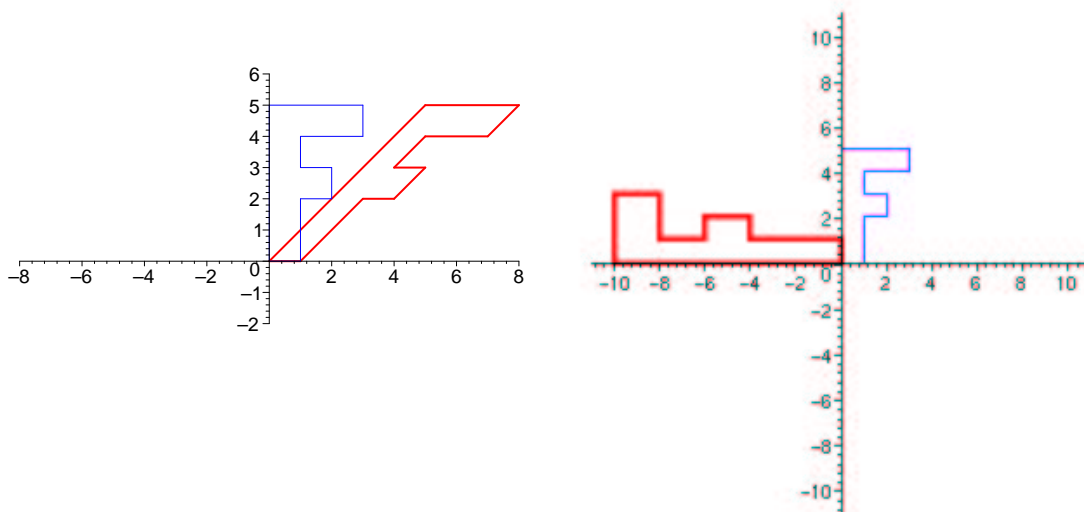


Homework Problems

- Let S be any set. Show that the set of all invertible maps $f : S \rightarrow S$ forms a group, where the operation is composition of maps.

REMARK: Many groups arise this way. If the set has some additional structure (linear space, metric space, topological space, etc.), the subgroup of invertible maps that preserves the structure are basic.

- If every element of a group has order two, show the group is abelian.
- Find all groups of order 6.
 - Find all groups of order 8.
- Show that the group of permutations of 3 elements is isomorphic to the dihedral group D_3 of symmetries of an equilateral triangle.
- Show that a linear map $L : \mathbb{R}^k \rightarrow \mathbb{R}^n$ takes straight lines to straight lines and parallel lines to parallel lines (although the images of parallel lines might coincide).
 - Let $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear map and $Q \in \mathbb{R}^2$ be the unit square: $0 < x < 1, 0 < y < 1$. If $\hat{Q} := A(Q)$ is its image, show that $\text{area}(\hat{Q}) = |\det A| \text{area}(Q)$. [More generally, the area of any region is magnified by $|\det A|$.]
 - For the following two figures find matrices A that gives the indicated linear transformation [here the light F is mapped to the dark F].

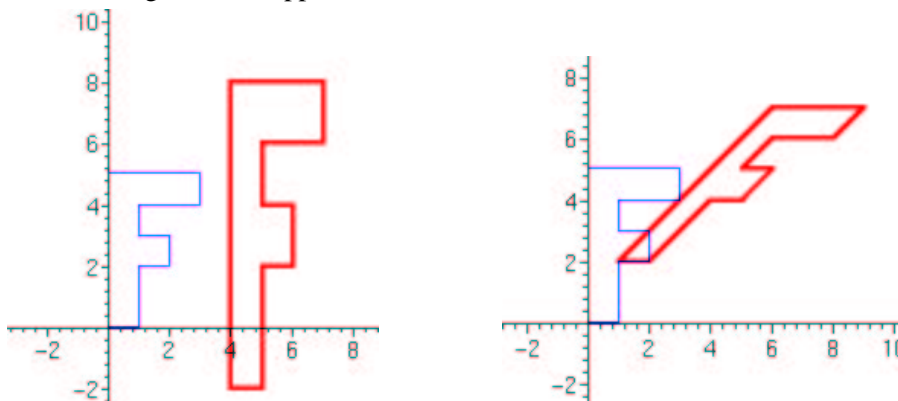


- In both of the above figures, draw the letter **F** that one gets using the *inverse* transformation, A^{-1} . Also compute A^{-1} .
- Linear maps $F(X) = AX$, where A is a matrix, have the property that $F(0) = A0 = 0$, so they necessarily leave the origin fixed. It is simple to extend this to include a translation to get the larger space of *affine maps*,

$$F(X) = V + AX,$$

where V is a vector. Note that $F(0) = V$.

