \) is continuous.
(iv) There exists \( \omega \in \mathbb{R} \) such that \( \| S(t)(f) - S(t)(g) \|_{\mathcal{X}} \leq e^{\omega t} \| f - g \|_{\mathcal{X}} \) for all \( t \geq 0 \) and \( f, g \in \mathcal{X}_0 \).

The next celebrated result of Crandall and Liggett shows how one can construct a semigroup by having an appropriate operator at hand.

**Theorem 2.4** (Thm. 1., [2], Cor. 5.4, [4]). For \( \omega \in \mathbb{R} \) let \( A \) be an \( \omega \)-dissipative operator on \( \mathcal{X} \) such that \( \text{ran} (I - \tau A) \supset D(A) \) holds for every \( \tau \in \left(0, \frac{1}{\omega}\right) \). Then there exists a strongly continuous semigroup \((S(t))_{t \geq 0}\) of type \( \omega \) on \( D(A) \). Moreover, for \( f \in D(A) \), we have the limit

\[
S(t)(f) = \lim_{k \to \infty} ((I - \frac{1}{k} A)^{-1})^k(f) \tag{2.5}
\]

which converges uniformly for \( t \) in bounded intervals.

In case of the theorem above we say that the operator \( A \) generates the semigroup \( S \). We note that the \( k^{th} \) power denotes the \( k \) times composition. Next we introduce the relevant results concerning the connection between semigroups of type \( \omega \) and abstract Cauchy problems with \( \omega \)-dissipative operators. For an operator \( A \) on \( \mathcal{X} \) we consider the abstract Cauchy problem

\[
\begin{align*}
\frac{d}{dt} u(t, \cdot) &= A(u(t, \cdot)), & t > 0 \\
u(0, \cdot) &= u_0(\cdot) \in \mathcal{X}_0.
\end{align*}
\tag{2.6}
\]

For the definition of integral and strong solutions, needed for the next theorem, we refer to Definition 5.5 in Ito and Kappel [4].
Theorem 2.5 (Thm. 5.6 and Thm. 5.8, [4]). Suppose that $A$ is an $\omega$-dissipative operator on $X$ generating the strongly continuous semigroup $S$ of type $\omega$. Suppose further that ran $(I - \tau A) \supset D(A)$ holds for all $\tau \in (0, \frac{1}{\omega})$. Then the following is true.

(a) For any $u_0 \in D(A)$, there exists a unique integral solution $u$ to problem (2.6) given by $u(t, \cdot) = (S(t)u_0)(\cdot)$ for all $t \geq 0$.

(b) For $\omega = 0$, the solution above is the unique strong solution.

Later, when studying the convergence of the spatial discretisations, we will need the following theorem as well (similar to Thm. 3.2, [5] and to Cor. 10.8, [4]).

Theorem 2.6 (Cor. 4.2, [1]). Let $\omega \geq 0$ and $A$ be an $\omega$-dissipative single-valued operator on $X$ satisfying $\text{ran}(I - \tau A) \supset D(A)$ for some $\tau \in \left(0, \frac{1}{\omega}\right)$ and let $S$ be the semigroup of type $\omega$ generated by $A$ on $D(A)$. Let $A_m : D(A_m) \subset X \to X$ be $\omega_m$-dissipative single-valued operators on $X$ satisfying ran $(I - \tau A_m) \supset D(A_m)$ for some $\tau \in \left(0, \frac{1}{\omega_m}\right)$ and for all $m \in \mathbb{N}$, and let $(S_m(t))_{t \geq 0}$ be the semigroup of type $\omega_m$ generated by $A_m$ on $X$. If

(i) there exists $\alpha \in [0, \infty)$ such that $0 \leq \omega, \omega_m \leq \alpha$,
(ii) $D(A) \subset D(A_m)$ for all $m \in \mathbb{N}$,
(iii) $\lim_{m \to \infty} A_m(f) \to A(f)$ for all $f \in D(A),$

then we have the limit

$$\lim_{m \to \infty} S_m(t)(f) = S(t)(f) \quad \text{for all} \quad f \in D(A), \quad (2.7)$$

where the convergence is uniform for $t$ in bounded intervals.

