CLASSIFICATION OF L_∞ ALGEBRAS ON A $2|1$-DIMENSIONAL SPACE

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Abstract. This article explores L_∞ algebra structures on a $2|1$-dimensional vector space. We do not give a complete classification in all cases, but do determine all structures which begin with either a nontrivial first or second order term. In particular, we determine all extensions of a super Lie algebra as an L_∞ algebra. The reader should note that our convention on the parities is the opposite of the usual one, because we define our structures on the symmetric coalgebra of the parity reversion of a space, so our $2|1$-dimensional L_∞ algebras correspond to the usual $1|2$-dimensional algebras.

1. Introduction

In [3], the authors have classified all L_∞ algebras of dimension less than or equal to 2, in [5], we constructed miniversal deformations for all L_∞ structures on a space of dimension $0|3$, and in [4], L_∞ algebras of dimension $1|2$ were classified.

The theory in the $2|1$ dimensional case is more complicated than $1|2$-dimensional algebras, because the space of n-cochains on a $1|2$ dimensional space has dimension $6|6$ for $n > 1$, while the space of n-cochains on a $2|1$- dimensional space has dimension $3n + 2|3n + 1$, making it more difficult to classify the nonequivalent structures. Accordingly, we will give a complete classification here of only those L_∞ algebras which are extensions of degree 1 coderivations, which are, as it turns out, equivalent to degree 1 coderivations, and those which are extensions of degree 2 coderivations, in other words, extensions of $\mathbb{Z}_2$-graded Lie algebras as L_∞ algebras.

We work in the framework of the parity reversion $W = \Pi V$ of the usual vector space V on which an L_∞ algebra structure is defined, because in the W framework, an L_∞ structure is simply an odd

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coderivation d of the symmetric coalgebra $S(W)$, satisfying $d^2 = 0$, in other words, it is an odd codifferential in the $\mathbb{Z}_2$-graded Lie algebra of coderivations of $S(W)$. As a consequence, when studying $\mathbb{Z}_2$-graded Lie algebra structures on V, the parity is reversed, so that a $2|1$-dimensional vector space W corresponds to a $1|2$-dimensional $\mathbb{Z}_2$-graded Lie structure on V. Moreover, the $\mathbb{Z}_2$-graded anti-symmetry of the Lie bracket on V becomes the $\mathbb{Z}_2$-graded symmetry of the associated coderivation d on $S(W)$.

A formal power series $d = d_1 + \cdots$, with $d_i \in L_i = \operatorname{Hom}(S^i(W), W)$ determines an element in $L = \operatorname{Hom}(S(W), W)$, which is naturally identified with $\operatorname{Coder}(S(W))$, the space of coderivations of the symmetric coalgebra $S(W)$. Thus L is a $\mathbb{Z}_2$-graded Lie algebra. An odd element d in L is a called a *codifferential* if $[d, d] = 0$. We also say that d is an L_∞ structure on W.

If $g = g_1 + \cdots \in \operatorname{Hom}(S(W), W)$, and $g_1 : W \to W$ is invertible, then g determines a coalgebra automorphism of $S(W)$ in a natural way, which we will denote by the same letter g. Moreover, every coalgebra automorphism is determined in this manner. Two codifferentials d and d' are said to be equivalent if there is a coalgebra automorphism g such that $d' = g^*(d) = g^{-1}dg$.

A detailed description of L_∞ algebras can be obtained in [6, 7]. The study of examples of L_∞ algebra structures in [3, 5, 4] may be useful to the reader in understanding how extensions and deformation theory work in practice, but we intend this article to be as self contained as possible. We shall use the following facts, which are established in [2, 1], to aid in the classification.

If d is an L_∞ structure on W, and d_N is the first nonvanishing term in d, then d_N is itself a codifferential, which we call the *leading term of* d, and we say that d is *an extension* of d_N. Define the cohomology operator D by $D(\varphi) = [\phi, d_N]$, for $\varphi \in L$. Then the following formula holds for any extension d of d_N as an L_∞ structure, and all $n \geq N$.

$$D(d_{n+1}) = -\frac{1}{2} \sum_{k=N+1}^{n} [d_k, d_{n+N-k+1}]. \tag{1}$$

Note that the terms on the right all have index less than $n + 1$. If a coderivation d has been constructed up to terms of degree m, satisfying equation (1) for $n = 1 \ldots m - 1$, then the right hand side of equation (1) for $n = m$ is automatically a cocycle. Thus d can be extended to the next level precisely when the cocycle given by the right hand side is trivial. There may be many nonequivalent extensions, because the term d_{n+1} which we add to extend the coderivation is only determined up to a cocycle. An extension d of d_N is given by any coderivation whose leading term is d_N, which satisfies equation (1) for every $m = N+1 \dots$. The theory here is parallel to the theory of formal deformations of an algebra structure; the extension of a codifferential d_N to a more complicated codifferential d resembles the process of extending an infinitesimal deformation to a formal one.

Classifying the extensions of d_N can be quite complicated. However, the following theorem often makes it easy to classify the extensions.

Theorem 1.1. *If the cohomology $H^n(d_N) = 0$, for $n > N$, then any extension of d_N to a L_∞ structure d is equivalent to the structure d_N.*

Before classifying the extensions of a codifferential d_N, we need to classify the codifferentials in L_N up to equivalence, that is, we need to study the *moduli space* of degree N codifferentials. A *linear automorphism* of $S(W)$ is an automorphism determined by an isomorphism $g_1 : W \to W$. If g is an arbitrary automorphism, determined by maps $g_i : S^i(W) \to W$, and W is finite dimensional, then g_1 is an isomorphism, so this term alone induces an automorphism of $S(W)$ which we call the *linear part* of g.

The following theorem simplifies the classification of equivalence classes of codifferentials in L_N.

Theorem 1.2. *If d and d' are two codifferentials in L_N, and g is an equivalence between them, then the linear part of g is also an equivalence between them.*

Thus we can restrict ourself to linear automorphisms when determining the equivalence classes of elements in L_N.

We will also use the following result.

Theorem 1.3. *Suppose that d and d' are equivalent codifferentials. Then their leading terms have the same degree and are equivalent.*

As a consequence of these theorems, we proceed to classify the codifferentials as follows. First, find all equivalence classes of codifferentials of degree N. For each equivalence class, study the equivalence classes of extensions of the codifferential.

Let us first establish some basic notation for the cochains. Suppose $W = \langle w_1, w_2, w_3\rangle$, with $w_1 =$, an odd element and $w_2, w_3 =$ even elements. If $I = \{i_1, i_2, i_3\}$ is a multi-index, with i_1 and i_2 either zero or one, let $w_I = w_1^{i_1} w_2^{i_2} w_3^{i_3}$. For simplicity, we will denote w_I simply by I. Then for $n \geq 1$,

$$(S^n(W))_e = \langle (0, p, n-p) | 0 \leq p \leq n \rangle, \qquad |(S^n(W))_e| = n + 1$$
$$(S^n(W))_o = \langle (1, q, n-q-1) | 0 \leq q \leq n-1 \rangle, \qquad |(S^n(W))_0| = n$$

If λ is a linear automorphism of $S(W)$, then in terms of the standard basis of W, its restriction to W has matrix

$$\lambda = \begin{pmatrix} q & 0 & 0 \\ 0 & r & t \\ 0 & s & u \end{pmatrix} \tag{2}$$

where $q(ru-st) \neq 0$. We will sometimes express λ by the submatrix $\left(\begin{smallmatrix} r & t \\ s & u \end{smallmatrix}\right)$. It is useful to note that for a linear automorphism

$$\lambda(w^I) = \lambda(w_1)^{i_1}\lambda(w_2)^{i_2}\lambda(w_3)^{i_3},$$

so that

$$\lambda(1,x,y) = \sum_{i=0}^{x}\sum_{j=0}^{y}(1, i+j, x+y-i-j)\binom{x}{i}\binom{y}{j} q r^i s^{x-i} t^j u^{y-j}. \tag{3}$$

Let $L_n := \operatorname{Hom}(S^n(W), W)$. Define

$$\varphi^I_j(w_J) = I!\delta^I_J w_j,$$

where $I! = i_1!i_2!i_3!$. If we let $|I| = i_1 + i_2 + i_3$, then $L_n = \langle \varphi^I_j, |I| = n\rangle$. If φ is odd, we denote it by the symbol ψ to make it easier to distinguish the even and odd elements. Then

$$(L_n)_e = \langle \varphi_1^{1,q,n-q-1}, \varphi_2^{0,p,n-p}, \varphi_3^{0,p,n-p} | 1 \leq q \leq n-1, 1 \leq p \leq n\rangle$$
$$(L_n)_o = \langle \psi_2^{1,q,n-q-1}, \psi_3^{1,q,n-q-1}, \psi_1^{0,p,n-p} | 1 \leq q \leq n-1, 1 \leq p \leq n\rangle,$$

so that $|L_n| = 3n+2|3n+1$.

2. Classification of Codifferentials

Let us compute the brackets of all odd cochains with each other.

$$[\psi_1^{0,p,n-p}, \psi_2^{1,q,m-q-1}] = \varphi_1^{1,p+q-1,n-p+m-q-1}p + \varphi_2^{0,p+q,n-p+m-q-1}$$
$$[\psi_1^{0,p,n-p}, \psi_3^{1,q,m-q-1}] = \varphi_1^{1,p+q,n-p+m-q-2}(n-p) + \varphi_3^{0,p+q,n-p+m-q-1}$$
$$[\psi_1^{0,p,n-p}, \psi_1^{0,q,m-q}] = 0 \qquad [\psi_2^{1,p,n-p-1}, \psi_2^{1,q,m-q-1}] = 0$$
$$[\psi_2^{1,p,n-p-1}, \psi_3^{1,q,m-q-1}] = 0 \qquad [\psi_3^{1,p,n-p-1}, \psi_3^{1,q,m-q-1}] = 0$$

Suppose that

$$d = \sum_{p=0}^{n} \psi_1^{0,p,n-p} a_p + \sum_{q=0}^{n-1} \psi_2^{1,q,n-q} b_q + \psi_3^{1,q,n-q} c_q, \tag{4}$$

where we sum over all odd codifferentials of degree n. Then using the above, we compute that

$$(5)\quad [d,d] = \sum_{p=0}^{n}\sum_{q=0}^{n-1} \varphi_1^{1,p+q-1,2n-p-q-1} p a_p b_q + \varphi_2^{0,p+q,2n-p-q-1} a_p b_q + \varphi_1^{1,p+q,2n-p-q-2}(n-p) a_p c_q + \varphi_3^{0,p+q,2n-p-q-1} a_p c_q$$

We claim that either all coefficients a_p must vanish or all coefficients b_q and c_q must vanish. For if p and q are the least indices for which a_p and b_q do not vanish, then there is only one term in the sum above of the form $\varphi_2^{0,p+q,2n-p-q-1}$, which would be a contradiction because its coefficient $a_p b_q$ must vanish.