- a) Show that the affine map F is invertible if and only if the linear map A is invertible – and give a formula for F^{-1} in terms of V and A^{-1} .
- b) Show that the set of invertible affine maps forms a group.
- c) For practice, find the vector V and the matrix A that describe each of the following mappings [here the light F is mapped to the dark F].



7. An $n \times n$ real matrix $R: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an *orthogonal matrix* if it preserves the (Euclidean) length of each vector $x \in \mathbb{R}^n$, that is,

$$\|Rx\| = \|x\| \quad \text{for all } x \in \mathbb{R}^n.$$

- a) If R and S are both $n \times n$ orthogonal matrices, show that their product, SR is also an orthogonal matrix [thus they form a *group*, $O(n)$, under multiplication].
 - b) Show that the following are equivalent:
 - i) R is an orthogonal matrix.
 - ii) $\langle Rx, Ry \rangle = \langle x, y \rangle$ for all x, y in \mathbb{R}^n .
 - iii) R is invertible and $R^{-1} = R^T$. [R^T is the transpose of R .]
 - iv) The columns of R are orthonormal vectors.
 - v) The rows of R are orthonormal vectors.
 - c) If $R \in O(n)$, show that $\det R = \pm 1$.
 - d) If λ is an eigenvalue of R , show that $|\lambda| = 1$.
 - e) If $R \in O(2)$ has $\det R = 1$, show that $R = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ for some real θ , so R is just a rotation through an angle θ . Based on this, it is useful to define the orthogonal matrices R^t for real t as $R^t := \begin{pmatrix} \cos t\theta & -\sin t\theta \\ \sin t\theta & \cos t\theta \end{pmatrix}$.
8. (Euler) Let $SO(3)$ be the set of 3×3 real orthogonal matrix with $\det R = 1$.
 - a) If $R \in SO(3)$, show that $\lambda = 1$ is an eigenvalue.
 - b) If $R \neq I$, let v be an eigenvector associated with $\lambda = 1$. Show that R is a rotation of \mathbb{R}^3 with v as the axis of rotation. What is the angle of rotation?
 - c) Define and use R^t (see Problem 7e) to show that the group of 3×3 real orthogonal matrices in $SO(3)$ is connected.

9. Some topological properties of $O(n)$.

- Show that the set of $n \times n$ orthogonal matrices is a compact set in \mathbb{R}^{n^2} .
- Use $\det R = \pm 1$ to show that $O(n)$ is not connected.
- Let $SO(n)$ be the subgroup of orthogonal matrices with $\det R = +1$. Show that $SO(n)$ is connected.
- Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the reflection across the hyperplane $x_1 = 0$, so $T : (x_1, x_2, \dots, x_n) \rightarrow (-x_1, x_2, \dots, x_n)$. Show that if $R \in O(n)$ with $\det R = -1$, then $R = TR_0$, where $R_0 \in SO(n)$. This reduces the study of the orthogonal group to that of $SO(n)$.

10. Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be an *isometry*, that is, it preserve the (Euclidean) distance between points in \mathbb{R}^n :

$$\|F(x) - F(y)\| = \|x - y\| \quad \text{for all } x, y \text{ in } \mathbb{R}^n.$$

- Show that the isometries of \mathbb{R}^n form a group.

More generally, the invertible isometries of any metric space form a group. The surjective assumption is needed as illustrated by the example of the Hilbert Space ℓ_2 . The right shift map $S : (x_1, x_2, \dots) \mapsto (0, x_1, x_2, \dots)$ is an isometry that is not surjective. Understanding a group of invertible isometries is a basic problem.

- If also $F(0) = 0$, show that F is a linear map. Thus in the definition of an orthogonal transformation (see the previous problem), one need not assume R is linear, only that it keeps the origin fixed. [This is not true for isometries of *complex* vector spaces. Example: complex conjugation is an isometry that is not a linear map.]
- Conclude that the group of isometries of \mathbb{R}^n with the Euclidean metric are the affine maps $F(x) = v + Rx$, where $v \in \mathbb{R}^n$ and R is an orthogonal transformation.

11. Let $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ be points in \mathbb{C}^n . Define the inner product and norm by the usual formulas $\langle z, w \rangle := z_1 \bar{w}_1 + z_2 \bar{w}_2 + \dots + z_n \bar{w}_n$, and $\|z\| := \sqrt{\langle z, z \rangle}$. Define a matrix U to be *unitary* if it preserves the norm:

$$\|Uz\| = \|z\| \quad \text{for all } z \in \mathbb{C}^n.$$

Let $U(n)$ be the set of $n \times n$ unitary matrices.