3. Discretisation schemes

To define the discrete problem (1.2), we consider problem (1.1) with an operator $F$ of a special form. Throughout the paper we suppose that $A$ is an $\omega$-dissipative operator on $X$ for some $\omega \geq 0$ with ran $(I - \tau A) \supset D(A)$ for some $\tau \in \left(0, \frac{1}{\omega}\right)$. We consider then problem (1.1) in the following form:

$$\begin{cases} F(u) = 0 & \text{for } u \in D(F), \\
u(0, \cdot) = u_0 \in D(A) & \text{given,} \\
(F(v))(t, x) := (\frac{\partial}{\partial t}v(t, \cdot) - A(v(t, \cdot)))(x) & \text{for } v \in D(F), \ t > 0, \ x \in \mathbb{R}^d. \end{cases} \quad (3.8)$$

According to Theorem 2.6 operator $A$ generates a semigroup $S$ of type $\omega$ on $D(A)$. In order to obtain an approximation to the exact solution $u$, i.e., to the semigroup $S$, we discretise the nonlinear evolution equation (3.8) both in space and time.

3.1. Discretisation in space

To obtain the spatially discretised solution we assume the following.

Assumption 3.1. We assume that there exist operators $A_m, m \in \mathbb{N}$ on $X$ such that

(a) $A_m$ is $\omega_m$-dissipative on $X$ for some $\omega_m \geq 0$ for each $m \in \mathbb{N},$
(b) $\text{ran} (I - \tau A_m) \supset D(A_m)$ for all $m \in \mathbb{N}$ and for some $\tau \in \left(0, \frac{1}{\omega_m}\right).$
(c) there exists $\alpha \in [0, \infty)$ such that $0 \leq \omega, \omega_{m} \leq \alpha$ for all $m \in \mathbb{N}$,
(d) $D(A) \subset D(A_{m})$ for all $m \in \mathbb{N}$,
(e) $\lim_{m \to \infty} A_{m}(f) = A(f)$ for all $f \in D(A)$.

The smallest possible value of $\alpha$ is denoted by $\beta$.

Assumption 3.1 and Theorem 2.4 imply that the operator $A_{m}$ is the generator of a semigroup $S_{m}$ for all $m \in \mathbb{N}$. Theorem 2.6 implies that these semigroups converge, that is, the limit (2.7) holds uniformly for $t$ in compact intervals.

From the numerical point of view this means that $A_{m}$ represents the approximation of $A$ by using some spatial discretisation scheme. For instance, if $A$ involves a spatial derivative, then $A_{m}$ stands for e.g. the finite difference approximation or the approximation by using finite discrete Fourier transform. In these cases $m$ refers to the number of spatial grid points or the number of Fourier coefficients, respectively. If the approximate generators $A_{m}$ converge to $A$, then the numerical solution will converge to the exact solution, too.

3.2. Discretisation in time

In order to get the fully discretised approximative solution to problem (1.1) we need to define problem (1.2), especially the operator $F_{n}$ in it.

3.2.1. Implicit Euler method

First we notice that Theorem 2.4 states that the solution $u$ to problem (1.1) has the form $u(t, \cdot) = (S(t)(u_{0})(\cdot))$ where $S$ is the semigroup generated by $A$. Formula (2.5) and Theorem 2.6 imply that

$$S(t)(u_{0}) = \lim_{m \to \infty} \lim_{k \to \infty} ((I - \frac{\tau}{A_{m}})^{-1})^{k}(u_{0}),$$

(3.9)

where the convergence is uniform for $t$ in compact intervals. We note that limit (2.5) in Theorem 2.4 and therefore formula (3.9) already contain a kind of time discretisation, namely, the implicit Euler method, that is, when the operator $S_{m}(t)$ is approximated by the operator $((I - \frac{\tau}{A_{m}})^{-1})^{k}$ for some $k \in \mathbb{N}$. For each $t \geq 0$ we fix now $K \in \mathbb{N}$ such that $K > \beta$, where $\beta$ is the smallest possible common bound on $\omega$ and $\omega_{m}$ from Assumption 3.1 and introduce the product spaces $X_{m} := \mathcal{S}^{K+1}$, $Y_{n} := \mathcal{S}^{K+1}$ endowed by some appropriate norms specified later. Then limit (3.9) motivates us how to define the fully discretised numerical solution $u_{k}$ for all $n \in \mathbb{N}$. Its $k^{th}$ component corresponds to the approximation of the solution at the $k^{th}$ time level, and has the form

$$u_{k} = ((I - \frac{1}{\tau A_{m}})^{-1})^{k}(u_{0}) = (I - \frac{\tau}{A_{m}})^{-1}(u_{k-1}) \quad \text{for} \quad k = 0, \ldots, K.$$  

(3.10)