As a consequence of this observation, we note that codifferentials of degree n fall into two distinct families, those of the *first kind*

$$(6)\qquad \sum_{q=0}^{n-1} \psi_2^{1,q,n-q} b_q + \psi_3^{1,q,n-q} c_q.$$

and those of the *second kind*

$$(7)\qquad d = \sum_{p=0}^{n} \psi_1^{0,p,n-p} a_p.$$

Moreover, any expression of either kind gives a codifferential. Thus we have determined all codifferentials of degree N. However, the process of classification requires that we determine the equivalence classes of codifferentials under the action of the automorphism group of the symmetric coalgebra, and we are a long way away from this classification at this stage.

Several things can be said in general. First, let us suppose that d is of the second kind. Then from the brackets computed already, we note that the odd d-cocycles are precisely the odd cochains of the second kind. The space of odd cocycles has dimension $n+1$, which means that the space of even d-coboundaries has dimension $2n$. Also, if φ is any even cocycle, then its bracket with d is an odd cocycle of the second kind. Precise computation of the cohomology depends on solving a linear system of equations whose coefficients depend on the coefficients in d.

Similarly, if d is of the first kind, then the odd d-cocycles are the ones of the first kind, and thus the dimension of the space of odd cocycles is $2n$, and the dimension of the space of even coboundaries is $n+1$. The bracket of any even cocycle with d is a cocycle of the first kind. The cohomology can be computed by solving a system of linear equations in coefficients depending on the coefficients of d.

Since there are $2n$ coefficients in a codifferential of the first kind, and $n+1$ coefficients in a codifferential of the second kind, there are potentially a lot of equivalence classes of codifferentials. The main strategy involved in classification is to reduce the number of independent variable to a manageable number.

It is useful to compute the brackets of even and odd cochains.

$$
\begin{aligned}
[\varphi_1^{1,p,m-p},\psi_1^{0,q,n-q}] =&\psi_1^{0,p+q,m+n-p-q}\\
[\varphi_2^{0,p,m-p},\psi_1^{0,q,n-q}] =&-\psi_1^{0,p+q-1,m+n-p-q}q\\
[\varphi_3^{0,p,m-p},\psi_1^{0,q,n-q}] =&\psi_1^{0,p+q,m+n-p-q-1}(-n+q)\\
[\varphi_1^{1,p,m-p},\psi_2^{1,q,n-q}] =&-\psi_2^{1,p+q,m+n-p-q}\\
[\varphi_2^{0,p,m-p},\psi_2^{1,q,n-q}] =&\psi_2^{1,p+q-1,m+n-p-q}(p-q)\\
[\varphi_3^{0,p,m-p},\psi_2^{1,q,n-q}] =&\psi_2^{1,p+q,m+n-p-q-1}(-n+q)+\psi_3^{1,p+q-1,m+n-p-q}p\\
[\varphi_1^{1,p,m-p},\psi_3^{1,q,n-q}] =&-\psi_3^{1,p+q,m+n-p-q}\\
[\varphi_2^{0,p,m-p},\psi_3^{1,q,n-q}] =&\psi_2^{1,p+q,m+n-p-q-1}(m-p)-\psi_3^{1,p+q-1,m+n-p-q}q\\
[\varphi_3^{0,p,m-p},\psi_3^{1,q,n-q}] =&\psi_3^{1,p+q,m+n-p-q-1}(m+n-p+q)
\end{aligned}
$$

Notice that bracket of any even cochain with an odd cochain of a certain type is an odd cochain of the same type. This is very important in what follows, because this fact means that there is no mixing of types occurring in the cohomology of a codifferential of a fixed type.

Let us call the degree of the leading term of a codifferential the order of that codifferential. We begin with a classification of all codifferentials of order 1.

3. Classification of codifferentials with $d_1 \neq 0$

Let us suppose that d is an odd, degree 1 codifferential of the first kind. Then $d=\psi_2^{1,0,0}a_1+\psi_3^{1,0,0}a_2$ for some constants a_1 and a_2. To see that d is equivalent to $d'=\psi_2^{1,0,0}$, let t and u be such that $a_1t+a_2u\neq 0$. Suppose that $g=\begin{pmatrix}a_1 & t\\ a_2 & u\end{pmatrix}$. Then

$$dg(w_1)=d(w_1)=w_2a_1+w_3a_2=g(w_2)=gd'(w_1).$$

Since $dg(w_2)=gd'(w_2)=0$ and $dg(w_3)=gd'(w_3)=0$, it follows that d' and d are equivalent. Thus every codifferential of the first kind is equivalent to $\psi_2^{1,0,0}$.

Now let us study the cohomology of the codifferential $d=\psi_2^{1,0,0}$. We define the coboundary operator D by $D(\varphi)=[\varphi,d]$. Then Computing brackets, we see that

$$\begin{aligned} D(\varphi_2^{0,p,n-p}) &= \psi_2^{1,p-1,n-p}p, \quad D(\psi_1^{0,p,n-p}) = \varphi_1^{1,p-1,n-p}p+\varphi_2^{0,p,n-p}\\ D(\varphi_3^{0,p,n-p}) &= \psi_3^{1,p-1,n-p}p, \quad D(\psi_2^{1,q,n-q-1})=0\\ D(\varphi_1^{1,q,n-q-1}) &= -\psi_2^{1,q,n-q-1}, \quad D(\psi_3^{1,q,n-q-1})=0 \end{aligned}$$

Note that for the n-cochains above, p ranges from 0 to n, while q ranges only from 0 to $n-1$. It is easy to see that $\psi_2^{1,q,n-q-1}$ and $\psi_3^{1,q,n-q-1}$ give a basis of the odd cocycles, and since both of these types are evidently coboundaries, of $\varphi_2^{0,p,n-p}$ and $\varphi_3^{0,p,n-p}$, resp., where $q=p-1$, all odd cocycles are coboundaries. Similarly, if we let $q=p-1$, then every even cocycle is a linear combination of elements of the form $\varphi_1^{1,p-1,n-p}p+\varphi_2^{0,p,n-p}$, and since these elements are coboundaries, it follows that all even cocycles are coboundaries. Thus the cohomology of d is zero, and we know by Theorem (1.1) that all extensions of d are equivalent to d. This completes the picture for codifferentials of the first kind of degree 1.

If d is a codifferential of the second kind of degree 1, it is of the form $d=\psi_1^{0,1,0}a_1+\psi_1^{0,0,1}a_2$. We show that it is equivalent to $d'=\psi_1^{0,1,0}$. For suppose that b_1 and b_2 are chosen so that $a_1b_1+a_2b_2=1$. Then if g is given by $\begin{pmatrix} b_1 & -a_2\\ b_2 & a_1\end{pmatrix}$, we have

$$\begin{aligned} dg(w_2) &= d(w_2b_1+w_3b_2)=w_1(a_1b_1+a_2b_2)=w_1=gd'(w_2)\\ dg(w_3) &= d(-w_2a_2+w_3a_1)=-a_2a_1+a_1a_2=0=gd'(w_3) \end{aligned}$$

Now, we study the cohomology induced by $d=\psi_2^{1,0,0}$. Calculating coboundaries, we have

$$\begin{aligned} D(\varphi_2^{0,p,n-p}) &= -\,\psi_1^{0,p,n-p}, & D(\psi_1^{0,p,n-p}) &= 0\\ D(\varphi_3^{0,p,n-p}) &= 0, & D(\psi_2^{1,q,n-q-1}) &= \varphi_1^{1,q,n-q-1}+\varphi_2^{0,q+1,n-q-1}\\ D(\varphi_1^{1,q,n-q-1}) &= \psi_1^{1,q+1,n-q-1}, & D(\psi_3^{1,q,n-q-1}) &= \varphi_3^{0,q+1,n-q-1} \end{aligned}$$

It is not difficult to see from this table that the cohomology of this codifferential is also zero. Thus every extension of a codifferential of the second kind is equivalent to the original codifferential.

The picture for codifferentials of degree 1 is very simple. First, the classification into equivalence classes is easy, and then, since the cohomology vanishes, the classification of extensions is immediate. There are exactly two equivalence classes of codifferentials of order 1.

We will study the odd codifferentials of degree two next, with an aim to classify all extensions of such codifferentials to L_∞ algebra structures. As a byproduct of this process, we will also give a complete classification of all $1|2$ dimensional $\mathbb{Z}_2$-graded Lie algebras, which are exactly the codifferentials of degree 2 on a $2|1$ dimensional space.

4. Codifferentials of Degree 2 of the First Kind

Let us suppose that $d=\psi_2^{1,1,0}x+\psi_3^{1,1,0}a+\psi_2^{1,0,1}b+\psi_3^{1,0,1}c$, and let us call the multi-index (x,a,b,c) the type of the codifferential. Let us say that a codifferential is of type (x,a,b,c) whenever it is equivalent to a codifferential of that type, so that the type of a codifferential is not unique. Our goal is to show that the equivalence classes of codifferentials reduce to only a few simple types. Let us first remark that if we express d as a matrix of the form $d=\begin{pmatrix}x&b\\a&c\end{pmatrix}$, then if $d'=g^{-1}dg$, then its matrix is simply the product of the matrices expressing g^{-1}, d and g, multiplied by the scalar q, where $g(w_1)=w_1q$.

If $x\neq0$ then by applying a simple coalgebra automorphism, one can assume that it is equal to one. Similarly, if $a\neq0$ one can assume it is also 1. Thus if both x and a are non zero, our codifferential is of type $(1,1,b,c)$. We will show later that we can express codifferentials of this type in an even simpler form, but first we examine what possibilities have not been covered by our considerations.