- Explicitly write the general 2×2 unitary matrix.
- State and prove the assertions analogous to Problem 7.

12. Some topological properties of $U(n)$.

- Show that the set of $n \times n$ unitary matrices is a compact set in \mathbb{C}^{n^2} .
- Show that $SU(n)$, the unitary matrices with $\det U = +1$, is a subgroup of $U(n)$.
- Is $U(n)$ connected? Is $SU(n)$ connected?

13. Say one associates the complex number $z = x + iy$ (x, y real) with the 2×2 real matrix

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix} = x \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + y \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Show that gives an isomorphism between the complex numbers and this family of matrices. Note that complex conjugation corresponds to taking the adjoint of the matrix.

14. The *quaternions* can be defined as expressions of the form $q = x + y\mathbf{i} + z\mathbf{j} + w\mathbf{k}$, where $x, y, z,$ and w are real numbers. They are added as vectors and multiplied using the rules $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$, $\mathbf{ij} = \mathbf{k} = -\mathbf{ji}$, $\mathbf{jk} = \mathbf{i} = -\mathbf{kj}$, $\mathbf{ki} = \mathbf{j} = -\mathbf{ik}$ and the usual distributive rules. Define the *conjugate* by $\bar{q} = x - y\mathbf{i} - z\mathbf{j} - w\mathbf{k}$.

a) Compute $|q|^2 := q\bar{q}$. Use this to show that every $q \neq 0$ has a multiplicative inverse. Thus show that the quaternions are a field – except they are *not* commutative under multiplication.

b) Let

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{i} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{j} = ? \quad \mathbf{k} = ? \quad (1)$$

be (complex) 2×2 matrices. Find \mathbf{j} and \mathbf{k} so that the set of matrices of the form $Q = x\mathbf{I} + y\mathbf{i} + z\mathbf{j} + w\mathbf{k}$ is isomorphic to the quaternions.

c) Prove that the *unit quaternions*, that is, those q with $|q|^2 = x^2 + y^2 + z^2 + w^2 = 1$, form a group under multiplication. Note that the unit quaternions can also be thought of as points on the unit sphere $S^3 \in \mathbb{R}^4$.

Prove that this group is isomorphic to SU_2 .

15. Consider the linear space of real $k \times n$ matrices, so maps from \mathbb{R}^n to \mathbb{R}^k .

a) Show that the following defines an inner product:

$$\langle A, B \rangle := \text{trace}(A^T B).$$

b) How should this be modified for complex matrices?

c) Show that the four matrices in equation (1) are orthogonal to each other.

16. Consider the set \mathcal{T} of upper triangular $n \times n$ matrices.

REMARK: Although some matrices, as $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, are not similar to diagonal matrices, a useful short theorem by Shur states that *given any square matrix matrix A there is an upper triangular matrix T and a unitary matrix U so that $A = UTU^{-1}$, so every A is similar to an upper triangular matrix.* While not needed for this problem, this will be helpful later on. It is often an adequate substitute for the more complicated Jordan normal form.

a) Show that the product ST of two upper triangular matrices is also upper triangular, and its diagonal elements are the products of the corresponding diagonal elements of S and T .

b) Show that $A \in \mathcal{T}$ is invertible if and only none of its diagonal elements are zero.

c) Show that the invertible upper triangular matrices form a multiplicative group.

17. If A is a square matrix (real or complex), we define $e^A := \sum_0^{\infty} \frac{A^k}{k!}$.

a) Show that the series always converges.

- b) Show that $e^{(s+t)A} = e^{sA}e^{tA}$ for all real or complex s, t .
- c) If $AB = BA$, show that $e^{A+B} = e^Ae^B$. Give an example showing this is false if A and B don't commute.
- d) If $A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$, compute e^A .
- e) If $A = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$, compute e^A .
- f) If $A = \begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix}$, compute e^A .
- g) If a square matrix A satisfies $A^2 = \alpha^2 I$, show that

$$e^A = (\cosh \alpha)I + \frac{\sinh \alpha}{\alpha}A.$$

How is this modified if $A^2 = -\alpha^2 I$?