Hence, with time step $\tau := \frac{1}{N}$, problem (1.2) contains the operator $F_{n}$ defined for all $v_{n} \in (D(A))^{K+1}$, $n \in \mathbb{N}$, as

$$\begin{cases} 
(F_{n}(v_{n}))_{0} := (v_{n}), \\
(F_{n}(v_{n}))_{k} := (v_{n})_{k} - (I - \tau A_{m})^{-1}((v_{n})_{k-1}), \quad \text{for all} \quad k = 1, \ldots, K,
\end{cases}$$

(3.11)

where $(v_{n})_{k} \in D(A)$ for all $k = 0, \ldots, K$. Since $\omega_{m} \leq \beta$ for all $m \in \mathbb{N}$, Remark 2.2 implies that for all $f, g \in D(A)$ and $m \in \mathbb{N}$ we have

$$||((I - \tau A_{m})^{-1}(f) - (I - \tau A_{m})^{-1}(g))||_{X} \leq \Lambda_{1}||f - g||_{X}$$

with \quad $\Lambda_{1} := \frac{1}{1 - \tau \beta}$.

We note that for dissipative operators $A_{m}$ we have $\omega_{m} = 0$, therefore, $\beta = 0$ and $\Lambda_{1} = 1$. 

5
3.2.2. Rational approximations

As we already mentioned in the Introduction rational approximations are well-known and widely investigated for linear operators, see Hairer and Wanner [6]. This motivated us to analyse them in an abstract framework for nonlinear operators as well. For a given $t \geq 0$ we choose $K \in \mathbb{N}$, fix $\tau = \frac{1}{K}$ and choose constants $z_0, z_i \in \mathbb{R}$, $c_i \in \mathbb{R}$, $v, v_i \in \mathbb{N}$ with $c_i > \beta \tau$ (i.e., $c_i K > \beta$). Then for all $f \in D(A)$ we define the rational approximations for nonlinear operators as

$$r(\tau A_m)(f) = z_0 f + \sum_{i=1}^{v} \sum_{j=1}^{v_i} z_i ((I - \frac{z_i}{c_i} A_m)^{-1})^j(f).$$  \hfill (3.12)

After replacing the term $(I - \tau A_m)^{-1}$ by $r(\tau A_m)$ in (3.10), we obtain the discrete problem

$$(u_n)_k = r(\tau A_m)^k(u_0) \quad \text{for} \quad k = 0, ..., K.$$  \hfill (3.13)

Due to Remark 2.2, the operators $(I - \frac{z_i}{c_i} A_m)^{-1} : D(A) \to D(A)$ exist for all $0 < z_i < \frac{1}{\beta}$, therefore, the operators $r(\tau A_m) : D(A) \to D(A)$ are well-defined for all $m \in \mathbb{N}$. Formulae (3.11) and (3.13) lead to the full discretisation scheme (1.2) with the operator $F_n$ defined for all $v_n \in (D(A))^k$ as

$$\begin{pmatrix}
(F_n(v_n))_0 = (v_n)_0, \\
(F_n(v_n))_k = (v_n)_k - r(\tau A_m)^k((v_n)_0) = (v_n)_k - r(\tau A_m)((v_n)_{k-1}) \quad \text{for} \quad k = 1, ..., K.
\end{pmatrix}$$  \hfill (3.14)

Remark 2.2 implies that for all $f, g \in D(A)$ and $m \in \mathbb{N}$ we have

$$\| (I - \frac{z_i}{c_i} A_m)^{-1}(f) - (I - \frac{z_i}{c_i} A_m)^{-1}(g) \|_X \leq \Lambda_i \| f - g \|_X$$

with $\Lambda_i := \frac{1}{1 - \frac{z_i}{c_i} \beta}$.  \hfill (3.15)

Hence, for all $f, g \in D(A)$ and $m \in \mathbb{N}$ we have the estimate

$$\| r(\tau A_m)(f) - r(\tau A_m)(g) \|_X \leq \|f - g\|_X + \sum_{i=1}^{v} \sum_{j=1}^{v_i} |z_i| \|((I - \frac{z_i}{c_i} A_m)^{-1})^j(f) - ((I - \frac{z_i}{c_i} A_m)^{-1})^j(g)\|_X.$$  \hfill (3.16)