If $x=0$, but $c\neq0$, then by interchanging the roles of w_2 and w_3 one can replace it with an equivalent one whose $\psi_2^{1,1,0}$ has nonzero coefficient. Similarly, if $x=0$ or $a=0$ and both b and c do not vanish, then by the same interchange, we can see that the codifferential has type $(1,1,b,c)$ as well. This observation leads to the following possibilities,

If $x\neq0$ but $a=0$, then we can assume that either $b=0$ or $c=0$ (or both). This gives the possible types $(1,0,0,c)$ or $(1,0,b,0)$. If $b\neq0$, then by a simple transformation, the type $(1,0,b,0)$ can be reduced to type $(1,0,1,0)$.

The only other types which could arise have both the x and c coefficients vanishing, so they are of type $(0,a,b,0)$. If both a and b do not vanish, they can be adjusted so we obtain type $(0,1,1,0)$, and if one of the two vanishes but the other does not, we obtain type $(0,1,0,0)$.

Actually, this myriad of types can be much reduced as we shall see shortly. Let us examine the type $(1,1,b,c)$ and show that in most cases it can be reduced to type $(1,0,0,c)$.

Let d be of type $(1,1,b,c)$. Then if $d'=g^{-1}dg$, we compute

$$d'=\begin{pmatrix} \frac{q(ru+bus-rt-cst)}{ru-ts} & \frac{q(ut+bu^2-t^2-ctu)}{ru-ts} \\ -\frac{q(sr+bs^2-r^2-crs)}{ru-ts} & -\frac{q(ts+bus-rt-cru)}{ru-ts} \end{pmatrix} \tag{8}$$

Now, either r and u both do not vanish, or s and t both do not. Let us assume the former, and put $x=s/r$ and $y=t/u$. Substituting in the matrix for d', we obtain

$$d'=\begin{pmatrix} \frac{q(1+bx-y-cxy)}{1-xy} & \frac{q(y+b-y^2-cy)\frac{u}{r}}{1-xy} \\ -\frac{q(x+bx^2-1-cx)\frac{r}{u}}{1-xy} & -\frac{q(xy+bx-y-c)}{1-xy} \end{pmatrix}$$

Our goal is to remove the off diagonal terms without violating the condition $xy\neq 1$. The terms vanish precisely when the equations $y+b-y^2-cy$ and $x+bx^2-1-cx$ are both equal to zero. When $b\neq 0$, these equations are quadratic in y and x respectively, with solutions

$$x_\pm=\frac{c-1\pm\sqrt{(1-c)^2+4b}}{2b} \qquad y_\pm=\frac{1-c\pm\sqrt{(1-c)^2+4b}}{2}.$$

Oddly enough, we compute $x_+y_+=x_-y_-=1$, which is just what we want to avoid. On the other hand, $x_+y_-=1$, if and only if $b=-\frac{(1-c)^2}{4}$. Assuming otherwise, we can eliminate the off diagonal terms, so that after applying a simple automorphism, we can reduce it to type $(1,0,0,c')$, where c' is given by some rational expression in b and c.

On the other hand, when $b=0$, then the quadratic in x reduces to a linear expression, which is zero when $x=\frac{1}{1-c}$. Of course, x is not well defined if $c=1$, so let us first assume otherwise. Now $y=0$ is a solution of the quadratic equality for y, and substituting the expressions for x and y into the first and fourth terms yields that our codifferential is equivalent to one of type $(1,0,0,c)$ where the c in this expression is the same as the c occurring in the type $(1,1,0,c)$. In fact, it is also clear that even when $b\neq 0$, one can reduce any expression of type $(1,1,b,c)$ to the type $(1,1,0,c')$ by choosing $y=y_+$, and $x=0$, except in the special case when $b=-\frac{(1-c)^2}{4}$.

Note that the case $b=0$ and $c=1$ is a special case of the equality $b=-\frac{(1-c)^2}{4}$, so all we have left is to consider the case where this equality holds. Then we certainly can set $y=y_+=\frac{1-c}{2}$, and the condition $1-xy\neq 0$ reduces to the inequality $\frac{c-1}{2}x+1\neq 0$. If we choose an arbitrary x so this inequality is satisfied, then it is easy to see that the first and the fourth coefficients of d' become, simply $q\frac{c+1}{2}$. This means that when $c \neq -1$, we can choose q to make the first and fourth coefficients of d' equal to 1, the third coefficient equal to 0, and by choosing r/u appropriately, the second coefficient equal to 1 as well. Thus we obtain an element of type $(1, 1, 0, 1)$ unless $c = -1$. One can check that in this case, we obtain $b = -1$, so the element has type $(1, 1, -1, -1)$, and also it is obvious from this argument that in this case d is equivalent to the codifferential $d = \psi_3^{1,1,0}$, so that in particular $\psi_3^{1,1,0}$ has type $(1, 1, -1, -1)$. It is also easy to show that both types $(1, 1, -1, -1)$ and $(1, 1, 0, 1)$ can never be reduced to type $(1, 0, 0, c)$.

We now proceed to show that the other special types, $(1, 0, 1, 0)$, $(0, 1, 1, 0)$ also can be reduced to type $(1, 0, 0, c)$.

Type $(1, 0, 1, 0)$ is the same as type $(1, 0, 0, 0)$ and type $(1, 1, 0, 0)$. To see this, apply the generic linear transformation g to produce d' as before, and we obtain

$$d' = \begin{pmatrix} \frac{q(r+s)u}{ru-ts} & \frac{q(t+u)u}{ru-ts} \\ -\frac{q(r+s)s}{ru-ts} & -\frac{q(t+u)s}{ru-ts} \end{pmatrix}$$

If we choose $q = 1/2$, $s = t = r = -1$ and $u = 1$, we obtain type $(1, 1, 0, 0)$, and if instead we choose $q = r = u = 1$, $s = 0$, and $t = -1$, we obtain type $(1, 0, 0, 0)$.

Type $(0, 1, 1, 0)$ is the same as type $(1, 0, 0, -1)$ and type $(1, 1, 0, -1)$. To see this, apply the generic linear transformation g and we obtain

$$d' = \begin{pmatrix} \frac{q(su-rt)}{ru-ts} & \frac{q(u^2-t^2)}{ru-ts} \\ -\frac{q(s^2-r^2)}{ru-ts} & -\frac{q(su-rt)}{ru-ts} \end{pmatrix}$$

Choose $t = -1$ and $q = r = u = s = 1$. Then this becomes $d' = \psi_2^{1,1,0} - \psi_3^{1,0,1}$ as desired. On the other hand, if $q = s = u = 1$, $r = 0$, and $t = -1$, then we obtain $d' = \psi_2^{1,1,0} + \psi_3^{1,1,0} - \psi_3^{1,0,1}$.

This completes the classification of types of codifferentials. We have one family $(1, 0, 0, c)$ and two special cases, $(1, 1, 0, 1)$ and $(0, 1, 0, 0)$ which cannot be reduced to elements of this family.

We show that an element of type $(1, 0, 0, c)$ is equivalent to one of type $(1, 0, 0, c')$ precisely when $c' = c^{\pm 1}$, so that the set of equivalence classes of codifferentials has a one parameter subfamily, parameterized by the unit disc in $\mathbb{C}$. To see this, apply the generic linear transformation and we obtain

$$d' = \begin{pmatrix} \frac{q(ru-cst)}{ru-ts} & -\frac{qtu(-1+c)}{ru-ts} \\ -\frac{qrs(-1+c)}{ru-ts} & \frac{q(-ts+rcu)}{ru-ts} \end{pmatrix}$$

It is interesting to note that when $c = 1$, the middle two terms drop out and thus $\psi_2^{1,1,0} + \psi_3^{1,0,1}$ is not equivalent to any codifferential of type $(1, 1, b, c)$. Otherwise, if $c \neq 0$, let $r = u = 0$ and $s = t = 1$ and $q = -1/c$ and we obtain type $(1, 0, 0, 1/c)$. To see that this is the only other type that could occur, note that to cancel the middle terms, we must have either $r = u = 0$ or $s = t = 0$, so the claim is obvious.

For later purposes let us label the codifferentials representing the equivalence classes of degree 2 codifferentials of the first kind as follows.

$$\begin{aligned} d_* &= \psi_3^{1,1,0} \\ d_\sharp &= \psi_2^{1,1,0} + \psi_3^{1,1,0} + \psi_3^{1,0,1} \\ d_c &= \psi_2^{1,1,0} + \psi_3^{1,0,1} c \end{aligned}$$

5. Cohomology of Codifferentials of degree 2 of the First Kind

For a degree 2 codifferential d, with cohomology operator $D = [\bullet, d]$, the dimension of the cohomology is given by $h_n = z_n - b_{n-1}$.

5.1. Cohomology of $d_* = \psi_3^{1,1,0}$.

The coboundaries of basic cochains for d_* are as follows:

$$\begin{aligned} D(\varphi_1^{1,q,n-q-1}) &= -\psi_3^{1,1+q,n-q-1} \\ D(\varphi_2^{0,p,n-p}) &= \psi_2^{1,1+p,n-p-1}(n-p) - \psi_3^{1,p,n-p} \\ D(\varphi_3^{0,p,n-p}) &= \psi_3^{1,1+p,n-p-1}(n-p) \\ D(\psi_1^{0,p,n-p}) &= \varphi_1^{1,1+p,n-p-1}(n-p) + \varphi_3^{0,1+p,n-p} \\ D(\psi_2^{1,q,n-q-1}) &= 0 \\ D(\psi_3^{1,q,n-q-1}) &= 0 \end{aligned}$$

From this table, we see that $\psi_2^{1,q,n-q-1}$, $\psi_3^{1,q,n-q-1}$ are cocycles for $q = 0 \dots n-1$, and $\varphi_3^{0,p,n-p} + \varphi_1^{1,p,n-p-1}(n-p)$ is a cocycle for $p = 0 \dots n$. Also, $\varphi_2^{0,n,0} + \varphi_3^{0,n-1,1}$ is a cocycle, so $z_n = n+2|2n$, which means that

$b_n = n+1|2n$. It follows that $h_n = 2|2$ for $n > 1$ and $h_1 = 3|2$. Moreover,

$$H^1 = \langle \psi_2^{1,0,0}, \psi_3^{1,0,0}, \varphi_3^{0,1,0}, \varphi_3^{0,0,1} + \varphi_1^{1,0,0}, \varphi_2^{0,1,0} + \varphi_3^{0,0,1} \rangle$$
$$H^n = \langle \psi_2^{1,0,n-1}, \psi_3^{1,0,n-1}, \varphi_2^{0,n,0} + \varphi_3^{0,n-1,1}, \varphi_3^{0,0,n} + \varphi_1^{1,0,n-1} n \rangle, \quad n > 1$$

Because the odd part of the cohomology of d_* does not vanish for $n > 2$, there are nontrivial extensions of d_*. We will discuss them later in the section on extensions of codifferentials of the first kind.