- h) If a square matrix A satisfies $A^3 = \alpha^2 A$, compute e^A . The resulting formula generalizes the previous part. What is the formula if $A^3 = -\alpha^2 A$?
- i) If $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, compute e^A . This is the general 2×2 matrix with trace zero. There are three cases, depending on the sign of $\det A$.
- j) Use the previous part to compute e^A for a general 2×2 real matrix. What about complex matrices?
18. a) If λ is an eigenvalue of A with corresponding eigenvector v , show that e^λ is an eigenvalue of e^A with the same eigenvector.
- b) If the matrix A is similar to the diagonal matrix D , so $A = S^{-1}DS$, compute e^A in terms of D and S .
- c) If the matrix A is similar to the upper-triangular matrix T , so $A = S^{-1}TS$, compute e^A in terms of T and S .
19. a) If A is a real anti-symmetric matrix and $R := e^A$, show that R is an orthogonal matrix and satisfies $\det R = +1$.
- b) Given any orthogonal matrix R with $\det R = 1$, can one always find an anti-symmetric matrix A so that $e^A = R$?
- c) If A is a complex self-adjoint matrix and $U := e^{iA}$, show that U is a unitary matrix.

20. Let $A = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$ be a 3×3 real anti-symmetric matrix.

- a) Show that

$$e^A = I + \frac{\sin \theta}{\theta}A + \frac{1 - \cos \theta}{\theta^2}A^2,$$

where $\theta^2 = a^2 + b^2 + c^2$. [Suggestion: Observe that A^3 has a special form.]

- b) Let $R = e^A$, so R is an orthogonal matrix. As in Problem 8 show that the axis of rotation of R is $v = (c, -b, a)$ while $\theta = \|v\|$ is the angle of rotation. Thus, given the axis and angle of rotation, this gives an explicit formula for the corresponding orthogonal matrix.

21. Given vectors v and x in \mathbb{R}^3 , $v \neq 0$, show there is a unique anti-symmetric matrix A so that $v \times x = Ax$. Here we use the vector product in \mathbb{R}^3 .

22. Show that $\det(e^A) = e^{\text{trace}(A)}$.

23. Let $A(t)$ be a square matrix whose elements depend smoothly on a real parameter t .

a) Compute the derivative: $dA^3(t)/dt$.

b) If $A(t)$ is invertible, find the formula for the derivative of $A^{-1}(t)$. Of course it will resemble the 1×1 case $-A'(t)/A^2(t)$.

c) If A is a constant matrix (so it does not depend on t), compute the derivative of e^{tA} .

24. a) If $M := \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$, find a constant matrix A so that $M = e^{tA}$ for all real t .

b) If $N := \begin{pmatrix} \cosh t & -\sinh t \\ \sinh t & \cosh t \end{pmatrix}$, show there is *no* constant matrix A so that $N = e^{tA}$.

25. Let $A(t)$ be a real anti-symmetric matrix that depends continuously on a real parameter t and $R(t)$ the (matrix) solution of the differential equation

$$\frac{dR(t)}{dt} = AR(t), \quad \text{with } R(0) = I.$$

Show that for each $t \in \mathbb{R}$, the matrix $R(t)$ is an orthogonal transformation with $\det R(t) = 1$.

26. Let A be a square constant matrix. Show that the (unique) solution $X(t)$ of the matrix differential equation

$$\frac{dX(t)}{dt} = AX(t), \quad \text{with } X(0) = I$$

is $X(t) = e^{tA}$.

27. a) If A is an invertible real matrix, show there is a unique positive definite matrix P and a unique orthogonal matrix R so that $A = PR$. [This is like writing a complex number in polar form $z = re^{i\theta}$].

If $\det A > 0$, one can use this to define $A^t := P^t R^t$ for all real t .

b) The set of all invertible real $n \times n$ real matrices with $\det A = 1$ is called $SL(n, \mathbb{R})$. Show that $SL(n, \mathbb{R})$ is a group and is connected.

28. a) If A is an invertible (possibly complex) matrix, show there is a unique positive definite matrix P and a unique unitary matrix U so that $A = PU$. [This generalizes writing a complex number in polar form $z = re^{i\theta}$].

b) Show that the group $GL(n, \mathbb{C})$ of invertible complex $n \times n$ matrices is connected.

29. In Problem 7 we defined an orthogonal transformation on \mathbb{R}^2 as a linear map that preserves the quadratic form $Q(x, y) := x^2 + y^2$. What can you say about the linear maps that preserve the indefinite quadratic form $Q(x, y) := x^2 - y^2$? [In Special Relativity one measures distance in spacetime (x, t) using $x^2 - c^2t^2$, where c is the speed of light. This is also related to finding all the linear transformations that preserve the wave operator $\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{c^2 \partial t^2}$.]

More generally, the orthogonal group of any non-singular quadratic form is the group of linear maps that preserve the form.