Thus, by introducing

$$Z := \|z_0\| + \sum_{i=1}^{v} \sum_{j=1}^{v_i} |z_i| \Lambda_i^j$$  \hfill (3.16)

we have that

$$\| r(\tau A_m)(f) - r(\tau A_m)(g) \|_X \leq Z \| f - g \|_X.$$  \hfill (3.17)
Remark 3.2. Since we will use it later, we show now that $Z \geq 1$ holds for the rational approximations defined in (3.12). First we note that the operator $r(\tau A_m)$ is meant to approximate the operator $S_m(\tau)$ which approximates the operator $S(\tau)$. Hence, we expect that $r(\tau A_m)$ should possess some of the properties of $S(\tau)$, one of them is $S(0) = I$. Therefore, it seems natural to expect that $r(0A) = I$ should hold. Then we have that the operator

$$r(0A_m) = z_0I + \sum_{i=1}^{V} \sum_{j=1}^{V_i} z_{ij} I$$

equals the identity operator on $\mathcal{X}$ if and only if

$$z_0 + \sum_{i=1}^{V} \sum_{j=1}^{V_i} z_{ij} = 1.$$ 

Then the triangular inequality implies that

$$1 \leq |z_0| + \sum_{i=1}^{V} \sum_{j=1}^{V_i} |z_{ij}|.$$ 

Since $c_i K > \beta t$, from (3.15) we have $\Lambda_{c_i} \geq 1$ for all $\beta \geq 0$, therefore we obtain that

$$Z = |z_0| + \sum_{i=1}^{V} \sum_{j=1}^{V_i} |z_{ij}| \geq |z_0| + \sum_{i=1}^{V} \sum_{j=1}^{V_i} |z_{ij}| \geq 1.$$ 

At the end of this section we present two basic examples, both being well-known for linear problems, for nonlinear rational approximations (3.12).

Example 3.3. (i) The choice $z_0 = 0$, $\nu = 1$, $\nu_1 = 1$, $c_1 = 1$ and $z_{11} = 1$ in (3.12) corresponds to the implicit Euler method with $r(\tau A_m) = (I - \tau A_m)^{-1}$. In case of implicit Euler the estimate (3.17) holds with $Z = \Lambda_{c_1}$.

(ii) The choice $z_0 = -1$, $\nu = 1$, $\nu_1 = 1$, $c_1 = 2$ and $z_{11} = 2$ gives the Crank-Nicolson method with $r(\tau A_m) = (I + \frac{\tau}{2} A_m)(I - \frac{\tau}{2} A_m)^{-1}$, since by using the identity $(I + \frac{\tau}{2} A_m)(I - \frac{\tau}{2} A_m)^{-1} = I$ we have

$$r(\tau A_m) = -I + 2(I - \frac{\tau}{2} A_m)^{-1} = (I - \frac{\tau}{2} A_m)^{-1} + (I - \frac{\tau}{2} A_m)^{-1} - I$$

$$= (I - \frac{\tau}{2} A_m)^{-1} + \frac{\tau}{2} A_m(I - \frac{\tau}{2} A_m)^{-1} = (I + \frac{\tau}{2} A_m)(I - \frac{\tau}{2} A_m)^{-1}.$$ 

4. Stability in the nonlinear case

In this section we show the $N$-stability of the numerical scheme (1.2), that is, $F_n(u_n)$ for $u_n \in D(F_n) \subset X_n$, where $F_n$ is defined in (3.14). First we endow the spaces $X_n = \mathcal{X}^{K+1}$ and $Y_n = \mathcal{X}^{K+1}$ by the following norms:

$$\|f\|_{X_n} := a_K \sum_{k=0}^{K} \|f_k\|_{\mathcal{X}} \quad \text{for} \quad f = (f_0, ..., f_K) \in X_n = \mathcal{X}^{K+1},$$

$$\|f\|_{Y_n} := \sum_{k=0}^{K} \|f_k\|_{\mathcal{X}} \quad \text{for} \quad f = (f_0, ..., f_K) \in Y_n = \mathcal{X}^{K+1}, \quad (4.18)$$
Therefore, there exists an index $k$ holds for all
\begin{equation}
\alpha_k = \begin{cases}
\frac{1}{K+1}, & \text{if } Z = 1, \\
\frac{Z - 1}{Z^{k+1} - 1}, & \text{if } Z > 1.
\end{cases}
\end{equation}
Now we are at the position to show $N$-stability property (1.3) of the general rational approximation schemes defined in (3.14).

**Theorem 4.1.** Suppose that $A$ is an $\omega$-dissipative operator on $\mathcal{X}$ for some $\omega \geq 0$. Suppose further that the operators $A_m$, $m \in \mathbb{N}$ satisfy Assumption 3.1. Then the numerical scheme (3.14) is $N$-stable with the stability constant $C = 1$.