5.2. **Cohomology of** $d_\sharp = \psi_2^{1,1,0} + \psi_3^{1,1,0} + \psi_3^{1,0,1}$**.** The coboundaries for $d_\sharp$ are given by

$$\begin{aligned}
D(\varphi_1^{1,q,n-q-1}) &= -\psi_2^{1,1+q,n-q-1} - \psi_3^{1,1+q,n-q-1} - \psi_3^{1,q,n-q} \\
D(\varphi_2^{0,p,n-p}) &= \psi_2^{1,1+p,n-p-1}(n-p) - \psi_2^{1,p,n-p}(n-1) - \psi_3^{1,p,n-p} \\
D(\varphi_3^{0,p,n-p}) &= \psi_3^{1,1+p,n-p-1}(n-p) - \psi_3^{1,p,n-p}(n-1) \\
D(\psi_1^{0,p,n-p}) &= \varphi_1^{1,p,n-p} n + \varphi_1^{1,1+p,n-p-1}(n-p) + \varphi_2^{0,1+p,n-p} \\
&\quad + \varphi_2^{0,p,n-p+1} + \varphi_3^{0,1+p,n-p} \\
D(\psi_2^{1,q,n-q-1}) &= 0 \\
D(\psi_3^{1,q,n-q-1}) &= 0
\end{aligned}$$

We already know that $\psi_2^{1,q,n-q-1}$ and $\psi_3^{1,q,n-q-1}$ give a basis of the $2n$ odd cocycles. First note that $\varphi_3^{0,1,0}$ and $\varphi_2^{0,1,0} + \varphi_3^{0,0,1}$ are a basis of the even 1-cocycles. Thus $z_1 = h_1 = 2|2$ and $b_1 = 2|3$.

For $n > 1$ it is easy to see that images of the cochains of the form $\varphi_3^{0,p,n-p}$ are a basis of the $n+1$-dimensional subspace of cochains of the form $\psi_3^{1,p,n-p}$. If we consider the subspace X spanned by elements of the form $\psi_2^{1,1+q,n-q-1}$ and $\psi_3^{1,p,n-p}$, then D maps the $(2n+1)$-dimensional space spanned by elements of the form $\varphi_1^{1,q,n-q-1}$ and $\varphi_3^{0,p,n-p}$ bijectively onto X.

If $p > 0$, it is clear that the image of $\varphi_2^{0,p,n-p}$ lies in X. Thus we obtain a cocycle as a sum of the element $\varphi_2^{0,p,n-p}$ and a unique linear combination of the elements $\varphi_3^{0,p,n-p}$ and $\varphi_1^{1,q,n-q-1}$. Thus there are n independent cocycles generated by these elements.

For $p = 0$, the image of $\varphi_2^{0,0,n}$ does not lie in X, so it cannot contribute to any cocycle. Thus we see that there are exactly n independent even cocycles. Thus $z_n = n|2n$ and $b_n = n+1|2n+2$.

This means that $h_n = z_n - b_{n-1} = 0$ if $n > 2$. Furthermore, $h_2 = 2|4 - 2|3 = 0|1$. It is easy to see that $\psi_2^{1,0,1}$ can be taken as the basis for H^2. Thus we have

$$
\begin{aligned}
H^1 &= \langle \psi_2^{1,0,0}, \psi_3^{1,0,0}, \varphi_3^{0,1,0}, \varphi_2^{0,1,0} + \varphi_3^{0,0,1} \rangle \\
H^2 &= \langle \psi_2^{1,0,1} \rangle \\
H^n &= 0, \qquad \text{if } n > 2
\end{aligned}
$$

5.3. **Cohomology of** $d_c = \psi_2^{1,1,0} + \psi_3^{1,0,1}c$**.** Since d_c is equivalent to $d_{1/c}$, we can assume that c lies in the unit circle. Thus we will assume that $|c| \leq 1$ in the following. The coboundaries are given by

$$
\begin{aligned}
D(\varphi_1^{1,q,n-q-1}) &= -\psi_2^{1,1+q,n-q-1} - \psi_3^{1,q,n-q}c \\
D(\varphi_2^{0,p,n-p}) &= \psi_2^{1,p,n-p}(p - 1 + c(n-p)) \\
D(\varphi_3^{0,p,n-p}) &= \psi_3^{1,p,n-p}(p + c(n-p-1)) \\
D(\psi_1^{0,p,n-p}) &= \varphi_1^{1,p,n-p}(p + c(n-p)) + \varphi_2^{0,1+p,n-p} + \varphi_3^{0,p,n-p+1}c \\
D(\psi_2^{1,q,n-q-1}) &= 0 \\
D(\psi_3^{1,q,n-q-1}) &= 0
\end{aligned}
$$

Let $Q_p = p + c(n-p-1)$. When $Q_p \neq 0$, then

$$
\xi_p = \varphi_2^{0,p+1,n-p-1} + \varphi_3^{0,p,n-p}c + \varphi_1^{1,p,n-p-1}Q_p, \quad p = 0 \ldots n-1,
$$

give n independent even cocycles, which are obviously coboundaries. In most cases, the ξ_p give a basis of the even cocycles. However, when $Q_p = 0$ we get an additional even cocycle $\varphi_3^{0,p,n-p}$ which is never a coboundary, and $\psi_3^{1,p,n-p}$ is no longer a coboundary. When this happens the even part of h_n increases by one, and the odd part of h_{n+1} also increases by one. Note that if $n > 1$, for most values of c it never happens that $Q_p = 0$. In fact, if c is not a nonpositive rational number, then Q_p is never zero when $n > 1$.

There is another source of possible even cocycles, given by the terms $\varphi_2^{0,0,n}$, which is a cocycle if $nc = 1$, and $\varphi_3^{0,n,0}$, which is a cocycle if $n = c$. When this happens, the even part of z_n increases by 1, so the odd part of b_n decreases by 1. Thus we see again that the even part of h_n and the odd part of h_{n+1} both increase by 1. Moreover, if $nc = 1$, then we need to add $\psi_2^{1,0,n}$ to the basis of H^{n+1} and if $n = c$, we need to add $\psi_3^{1,n,0}$ to the basis. If c or its reciprocal is not a positive integer, then neither of these two cases hold.

5.3.1. *Cohomology for generic values of* c. Let us say that c is generic if it is not a nonpositive rational number, nor is it or its reciprocal a positive integer. If c is generic, and $n > 1$, then ξ_p are the only even cocycles, so that $z_n = n|2n$ and thus we have $b_n = n+1|2n+2$. It follows that $h_n = 0|0$ for $n > 2$. For $n = 1$, we always have $Q_0 = 0$, so we have two even cocycles, $\varphi_2^{0,1,0}$ and $\varphi_3^{0,0,1}$, which along with the two odd cocycles $\psi_2^{1,0,0}$ and $\psi_3^{1,0,0}$ generically form a basis for H^1. Thus $z_1 = h_1 = 2|2$ in the generic case.

Also, in the generic case, we obtain $b_1 = 2|3$. Since $z_2 = 2|4$, it follows that $h_2 = 0|1$. In fact, $\varphi_3^{1,0,1}$ can be taken as a basis for H^2. What this says is that you can deform d_c in the direction of the family.

Thus we conclude that for generic values of c we have

$$\begin{aligned} H^1 &= \langle \psi_2^{1,0,0}, \psi_3^{1,0,0}, \varphi_2^{0,1,0}, \varphi_3^{0,0,1} \rangle \\ H^2 &= \langle \psi_3^{1,0,1} \rangle \\ H^n &= 0, \qquad \text{if } n > 2 \end{aligned}$$

5.3.2. *Cohomology for the special value $c = 1$.* In this case both $nc = 1$ and $n = c$ hold for $n = 1$. Thus we obtain two additional 1-cohomology classes, given by $\varphi_2^{0,0,1}$ and $\varphi_3^{0,1,0}$ and $h_1 = z_1 = 4|2$. Thus $b_1 = 2|1$, so $h_2 = z_2 - b_1 = 2|4 - 2|1 = 0|3$. Thus for $c = 1$ we have

$$\begin{aligned} H^1 &= \langle \psi_2^{1,0,0}, \psi_3^{1,0,0}, \varphi_2^{0,1,0}, \varphi_3^{0,0,1}, \varphi_2^{0,0,1}, \varphi_3^{0,1,0} \rangle \\ H^2 &= \langle \psi_3^{1,0,1}, \psi_2^{1,0,1}, \psi_3^{1,1,0} \rangle \\ H^n &= 0, \qquad \text{if } n > 2 \end{aligned}$$

This suggests that somehow there are additional directions in which the codifferential can be deformed, and we will comment on this later.

5.3.3. *Cohomology when $1/c \neq 1$ is a positive integer.* Let $m = 1/c$. Then we have

$$\begin{aligned} H^1 &= \langle \psi_2^{1,0,0}, \psi_3^{1,0,0}, \varphi_2^{0,1,0}, \varphi_3^{0,0,1} \rangle \\ H^2 &= \langle \psi_3^{1,0,1} \rangle \\ H^m &= \langle \varphi_2^{0,0,m} \rangle \\ H^{m+1} &= \langle \psi_2^{1,0,m} \rangle \\ H^n &= 0, \qquad \text{otherwise} \end{aligned}$$

except when $c = 1/2$, in which case, since $m = 2$, $H^2 = \langle \psi_3^{1,0,1}, \varphi_2^{0,0,2} \rangle$.

5.3.4. *Cohomology when $c = 0$.* This case is special because $Q_0 = 0$ for all n. Thus we always have the even cohomology class $\varphi_3^{0,0,n}$, and the odd cohomology class $\psi_3^{1,0,n-1}$. Since Q_0 is zero when $n = 1$ in all cases, H^1 is not changed from the generic pattern. Thus

$$H^1 = \langle \psi_2^{1,0,0}, \psi_3^{1,0,0}, \varphi_2^{0,1,0}, \varphi_3^{0,0,1} \rangle$$
$$H^n = \langle \psi_3^{1,0,n-1}, \varphi_3^{0,0,n} \rangle, \qquad \text{if } n > 1$$

5.3.5. *Cohomology when c is a negative rational number.* Let us rewrite the equality $Q_p = 0$ in the form $p = \frac{(n-1)c}{c-1}$ When $-1 \le c < 0$ is rational, note that $0 < c/(c-1) \le \frac{1}{2}$, so that $Q = 0$ has an integral solution for p with $0 < p < n-1$ for infinitely many values of n. In fact, suppose that $\frac{c}{c-1} = \frac{r}{s}$, expressed as a fraction in lowest terms. Then $\frac{(n-1)c}{c-1}$ is a positive integer precisely when $n = ks+1$ for a positive integer k, in which case we have $kr = \frac{(n-1)c}{c-1}$, and $0 < kr < n-1$. From this, we can calculate the table of cohomology of d_c as follows.

$$H^1 = \langle \psi_2^{1,0,0}, \psi_3^{1,0,0}, \varphi_2^{0,1,0}, \varphi_3^{0,0,1} \rangle$$
$$H^2 = \langle \psi_3^{1,0,1} \rangle$$
$$H^{ks+1} = \langle \varphi_3^{0,kr,k(s-r)+1} \rangle$$
$$H^{ks+2} = \langle \psi_3^{1,kr,k(s-r)+1} \rangle$$
$$H^n = 0, \qquad \text{otherwise}$$

5.3.6. *The Moduli Space of Codifferentials of the First Kind.* Let us consider now only the deformations of these various types of codifferentials of degree 2 only as graded Lie algebras, *i.e.*, consider only H^2. Consider the following table of codifferentials and bases of the odd part of the second cohomology group:

Type	$(0,1,0,0)$	$(1,1,0,1)$	$(1,0,0,1)$	$(1,0,0,c)$
d	d_*	$d_\sharp$	d_1	d_c
$(H^2)_o$	$\psi_2^{1,0,1}, \psi_3^{1,0,1}$	$\psi_2^{1,0,1}$	$\psi_2^{1,0,1}, \psi_3^{1,0,1}, \psi_3^{1,1,0}$	$\psi_3^{1,0,1}$

There are three special cases, and the generic pattern. Note that even though the dimension of H^2 is not generic for d_0, the extra dimension is even, so does not contribute to the deformations over $\mathbb{C}$.

Clearly, there is only one family of codifferentials, so what is going on with the extra degrees of freedom in the cohomology? To understand this better, let us examine the moduli space of codifferentials of degree 2 in some more detail. Here we use the term moduli space in the following sense. The space of all codifferentials of degree 2 is a variety in a 4 dimensional complex space, preserved under the action of the group of linear automorphisms of the symmetric coalgebra. A quotient space of a variety by such a group action is called a moduli space, The structure of such moduli spaces can be very strange, from a topological point of view.

Let us parameterize our moduli space by types, and note that since type $(1, 0, 0, c)$ is the same as type $(1, 0, 0, 1/c)$, it is natural to think of the moduli space as the unit disc in $\mathbb{C}$, with an identification of the upper semicircle with the bottom. Then every point except 1 and -1 have neighborhoods which are discs, but 1 and -1 are orbifold points of degree 2. Of course, we are really describing the action of the group generated by the transformations $\{z \to z, z \to 1/z\}$ on the Riemann sphere, and identifying our standard points of the moduli space with the resulting images.

We should like to have some notion of neighborhood of a point in our moduli space, and the natural notion is to consider two elements of the moduli space to be close if they have inverse images which are close in the space of codifferentials. Of course, since any codifferential is equivalent to any multiple of itself, this would make all codifferentials close, so we have to be a bit more careful in our definition.

Consider the standard representatives of the equivalence classes of codifferentials, which are either $(1, 0, 0, c)$, $(1, 1, 0, 1)$ and $(0, 1, 0, 0)$. Let P and Q be equivalence classes. Let us say that Q is ϵ close to P if Q is among the types which occur by adding coordinates to the standard representation of P of absolute value no larger than ϵ. Then P is said to be infinitesimally close to Q if Q is epsilon close to P for all positive values of ϵ.

For most of our points, the notion of neighborhood we have just described yields no surprises. For any standard point P of type $(1, 0, 0, c)$ with $c \neq 1$, ϵ neighborhoods of P for small values of ϵ correspond to standard points $(1, 0, 0, c')$ with c' close to c.

However, for $c = 1$, things are quite different. As we have described before, the one parameter family $(1, 1, b, c)$ with $b = -\frac{1}{4}(c-1)^2$, contains types $(1, 0, 0, 1)$ and $(0, 1, 0, 0)$ for two special values of c, but gives type $(1, 1, 0, 1)$ otherwise. It follows that d_1 and d_* are infinitesimally close to $d_\sharp$. One can check that for ϵ small enough, a neighborhood of d_1 contains only this extra point, along with the points one would usually expect. It is hard to reconcile the fact that for the codifferential d_1, the dimension of the cohomology is 3. One might expect one extra dimension for the deformation in the $d_\sharp$ direction, but two extra dimensions are obtained instead.

Recall that type $(1,1,0,a)$ is the same as type $(1,0,0,a)$ when $a\neq 1$. This means that a neighborhood of $d_\sharp$ looks just like a neighborhood of d_1) (minus point d_1). Note that although d_1 is infinitesimally close to $d_\sharp$, the converse is not true. Notice that the dimension of the cohomology for $d_\sharp$ is just 1, corresponding to the fact that any small deformation of this codifferential just gives an ordinary element in the main family.

Finally, consider type $(0,1,0,0)$. Note that $(0,1,\epsilon,0)$ is the same as type $(1,0,0,-1)$, so d_* is infinitesimally close to d_{-1}. It is easy to see that type $(0,1,\epsilon_1,\epsilon_2,0)$ is the same as type $(1,1,\epsilon_1/\epsilon_2^2,0)$, if $\epsilon_2\neq 0$. One also sees that type $(1,0,0,c)$ is the same as type $(1,1,-\frac{c}{(c+1)^2},0)$, if $c\neq\pm1$. When $c=1$, we obtain type $(1,1,-\frac{1}{4},0)$ which is the same as type $(1,1,0,1)$. Thus d_* is infinitesimally close to every element of the moduli space except d_1. Note that the cohomology has odd dimension 2, and the type $(0,1,\epsilon_1,\epsilon_2)$ corresponds to adding a small cocycle to d_*.

6. Codifferentials of degree 2 of the second kind

A codifferential of degree 2 of the second kind is of the form

$$d=\psi_1^{0,2,0}a+\psi_1^{0,1,1}b+\psi_1^{0,0,2}c,$$

which we will say is of type (a,b,c). If either a or b is nonzero, then it is clearly equivalent to a codifferential of type $(1,b',c')$, for some b' and c'. Note that the only type which cannot be reduced in this way is type $(0,b,0)$, which is clearly also of type $(0,1,0)$. However, let us examine type $(0,1,0)$ to see what it is equivalent to. Applying a standard linear automorphism, we obtain that $d=\psi_1^{0,1,1}$ is equivalent to any codifferential of the form

$$d'=\psi_1^{0,2,0}\frac{2rs}{q}+\psi_1^{0,1,1}\frac{ru+ts}{q}+\psi_1^{0,0,2}\frac{2tu}{q},\qquad ru-ts\neq 0, q\neq 0.$$

If we set $q=2rs$, $x=u/s$ $y=s/r$, then we obtain type $(1,b,c)$ where $2b=x+y$, $c=xy$, and we must avoid the condition $xy=1$. But this occurs exactly when $b^2=c$. Thus type $(0,1,0)$ is equivalent to type $(1,b,c)$ whenever $b^2\neq c$. It is also clear that type $(0,1,0)$ is not equivalent to type $(1,0,0)$.

Let us next study type $(1,0,0)$. Applying a linear automorphism, we see that$\psi_1^{0,2,0}$ is equivalent to codiffentials of the form

$$d'=\psi_1^{0,2,0}\frac{r^2}{q}+\psi_1^{0,1,1}\frac{rt}{q}+\psi_1^{0,0,2}\frac{t^2}{q}.$$

If we set $r^2 = q$, then we see that this d is equivalent to any codifferential of the form $(1, b, b^2)$, exactly the types not covered by the first case.

Thus there are only two types of codifferentials, represented by $\psi_1^{0,2,0}$ and $\psi_1^{0,1,1}$. Let us study the second type first.

6.1. **Type** $(0,1,0)$**.** Let $D(\varphi) = [\varphi, \psi_1^{0,1,1}]$. Then we obtain the following table of coboundaries.

$$\begin{aligned}
D(\varphi_1^{1,q,n-q-1}) &= \psi_1^{0,1+q,n-q}\\
D(\varphi_2^{0,p,n-p}) &= -\psi_1^{0,p,n-p+1}\\
D(\varphi_3^{0,p,n-p}) &= -\psi_1^{0,p+1,n-p}\\
D(\psi_1^{0,p,n-p}) &= 0\\
D(\psi_2^{1,q,n-q-1}) &= \varphi_1^{1,q,n-q} + \varphi_2^{0,q+1,n-q}\\
D(\psi_3^{1,q,n-q-1}) &= \varphi_1^{1,q+1,n-q-1} + \varphi_3^{0,q+1,n-q}
\end{aligned}$$

Then we have $n+1$ odd cocycles of the form $\psi_1^{0,p,n-p}$, and $2n$ even cocycles of the form $\varphi_1^{1,q,n-q} + \varphi_3^{0,q,n-q-1}$ and $\varphi_1^{1,q,n-q-1} + \varphi_2^{0,q+1,n-q-1}$ This means $z_n = 2n|n+1$, so $b_n = 2n|n+2$, and $h_n = 2|0$ if $n > 1$. We have

$$\begin{aligned}
H^1 &= \langle \psi_1^{0,0,1}, \psi_1^{0,1,0}, \varphi_1^{1,0,0} + \varphi_3^{0,0,1}, \varphi_1^{1,0,0} + \varphi_2^{0,1,0} \rangle\\
H^n &= \langle \varphi_1^{1,0,n-1} + \varphi_3^{0,0,n}, \varphi_1^{1,n-1,0} + \varphi_2^{0,n,0} \rangle, \qquad \text{if } n > 1
\end{aligned}$$

and all cohomology for $n > 1$ is even. Thus we don't obtain any deformations in the Lie Algebra direction.