**Proof.** Since operators $A_m$ are $\omega_m$-dissipative on $\mathcal{X}$ for all $m \in \mathbb{N}$, formula (3.17) implies that
\[
\|r(\tau A_n)(f) - r(\tau A_m)(g)\|_X \leq Z\|f - g\|_X
\]
for all $f, g \in D(A)$ and $m \in \mathbb{N}$, where $Z$ is defined in (3.16). We have for all $v_n, z_n \in (D(A))^{k+1}$ that
\[
\|(v_n)_0 - (z_n)_0\|_X = \|(F_n(v_n)_0) - (F_n(z_n)_0)\|_X, \\
\|(v_n)_1 - (z_n)_1\|_X \leq \|(F_n(v_n)_1) - (F_n(z_n)_1)\|_X + \|r(\tau A_n)((v_n)_0) - r(\tau A_m)((z_n)_0)\|_X \\
\leq \|(F_n(v_n)_1) - (F_n(z_n)_1)\|_X + Z\|(v_n)_0 - (z_n)_0\|_X, \\
\|(v_n)_2 - (z_n)_2\|_X \leq \|(F_n(v_n)_2) - (F_n(z_n)_2)\|_X + \|r(\tau A_n)((v_n)_1) - r(\tau A_m)((z_n)_1)\|_X \\
\leq \|(F_n(v_n)_2) - (F_n(z_n)_2)\|_X + Z\|(v_n)_1 - (z_n)_1\|_X, \\
= \|(F_n(v_n)_2) - (F_n(z_n)_2)\|_X + Z\|(v_n)_1 - (z_n)_1\|_X.
\]
Therefore, there exists an index $\ell \in \mathbb{N}$ such that
\begin{equation}
\|(v_n)_k - (z_n)_k\|_X \leq \sum_{j=0}^{\ell} Z^{\ell-j}\|(F_n(v_n)_j) - (F_n(z_n)_j)\|_X
\end{equation}
holds for all $k = 0, ..., \ell$. The definition (3.14) of $F_n$ and the estimate (4.20) yields
\[
\|(v_n)_{\ell+1} - (z_n)_{\ell+1}\|_X \leq \|(F_n(v_n))_{\ell+1} - (F_n(z_n))_{\ell+1}\|_X + \|r(\tau A_n)((v_n)_{\ell}) - r(\tau A_m)((z_n)_{\ell})\|_X \\
\leq \|(F_n(v_n))_{\ell+1} - (F_n(z_n))_{\ell+1}\|_X + Z\|(v_n)_{\ell} - (z_n)_{\ell}\|_X, \\
\leq \|(F_n(v_n))_{\ell+1} - (F_n(z_n))_{\ell+1}\|_X + Z\sum_{j=0}^{\ell} Z^{\ell-j}\|(F_n(v_n))_j - (F_n(z_n))_j\|_X \\
= \sum_{j=0}^{\ell+1} Z^{\ell+1-j}\|(F_n(v_n))_j - (F_n(z_n))_j\|_X.
\]
By induction we obtain that (4.20) holds for all $k \in \mathbb{N}$, which we repeat here for further references:
\begin{equation}
\|(v_n)_k - (z_n)_k\|_X \leq \sum_{j=0}^{k} Z^{k-j}\|(F_n(v_n))_j - (F_n(z_n))_j\|_X \quad \text{for all } k \in \mathbb{N}.
\end{equation}
From this point we have two cases: \( Z = 1 \) and \( Z > 1 \).

**The case \( Z = 1 \).** Estimate (4.21) has now the form

\[
\| (v_n)_k - (z_n)_k \|_X \leq \sum_{j=0}^{k} \| (F_n(v_n))_j - (F_n(z_n))_j \|_X \quad \text{for all} \quad k \in \mathbb{N}. \tag{4.22}
\]

Inserting (4.22) into the definition (4.18) of the norm leads to the estimate

\[
\| v_n - z_n \|_X = \frac{1}{K + 1} \sum_{k=0}^{K} \| (v_n)_k - (z_n)_k \|_X
\]

\[
\leq \frac{1}{K + 1} \sum_{k=0}^{K} \sum_{j=0}^{k} \| (F_n(v_n))_j - (F_n(z_n))_j \|_X
\]

\[
= \frac{1}{K + 1} \sum_{k=0}^{K} (K + 1 - j) \| (F_n(v_n))_j - (F_n(z_n))_j \|_X
\]

\[
\leq \frac{1}{K + 1} \sum_{k=0}^{K} (K + 1) \| (F_n(v_n))_k - (F_n(z_n))_k \|_X
\]

\[
= \sum_{k=0}^{K} \| (F_n(v_n))_k - (F_n(z_n))_k \|_X = \| F_n(v_n) - F_n(z_n) \|_{v_n}.
\]

This yields \( N \)-stability with \( C = 1 \).