6.2. **Type** $(1,0,0)$**.** Let $D(\varphi) = [\varphi, \psi_1^{0,2,0}]$. Then we obtain the following table of coboundaries.

$$\begin{aligned}
D(\varphi_1^{1,q,n-q-1}) &= \psi_1^{0,2+q,n-q-1}\\
D(\varphi_2^{0,p,n-p}) &= -2\psi_1^{0,p+1,n-p}\\
D(\varphi_3^{0,p,n-p}) &= 0\\
D(\psi_1^{0,p,n-p}) &= 0\\
D(\psi_2^{1,q,n-q-1}) &= 2\varphi_1^{1,q+1,n-q-1} + \varphi_2^{0,q+2,n-q-1}\\
D(\psi_3^{1,q,n-q-1}) &= \varphi_3^{0,q+2,n-q-1}
\end{aligned}$$

Besides the obvious $n+1$ odd cocycles $\psi_1^{0,p.n-p}$ and $n+1$ even ones $\varphi_3^{0,p,n-p}$, we also have n more even cocycles

$$2\varphi_1^{1,q,n-q-1} + \varphi_2^{0,q+1,n-q-1},$$

so $z_n = 2n+1|n+1$ and $h_n = 3|1$, if $n > 1$. In fact, it is easily seen that

$$H^1 = \langle \psi_1^{0,0,1}, \psi_1^{0,1,0}, \varphi_3^{0,0,1}, \varphi_3^{0,1,0}, 2\varphi_1^{1,0,0} + \varphi_2^{0,1,0} \rangle$$
$$H^n = \langle \psi_1^{0,0,n}, \varphi_3^{0,0,n}, \varphi_3^{0,1,n-1}, 2\varphi_1^{1,0,n-1} + \varphi_2^{0,1,n-1}, \rangle, \quad \text{if } n > 1$$

Let us think about the moduli space of two points given by these codifferentials. Note that type $(1,0,\epsilon)$ is the same as type $(0,1,0)$ for any nonzero value of ϵ. Thus type $(1,0,0)$ is infinitesimally close to $(0,1,0)$, but not the other way around. It is not surprising, therefore to see that type $(1,0,0)$ has a nontrivial deformation as a Lie algebra.

7. Extensions of Codifferentials of Degree 2

We now consider how to extend a codifferential d_2 of degree 2 to a more general codifferential. Let G be the subgroup of automorphisms of $S(W)$ fixing d_2, and G_1 be the subgroup of G consisting of linear automorphisms. The groups G and G_1 act on the set of nonzero cohomology classes. We shall say that two cohomology classes δ_1 and δ_2 are equivalent if there is an element f in G such that $f^*(\delta_1) = \delta_2$, and linearly equivalent if f lies in G_1.

Any automorphism g can be expressed in the form

$$g = \lambda \prod_{k=2}^{\infty} \exp(\alpha_k),$$

where $\alpha_k \in L_k$ is a coderivation. Moreover, $g^* = \prod_{k=\infty}^{2} \exp(-\operatorname{ad}_{\alpha_k})\lambda^*$, where $\operatorname{ad}_{\alpha_k}(\beta) = [\alpha_k, \beta]$, and $\lambda^*(\beta) = \lambda^{-1}\beta\lambda$.

Suppose that $d = d_2 + d_k + d_{k+1} + \cdots$ is an extension of d_2. The following theorem will help to classify such extensions.

Theorem 7.1. *Suppose that $d = d_2 + d_k + d_{k+1} + \cdots$ is a codifferential. Then d_k is a cocycle with respect to the coboundary operator $D = [\bullet, d_2]$. Moreover,*

1. *d is equivalent to a codifferential whose first nonzero term after d_2 is of higher order than k iff d_k is a D-coboundary,.*
2. *If $d' = d_2 + d'_k + d'_{k+1} + \cdots$ is equivalent to d, then the cohomology classes of d'_k and d_k are linearly equivalent.*

Proof. Since $[d,d]=2D(d_k)+\cdots$, it follows that $D(d_k)=0$. If $d_k=D(\alpha)$, then applying $g=\exp(\alpha)$ to d, we obtain $g*(d)=d_2+d_k-D(\alpha)+\cdots$, so we have eliminated the term d_k. On the other hand, suppose that $g^*(d)=d_2+d'_k+d'_{k+1}+\cdots$, for some $g=\lambda\exp(\beta_2)\cdots$. Since we must have $\lambda^*(d_2)=d_2$, we compute

$$\begin{aligned} g^*(d)=&d_2+\lambda^*(d_k)-D(\beta_2)-\cdots-D(\beta_{k-1})\\ &+\tfrac{1}{2}[\beta_2,D(\beta_2)]+\cdots \end{aligned}$$

It follows that $D(\beta_i)=0$ for $i<k-1$ and $-D(\beta_{k-1})+\lambda^*(d_k)=d'_k$. If $d'_k=0$, this says that $\lambda^*(d_k)$ is a coboundary, and since the coboundary map commutes with automorphisms, we see that d_k is also a coboundary. It is also clear that the cohomology class of $\lambda^*(d_k)$ coincides with that of d'_k. □

Let us say that a codifferential d is standard if it is of the form $d_2+d_k+d_{k+1}+\cdots$, where d_k is a nontrivial cocycle for d_2. By the first part of this theorem, every nontrivial extension of d_2 is equivalent to a standard codifferential. For a codifferential in standard form, let us refer to the cohomology class of d_k as the secondary term of d.

In general, we don't expect d_2+d_k to be a codifferential for an extension $d=d_2+d_k+d_{k+1}+\cdots$ of d_2. However, for the examples which arise in this paper, it turns out to be true. We state a theorem which is useful in characterizing the extensions of a codifferential of the form $d=d_2+d_k$.

Theorem 7.2. *Suppose $d=d_2+d_k$ is a codifferential. Let D_1 be the coboundary operator determined by d_2, and D_2 be the one given by d_k. Let $d'=d_2+d_k+d_l+\cdots$ be an extension of d, where $l>k$. Let $\delta=d_l$ Then*

1. *δ is a D_1-cocycle. Moreover $D_2(\delta)$ is a D_1-coboundary.*
2. *If δ is a D_1-coboundary, then d' is equivalent to an extension whose third term has degree larger than l.*
3. *If $H^n(D_1)=0$ for $n>k$, then any extension d' of d is equivalent to d.*
4. *If $\delta=D_2(\eta)$ for some D_1-cocycle η, then d' is equivalent to an extension whose third term has degree larger than l.*
5. *If every cocycle δ whose order is larger than k for which $D_2(\delta)$ is a D_1-coboundary is of the form $\delta=D_2(\eta)$ for some D_1-cocycle η, then any extension of d is equivalent to d.*

A useful generalization this theorem is as follows.

Theorem 7.3. *Let $d_e=d_2+\delta$ be a codifferential, where $\delta=d_3+\cdots+d_k$ and D, D_e be the coboundary operators determined by d_2 and d_e,* *respectively. Suppose that every D-cocycle α of degree greater than k extends to a D_e-cocycle, and that $H^n(d_e) = 0$ for $n > k$. Then every extension of d is equivalent to d.*

For each of the codifferentials, we will study the nontrivial extensions. Such extensions exist only when the odd part of H^n does not vanish for some $n > 2$. Note that ordinarily, when considering extensions, we have to construct them term by term, because a finite number of terms may not determine a codifferential. However, in our examples, the cocycles all have trivial brackets with respect to each other, so there is never any question about whether adding a cocycle gives a codifferential.

Let us state a conjecture which we will use to classify the codifferentials. Let us say that an automorphism g is formal if its linear term is the identity, in other words, $g = \prod_{k=2}^{\infty} \exp(\beta_k)$. Then two codifferentials are said to be formally equivalent if there is a formal automorphism expressing an equivalence between them.

Conjecture 7.4. *Suppose that $d = d_2 + d_k$ is a codifferential, $d_e = d_2 + d_k + d_l + \cdots$ and $d'_e = d_2 + d_k + d'_l + \cdots$. If d'_e is formally equivalent to d_e, then $d'_e - d_e$ is a coboundary with respect to the coboundary operator $D = [\bullet, d]$.*

In particular, this conjecture implies that if d_l is not the leading term of a coboundary with respect to D, then d_e is not equivalent to d.

7.1. **Extensions of the codifferential $\psi_1^{0,2,0}$ of type(1,0,0).** Let

$$d = \psi_1^{0,2,0}. \tag{10}$$

Suppose that λ is a linear automorphism of d. It is easy to check that the condition $\lambda^*(d) = d$ is equivalent to $s = 0$ and $u^2 = 1$ in the standard expression (equation (2)) for λ.

For $k > 2$, we can extend d to the codifferential

$$d_e = \psi_1^{0,2,0} + \psi_1^{0,0,k} a. \tag{11}$$

When a is a nonzero number, by using a diagonal automorphism of d, we see that d_e is equivalent to the codifferential $\psi_1^{0,2,0} + \psi_1^{0,0,k}$, so we may assume that $a = 1$. If λ is an automorphism of d, then

$$\lambda^*(\psi_1^{0,0,k}) = \sum_{x=0}^{k} \psi_1^{0,x,k-x} \frac{t^x u^{k-x}}{q}. \tag{12}$$

In order for λ to be the linear part of a generalized automorphism of d_e, $\lambda^*(\psi_2^{0,0,k})$ must be D-cohomologous to $\psi_2^{0,0,k}$. This can only happen if $u^k/q = 1$ and $t = 0$. Thus λ must be diagonal, and it follows that λ

actually lies in the linear automorphism group of d_e. In other words, $\lambda^*(d_e) = d_e$.

We compute

$$\begin{aligned}
D_e(\varphi_3^{0,0,n}) &= -\psi_1^{0,0,k+n-1}k\\
D_e(\varphi_3^{0,1,n-1}) &= -\psi_1^{0,1,k+n-2}k\\
D_e(2\varphi_1^{1,0,n-1} + \varphi_2^{0,1,n-1}) &= 2\psi_1^{0,0,k+n-1}
\end{aligned}$$

Thus the first and the third d-cocycle combine to give the cocycle $2\varphi_3^{0,0,n} + 2\varphi_1^{1,0,n-1}k + \varphi_2^{0,1,n-1}k$, while the second extends to the D_e-cocycle $2\varphi_3^{0,1,n-1} - \varphi_2^{0,0,n+k-2}$. Also, we note that $\psi_1^{0,0,n}$ is a coboundary for $n \geq k$.

Thus we have no higher order cohomology, so every extension of d_e is equivalent to it.

The cohomology of d_e is given by

$$\begin{aligned}
H^1 &= \langle \psi_1^{0,0,1}, \psi_1^{0,1,0}, 2\varphi_3^{0,0,1} + 2\varphi_1^{1,0,0}k + \varphi_2^{0,1,0}k, 2\varphi_3^{0,1,0} - \varphi_2^{0,0,k-1}\rangle\\
H^n &= \langle \psi_1^{0,0,n}, 2\varphi_3^{0,0,n} + 2\varphi_1^{1,0,n-1}k + \varphi_2^{0,1,n-1}k, 2\varphi_3^{0,1,n-1} - \varphi_2^{0,0,n+k-2}\rangle,\\
&\quad \text{if } 1 < n < k\\
H^n &= \langle 2\varphi_3^{0,0,n} + 2\varphi_1^{1,0,n-1}k + \varphi_2^{0,1,n-1}k, 2\varphi_3^{0,1,n-1} - \varphi_2^{0,0,n+k-2}\rangle,\\
&\quad \text{if } n \geq k
\end{aligned}$$

7.2. **Extensions of $d_c = \psi_2^{1,1,0} + \psi_3^{1,0,1}c$.** Let us first consider what the linear automorphism group of d_c consists of. It is not hard to check that if λ is a linear automorphism of $S(W)$ is given by the standard form (equation (2)) then

$$\begin{aligned}
\lambda^*(d_c) =& \psi_2^{1,1,0}\left(\tfrac{q(ru-cst)}{ru-st}\right) + \psi_3^{1,1,0}\left(\tfrac{(c-1)qrs}{ru-st}\right)\\
&+\psi_3^{1,0,1}\left(\tfrac{(1-c)qtu}{ru-st}\right) + \psi_3^{1,0,1}\left(\tfrac{q(cur-st)}{ru-st}\right).
\end{aligned}$$

Assuming that $c \neq 1$ (which does not interest us because there are no nontrivial extensions of d_1), we observe that only two cases occur. First, we could have $s = t = 0$ and $q = 1$. Then λ is diagonal. Secondly, we could have $u = r = 0$, $q = -1$ and $c = -1$. Thus the second case only occurs for a special value of c. This special case is of particular interest to us, so we will consider it separately.

7.2.1. *Extensions of d_c, when $1/c > 1$ is a positive integer.* This case is not very complicated, since if we let $m = 1/c$, then a nontrivial extension of d_c must be of the form

$$d_e = \psi_2^{1,1,0} + \psi_3^{1,0,1}c + \psi_2^{1,0,m}a$$

It is easy to see, using a diagonal linear transformation, that we can take $a = 1$. Note that a linear automorphism λ of d preserves d_e precisely when $u^m = r$. (Remember that λ is diagonal and $q = 1$.)

Let us study the cohomology of d_e. First, note that

$$D_e(\varphi_2^{0,1,0}) = \psi_2^{1,0,m}$$
$$D_e(\varphi_3^{0,0,1}) = -\psi_2^{1,0,m}m,$$

so that the two 1-cohomology classes for d_c are replaced by the single cohomology class $\varphi_2^{0,1,0}m + \varphi_3^{0,0,1}$, and $\psi_2^{1,0,m}$ becomes a coboundary for the cohomology. Note that $\varphi_2^{0,0,m}$ remains a cocycle for D_e. Thus the cohomology for D_e is given by

$$H^1 = \langle \psi_2^{1,0,0}, \psi_3^{1,0,0}, \varphi_2^{0,1,0}m + \varphi_3^{0,0,1} \rangle$$
$$H^2 = \langle \psi_3^{1,0,1} \rangle$$
$$H^m = \langle \varphi_2^{0,0,m} \rangle$$
$$H^n = 0, \qquad \text{otherwise}$$

except when $c = 1/2$, in which case, since $m = 2$, $H^2 = \langle \psi_3^{1,0,1}, \varphi_2^{0,0,2} \rangle$. By Theorem (7.3), it follows that any extension of d_e is equivalent to d_e, so we have found all nonequivalent extensions of d_c.

7.3. **Extensions of $d_0 = \psi_2^{1,1,0}$.** If we take an extension of the form $\psi_2^{1,1,0} + \psi_3^{1,0,k}a$, then as usual, we can assume $a = 1$, so we may as well assume that our extended codifferential is

$$d_e = \psi_2^{1,1,0} + \psi_3^{1,0,k}.$$

The linear part λ of a generalized automorphism of d_e must satisfy $u^{k-1} = 1$, and as in the previous case, it is a diagonal automorphism with $q = 1$. Moreover $\lambda^*(d_e) = d_e$. Note that We note that

$$D_e(\varphi_2^{0,1,0}) = 0$$
$$D_e(\varphi_3^{0,0,n}) = \psi_3^{1,0,k+n-1}(n - k)$$

so that $\varphi_2^{0,1,0}$ remains a cocycle, but with the exception of $n = k$, the cochains $\varphi_3^{0,0,n}$ give rise to coboundaries, and are no longer cocycles. Also, $\psi_3^{1,0,m}$ is a D_e-coboundary for $m \geq k$ except for $m = 2k - 1$. Thus, after applying an appropriate automorphism, we can assume that the third order term in any nontrivial extension of d_e is of the form $\psi_3^{1,0,2k-1}a$, for some nonzero a, and we have the following candidate for a possible nontrivial extension of d_e.

$$d_{0,a} = \psi_2^{1,1,0} + \psi_3^{1,0,k} + \psi_3^{1,0,2k-1}a.$$

Let us consider which linear automorphisms preserve d_0. It is easy to see that if a linear automorphism λ is given by equation (2), then $\lambda^*(d_0) = d_0$ precisely when $q = 1$ and $s = t = 0$. Thus λ is diagonal. Moreover, $\lambda^*(\psi_3^{1,0,k}) = \psi_3^{1,0,k} u^{k-1}$, and the only way in which this can be cohomologous to $\psi_3^{1,0,k}$ is if $u^{k-1} = 1$, and in this case, it is also easy to check that $\lambda^*(d_{0,a}) = d_{0,a}$. Thus, the coefficient a cannot be eliminated by a linear automorphism. Suppose that d_e were equivalent to a codifferential with leading part $d_{0,a}$. Then since the linear part fixes $d_{0,a}$, there is a formal equivalence between d_e and a codifferential with leading part $d_{0,a}$. But then $\psi_3^{1,0,2k-1}$ would be the leading term of a coboundary with respect to D_e, by Theorem (7.4), and since this is not true, we see that $d_{0,a}$ is a nontrivial extension of d_e. Moreover, two codifferentials with different values of a are not equivalent.

Because of this, we would expect that $\psi_3^{1,0,2k-1}$ remains a cohomology class for $D_{0,a} = [\bullet, d_{0,a}]$, and this fact is easily checked. Moreover, $\varphi_2^{0,1,0}$ still remains a cocycle. Thus the cohomology for $d_{0,a}$ is given by

$$\begin{aligned}
H^1 &= \langle \psi_2^{1,0,0}, \psi_3^{1,0,0}, \varphi_2^{0,1,0} \rangle \\
H^n &= \langle \psi_3^{1,0,n-1} \rangle, \qquad \text{if } 1 < n \le k \\
H^{2k} &= \langle \psi_3^{1,0,2k-1} \rangle \\
H^n &= 0, \qquad \text{otherwise}
\end{aligned}$$

By Theorem (7.3), we again see that any extension of $d_{0,a}$ is equivalent to $d_{0,a}$. Thus we have classified all extensions of d_0.

7.4. **Extensions of d_c, when c is a negative rational number.** Recall that in this case, $\frac{c}{c-1} = \frac{r}{s}$ where $1 \le r < s$. Let us add a cocycle of the form $\psi_3^{1,kr,k(s-r)+1} a$, and it is easily seen that we can choose $a = 1$. Thus we have

$$d_e = \psi_2^{1,1,0} + \psi_3^{1,0,1} c + \psi_3^{1,kr,k(s-r)+1}.$$

$$D_e(\varphi_3^{0,lr,l(s-r)+1}) = \psi_3^{1,(l+k)r,(l+k)(s-r)+1}(l-k)(s-r),$$

which does not vanish unless $l = k$, and we obtain the following table for the cohomology of d_e.

$$\begin{aligned}
&H^1 = \langle \psi_2^{1,0,0}, \psi_3^{1,0,0}, \varphi_2^{0,1,0}(r-s) + \varphi_3^{0,0,1} r\rangle \\
&H^{ms+2} = \langle \psi_3^{1,mr,m(s-r)+1}\rangle \qquad \text{if } 0 \le m < k \\
&H^{ks+1} = \langle \varphi_3^{0,kr,k(s-r)+1}\rangle \\
&H^{2ks+2} = \langle \psi_3^{1,2kr,2k(s-r)+1}\rangle \\
&H^n = 0, \qquad \text{otherwise}
\end{aligned}$$

Thus the only candidate for an extension of d_e is

(13) $$d_{c,e} = \psi_2^{1,1,0} + \psi_3^{1,0,1} c + \psi_3^{1,kr,r(s-r)+1} + \psi_3^{1,2kr,2k(s-r)+1} a,$$

where a is a parameter which, as we will see, cannot be eliminated. First, note that $\varphi_2^{0,1,0}(r-s)+\varphi_3^{0,0,1} r$ remains a cocycle for $d_{c,a}$. Let $\varphi = \varphi_3^{0,kr,k(s-r)+1}$. Since $D_{c,a}(\varphi) = -\psi_3^{1,3kr,3k(s-r)+1} k(s-r)a$, φ no longer generates H^{ks+1}. However, φ extends to a cocycle φ', which is given by a power series with leading term φ. Recall that for a codifferential which is not of fixed degree, the spaces H^n do not make sense in the usual manner; rather H inherits a filtration from the natural filtration on L, and H^n is a quotient space of the n-th filtered part by the $n+1$-st, and in this sense, we have $H^{ks+1} = \langle \varphi' \rangle$. Note that other than this change, all of the cohomology for $d_{c,a}$ remains the same as for d_e.