**The case \( Z > 1 \).** From formula (4.21) we obtain the estimate

\[
\| (v_n)_k - (z_n)_k \|_X \leq \sum_{j=0}^{k} Z^{k-j} \| (F_n(v_n))_j - (F_n(z_n))_j \|_X \quad \text{for all} \quad k \in \mathbb{N}. \tag{4.24}
\]

In the same manner as before, we insert (4.24) into the definition (4.18) and obtain

\[
\| v_n - z_n \|_X = \frac{Z - 1}{Z^{K+1} - 1} \sum_{k=0}^{K} \| (v_n)_k - (z_n)_k \|_X
\]

\[
\leq \frac{Z - 1}{Z^{K+1} - 1} \sum_{k=0}^{K} \sum_{j=0}^{k} Z^{k-j} \| (F_n(v_n))_j - (F_n(z_n))_j \|_X
\]

\[
= \frac{Z - 1}{Z^{K+1} - 1} \sum_{k=0}^{K} Z^{K+1-k} \| (F_n(v_n))_k - (F_n(z_n))_k \|_X
\]

\[
\leq \frac{Z - 1}{Z^{K+1} - 1} \sum_{k=0}^{K} Z^{K+1-k} \| (F_n(v_n))_k - (F_n(z_n))_k \|_X
\]

\[
= \sum_{k=0}^{K} \| (F_n(v_n))_k - (F_n(z_n))_k \|_X = \| F_n(v_n) - F_n(z_n) \|_{v_n}.
\]

This yields \( N \)-stability with \( C = 1 \) in this case as well. \( \square \)
Remark 4.2. We briefly show that Theorem 4.1 remains valid if the norms are defined different from (4.18).

(a) We endow the spaces $X_n = X^{K+1}$ and $Y_n = X^{K+1}$ with the following norms:

$$
\|f\|_{X_n} := a_K \sup_{k=0,\ldots,K} \|f_k\|_X \quad \text{for} \quad f = (f_0, \ldots, f_K) \in X_n,
$$

$$
\|f\|_{Y_n} := \sup_{k=0,\ldots,K} \|f_k\|_X \quad \text{for} \quad f = (f_0, \ldots, f_K) \in Y_n,
$$

where $a_K$ is defined as before in (4.19) and $f_k \in X$ for all $k = 0, \ldots, K$. The proof of Theorem 4.1 has to be changed only at the last estimates (4.21) and (4.25), respectively.

Estimate (4.21) implies for $Z = 1$ that

$$
\|v_n - z_n\|_{X_n} = \frac{1}{K+1} \sup_{k=0,\ldots,K} \|(v_n)_k - (z_n)_k\|_X
\leq \frac{1}{K+1} \sup_{k=0,\ldots,K} \left( \sum_{j=0}^{k} \|(F_n(v_n))_j - (F_n(z_n))_j\|_X \right)
\leq \frac{1}{K+1} \sup_{k=0,\ldots,K} \left( \sum_{j=0}^{k} \sup_{j=0,\ldots,K} \|(F_n(v_n))_j - (F_n(z_n))_j\|_X \right)
\leq \frac{1}{K+1} \left( k + 1 \right) \|F_n(v_n) - F_n(z_n)\|_{Y_n}
= \frac{1}{K+1} \left( k + 1 \right) \|F_n(v_n) - F_n(z_n)\|_{Y_n} = \|F_n(v_n) - F_n(z_n)\|_{Y_n}.
$$

This yields $N$-stability with $C = 1$ indeed.

Estimate (4.21) implies for $Z > 1$ that

$$
\|v_n - z_n\|_{X_n} = \frac{Z-1}{Z^{K+1} - 1} \sup_{k=0,\ldots,K} \|(v_n)_k - (z_n)_k\|_X
\leq \frac{Z-1}{Z^{K+1} - 1} \left( \sum_{j=0}^{k} Z^{k-j} \|(F_n(v_n))_j - (F_n(z_n))_j\|_X \right)
\leq \frac{Z-1}{Z^{K+1} - 1} \left( \sum_{j=0}^{k} Z^{k-j} \sup_{j=0,\ldots,K} \|(F_n(v_n))_j - (F_n(z_n))_j\|_X \right)
\leq \frac{Z-1}{Z^{K+1} - 1} \left( \sum_{j=0}^{k} Z^{k-j} - 1 \right) \|(F_n(v_n)) - (F_n(z_n))\|_{Y_n}
= \frac{Z-1}{Z^{K+1} - 1} \left( \sum_{j=0}^{k} Z^{k-j} - 1 \right) \|(F_n(v_n)) - (F_n(z_n))\|_{Y_n} = \|(F_n(v_n)) - (F_n(z_n))\|_{Y_n}.
$$

Which yields $N$-stability again with $C = 1$. 

10
Now we endow the spaces \( X_n = \mathcal{X}^{K+1} \) and \( Y_n = \mathcal{X}^{K+1} \) with the following norms:

\[
\|f\|_{X_n} := \frac{1}{Z^K} \sup_{k=0,...,K} \|f_k\|_X \quad \text{for} \quad f = (f_0, ..., f_K) \in X_n = \mathcal{X}^{K+1},
\]

\[
\|f\|_{Y_n} := \sum_{k=0}^K \|f_k\|_X \quad \text{for} \quad f = (f_0, ..., f_K) \in Y_n = \mathcal{X}^{K+1}.
\]

Using (4.21) for \( Z = 1 \) one obtains now the following estimate instead of (4.23) in the proof of Theorem 4.1:

\[
\|v_n - z_n\|_{X_n} = \sup_{k=0,...,K} \|v_n)_k - (z_n)_k\|_X \leq \sup_{k=0,...,K} \left( \sum_{j=0}^k \|F_n(v_n))_j - (F_n(z_n))_j\|_X \right)
\]

\[
= \sum_{j=0}^k \|F_n(v_n))_j - (F_n(z_n))_j\|_X = \|F_n(v_n) - F_n(z_n)\|_{Y_n}
\]

meaning again \( N \)-stability with \( C = 1 \).

Using (4.21) we have for \( Z > 1 \) that

\[
\|v_n - z_n\|_{X_n} = \frac{1}{Z^K} \sup_{k=0,...,K} \|v_n)_k - (z_n)_k\|_X \leq \frac{1}{Z^K} \sup_{k=0,...,K} \sum_{j=0}^k Z^{k-j}\|F_n(v_n))_j - (F_n(z_n))_j\|_X
\]

\[
\leq \frac{1}{Z^K} Z^K \sup_{k=0,...,K} \sum_{j=0}^k \|F_n(v_n))_j - (F_n(z_n))_j\|_X
\]

\[
\leq \frac{1}{Z^K} Z^K \|F_n(v_n) - F_n(z_n)\|_{Y_n} = \|F_n(v_n) - F_n(z_n)\|_{Y_n}.
\]

Which yields again \( N \)-stability with \( C = 1 \).

5. Stability in the linear case

In this section we show how our results apply for the linear case. We take the same setting (spaces and norms) as defined by Sanz-Serna and Palencia in [8] for linear operators. Our aim is to show that Example 3.1 in [8], that is, the classical Lax–Richtmyer theory, follows from our recent results for rational approximations defined by formula (3.12).

Let \( F : D(F) \subset X \rightarrow Y \) be the operator defined in (3.8), where \( A \) is now a linear operator on the Banach space \( \mathcal{X} \). As in [8], let the spaces \( X_n = \mathcal{X}^{K+1}, Y_n = \mathcal{X}^{K+1} \) be endowed by the norms

\[
\|f\|_{X_n} = \sup_{k=0,...,K} \|f_k\|_X \quad \text{for all} \quad f \in X_n \quad \text{and}
\]

\[
\|f\|_{Y_n} = \sum_{k=0}^K \|f_k\|_X \quad \text{for all} \quad f \in Y_n,
\]

11
respectively, that is, the case (b) in Remark 4.2 without the multiplication by \(a_K\). Then condition (1.3) of the N-stability of the operator \(F_n : D(F_n) \subset X_n \rightarrow Y_n\), reduces to the estimate \(\|F_n^{-1}\|_{Y_n \rightarrow X_n} \leq \bar{C}\) for some constant \(\bar{C} > 0\). Let \(F_n\) be defined as in (3.14) with the linear operators \(A_m\), \(m \in \mathbb{N}\), satisfying Assumptions 3.1, and the rational approximation \(r\) defined in (3.12). In this case we have

\[
F_n = \begin{pmatrix}
I & 0 & 0 & \ldots & 0 \\
-\tau A_m & I & 0 & \ldots & 0 \\
0 & -\tau A_m & I & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & -\tau A_m & I
\end{pmatrix}
\]

and

\[
F_n^{-1} = \begin{pmatrix}
I & 0 & 0 & 0 & \ldots & 0 \\
\tau A_m & I & 0 & 0 & & 0 \\
\tau A_m^2 & \tau A_m & I & 0 & \ldots & 0 \\
\tau A_m^3 & \tau A_m^2 & \tau A_m & I & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
\tau A_m^K & \tau A_m^{K-1} & \tau A_m^{K-2} & \ldots & \tau A_m & I
\end{pmatrix}
\]

which are exactly the same operator matrices presented in [8]. The norm of \(F_n^{-1}\) can be estimated as

\[
\|F_n^{-1}\|_{Y_n \rightarrow X_n} = \sup_{f \in Y_n \atop \|f\|_{Y_n} = 1} \|F_n^{-1} f\|_{X_n}
\]

\[
= \sup_{f \in Y_n \atop \|f\|_{Y_n} = 1} \left\| \begin{pmatrix}
f_0 \\
\tau A_m f_0 + f_1 \\
\tau A_m^2 f_0 + \tau A_m f_1 + f_2 \\
\tau A_m^3 f_0 + \tau A_m^2 f_1 + \tau A_m f_2 + f_3 \\
\vdots \\
\tau A_m^K f_0 + \tau A_m^{K-1} f_1 + \tau A_m^{K-2} f_2 + \cdots + \tau A_m f_K + f_{K+1}
\end{pmatrix} \right\|_{X_n}
\]

\[
\leq \sup_{j=0,\ldots,K} \|\tau A_m^j\|_{X \rightarrow X} \sup_{f \in Y_n \atop \|f\|_{Y_n} = 1} \|f_{j+1}\|_{X}
\]

\[
\leq \sup_{j=0,\ldots,K} \|\tau A_m^j\|_{X \rightarrow X} \sup_{f \in Y_n \atop \|f\|_{Y_n} = 1} \sum_{j=0}^K \|f_j\|_{X}
\]

\[
= \sup_{j=0,\ldots,K} \|\tau A_m^j\|_{X \rightarrow X} \sup_{f \in Y_n \atop \|f\|_{Y_n} = 1} \|f\|_{Y_n} = \sup_{j=0,\ldots,K} \|\tau A_m^j\|_{X \rightarrow X}.
\]
Hence, one obtains the following stability condition: There should exists a constant $\tilde{C} > 0$ such that
\[
\sup_{k=0,\ldots,K} \|r(\tau A_m)^k\|_{X \to X} \leq \tilde{C}
\] (5.26)
holds for all $\tau = \frac{\tau}{K}$ for each fixed $\tau \geq 0$ time level. For a fixed $K \in \mathbb{N}$, this is the usual definition of Lax–Richtmyer stability obtained in [8] as well. Since formula (3.17) corresponds to $\|r(\tau A_m)^k\|_{X \to X} \leq Z$ for linear operators, we have that
\[
\sup_{k=0,\ldots,K} \|r(\tau A_m)^k\|_{X \to X} \leq \sup_{k=0,\ldots,K} \|r(\tau A_m)^k\|_{X \to X} \leq \sup_{k=0,\ldots,K} Z_k = Z^K,
\] (5.27)
that is, in this case the stability criterion (5.26) holds with $\tilde{C} := Z^K$ for each fixed $K \in \mathbb{N}$. We note that if the norms are defined as in Remark 4.2(b), then (5.27) is the same result as stated in Theorem 4.1.

6. Conclusion

We considered nonlinear evolution equations whose solution is given by a nonlinear semigroup. We showed that the definition of nonlinear semigroups already contains a sort of time discretisation, the implicit Euler method, which leads to $\mathcal{N}$-stable discrete problems when applied together certain convergent space discretisations. Moreover, we proposed a more general time discretisation, being the nonlinear counterpart of the rational approximations in the linear case, and showed its $\mathcal{N}$-stability as well. As a next step, the analysis of the concept when the operators $A_m$ do not act on $X$ but on the Banach spaces $X_m$, converging in some sense to $X$, is the topic of our forthcoming work.

Acknowledgement

I. Faragó and I. Fekete were supported by the Hungarian Research Grant OTKA-112157.

References