In order to apply the conjecture to show that the extensions given by $d_{c,a}$ are all nonequivalent, we need to study the linear automorphisms preserving d_c. It is easy to see that if λ is given by equation (2) then $\lambda^*(d_c) = d_c$ can occur only under the following circumstances. First, if $q = 1$, and $s = t = 0$, in which case λ is simply a diagonal matrix. In that case, it is easy to check that The second possibility only occurs if $c = -1$, and in that case, we have an additional solution $q = -1$, $x = u = 0$, so λ interchanges the roles of w_2 and w_3. The second case is a bit tricky, so we will deal with the special case $c = -1$ separately.

7.4.1. *The case $c \neq -1$.* In this case, if $\lambda^*(d_c) = d_c$, we have $\lambda = \operatorname{diag}(1, x, u)$, and we can compute that

$$\lambda^*(\psi_3^{1,kr,k(s-r)+1}) = \psi_3^{1,kr,k(s-r)+1} x^{kr} u^{k(s-r)}.$$

This cochain is cohomologous to $\psi_3^{1,kr,k(s-r)+1}$ precisely when it is equal to it, so that $x^{kr}u^{k(s-r)} = 1$, and $\lambda^*(d_{c,a}) = d_{c,a}$. Thus we can restrict ourselves to formal equivalences, and as in the case of $d_{0,a}$, we conclude that the extensions of d_e determined by different values of a are all nonequivalent.

7.4.2. *The case* $c = -1$. Note that for the first type of automorphism, all the remarks made before hold as well. Thus we need only consider what the second type of automorphism does.

7.5. **Extensions of** $d_* = \psi_3^{1,1,0}$**.** First let us consider when a linear automorphism λ, given in standard form by equation (2), preserves d_*. It is easy to see that $\lambda^*(d_*) = d_*$ precisely when $t = 0$ and $u = qr$. Because the cohomology of d_* can be represented by cocycles of type $\delta = \psi_3^{1,0,l}$ and $\gamma = \psi_2^{1,0,k}$, every extension of d_* is equivalent to one where all added terms are of one of these two types. With a little work, one can show that

$$\lambda^*(\delta) = \sum_{x=0}^{l} \psi_3^{1,x,l-x} s^x u^{l-x-1} q = \sum_{x=0}^{l} \psi_3^{1,x,l-x} \left(\tfrac{s}{u}\right)^x u^{l-1} q \tag{14}$$

$$\lambda^*(\gamma) = \sum_{x=0}^{k} \psi_2^{1,x,k-x} a \left(\tfrac{s}{u}\right)^x \tfrac{qu^k}{r} - \psi_3^{1,x,k-x} a \left(\tfrac{s}{u}\right)^{x+1} \tfrac{qu^k}{r}. \tag{15}$$

We will show that any extension of d_* is equivalent to an extension of a codifferential of the form

$$d_{k,l} = \psi_3^{1,1,0} + \psi_2^{1,0,k} a + \psi_3^{1,0,l} b.$$

For the moment, we make no assumptions about which of k and l is the larger. It is usually true that if a and b are nonzero, they can be taken to be 1. There is one exception to this statement, and that is the case when $k + 1 = 2l$. We will discuss this special case in more detail later. For the moment, it will be more convenient for us to leave the coefficients a and b undetermined.

In order to determine the leading terms of coboundaries for the coboundary operator $D_{k,l}$ given by $d_{k,l}$ consider the extended coboundary formulae:

$$\begin{aligned}
D_{k,l}(\varphi_1^{1,q,n-q-1}) &= -\psi_3^{1,1+q,n-q-1} - \psi_2^{1,q,k+n-q-1} a - \psi_3^{1,q,l+n-q-1} b \\
D_{k,l}(\varphi_2^{0,p,n-p}) &= \psi_2^{1,p+1,n-p-1}(n-p) - \psi_3^{1,p,n-p} \\
&\quad + \psi_2^{1,p-1,k+n-p} ap + \psi_2^{1,p,l+n-p-1} b(n-p) \\
D_{k,l}(\varphi_3^{0,p,n-p}) &= \psi_3^{1,p+1,n-p-1}(n-p) - \psi_2^{1,p,k+n-p-1} ak \\
&\quad + \psi_3^{1,p-1,k+n-p} ap + \psi_3^{1,p,l+n-p-1} b(n-p-l)
\end{aligned}$$

From the coboundary formulas above, we can establish the following *recursion formulas.*

$$D_{k,l}(\varphi_2^{0,0,v+1}) = \psi_2^{1,1,v}(v+1) - \psi_3^{1,0,v+1} + \psi_2^{1,0,v+l} b(v+1) \tag{16}$$

$$D_{k,l}(\varphi_3^{0,p,v+1} - \varphi_1^{1,p,v}k) = \psi_3^{1,p+1,v}(v+k+1) + \psi_3^{1,p-1,v+k+1}ap + \psi_3^{1,p,v+l}b(v+k+1-l) \tag{17}$$

$$\begin{aligned} D_{k,l}(\varphi_2^{0,p+1,v+1} - \varphi_1^{1,p,v+1} + \varphi_2^{0,p,v+l+1}b) = \\ \psi_2^{1,p+2,v}(v+1) + \psi_2^{1,p+1,v+l}b(l+2+2v) + \psi_2^{1,p,v+k+1}a(p+2) \\ + \psi_2^{1,p,v+2l}b^2(l+1+v) + \psi_2^{1,p-1,v+k+1+l}abp \end{aligned} \tag{18}$$

If $\eta - \xi$ is a $D_{k,l}$ coboundary, then let us denote this by $\eta \sim \xi$. The recursion formulas above allow us to conclude the following *reduction formulas*.

$$\psi_3^{1,p+1,v} \sim -\psi_3^{1,p,v+l}b\tfrac{v+k+1-l}{v+k+1} - \psi_3^{1,p-1,v+k+1}a\tfrac{p}{v+k+1} \tag{19}$$

$$\begin{aligned} \psi_2^{1,p+2,v} \sim & -\psi_2^{1,p+1,v+l}b\tfrac{l+2(v+1)}{v+1} - \psi_2^{1,p,v+k+1}a\tfrac{p+2}{v+1} \\ & - \psi_2^{1,p,v+2l}b^2\tfrac{l+v+1}{v+1} - \psi_2^{1,p-1,v+k+1+l}ab\tfrac{p}{v+1} \end{aligned} \tag{20}$$

$$\psi_2^{1,1,v} \sim \psi_3^{1,0,v+1}\tfrac{1}{v+1} - \psi_2^{1,0,v+l}b \tag{21}$$

The first reduction formula does not hold when $a = 0$; instead, we have the simpler reduction formula

$$\psi_3^{1,p+1,v} \sim -\psi_3^{1,p,v+l}b. \tag{22}$$

These formulas show us to reduce any cochain of the form $\psi_2^{0,p,n-p-1}$ or $\psi_3^{0,p,n-p-1}$ to a cochain where the middle index of each term is zero, modulo a coboundary.

If we are considering an extension of $d_{k,l}$, we know that it can be reduced to one of the form

$$d = \psi_3^{1,1,0} + \psi_2^{1,0,k}a + \psi_3^{1,0,l}b + \psi_2^{1,0,m}a_m + \psi_3^{1,0,m}b_m,$$

where $a_m = 0$ if $m \le k$ and $b_m = 0$ if $m \le l$. Thus we really are interested in when we can get rid of terms of the form $\psi_2^{1,0,m}$ and $\psi_3^{1,0,m}$ in the expression above, which can be done only when they appear as leading terms in $D_{k,l}$ coboundaries. The reduction formulas allow us to add coboundary terms to a coboundary to reduce terms of the form $\psi_2^{1,q,n-q-1}$ and $\psi_3^{1,q,n-q-1}$ to terms of the form $\psi_2^{1,0,m}$ and $\psi_3^{1,0,m}$, for certain values of m. This will help us to determine which terms of this form are leading terms in coboundaries.

We will be studying the coboundaries of the even cochains

$$\begin{aligned} \varphi_n &= \varphi_1^{1,0,n-1}n + \varphi_3^{0,0,n} \\ \varphi_n' &= \varphi_2^{0,n,0} + \varphi_3^{0,n-1,1}, \end{aligned}$$

which are a basis of the even part of $H^n(d_*)$. What we expect is that these D_*-cocycles will give rise to new $D_{k,l}$-coboundaries, which will allow us to eliminate certain terms in the expressions above which were not leading terms for D_*-coboundaries, but which are leading terms for $D_{k,l}$-coboundaries. We have

$$D_{k,l}(\varphi_n) = -\psi_2^{1,0,k+n-1}a(k+n) - \psi_3^{1,0,l+n-1}bl \tag{23}$$

$$\begin{aligned} D_{k,l}(\varphi'_n) = &-\psi_2^{1,n-1,k}a(k-n) + \psi_3^{1,n-2,k+1}a(n-1) \\ &+\psi_3^{1,n-1,l}b(1-l). \end{aligned} \tag{24}$$

Two extensions of d_* can only be equivalent if λ^* applied to the secondary term of the first extension differs from the secondary term in the the second extension by a coboundary for some linear automorphism λ. Thus we first need to consider the action of the linear automorphism group on the D_*-cohomology. The secondary term can be taken to be of the form $\psi_2^{1,0,k}a + \psi_3^{1,0,k}b$, because every D_*-cohomology class of degree k can be represented by a D_*-cocycle of this form. If a or b vanish, then applying a diagonal automorphism one sees easily that the other coefficient can be taken to be 1, and similarly, if both do not vanish, they can both be taken to be 1. Now

$$\lambda^*(\psi_2^{1,0,k} + \psi_3^{1,0,k}) = \sum_{x=0}^{k} \psi_2^{1,x,k-x}\left(\tfrac{s}{u}\right)^x \tfrac{qu^k}{r} + \psi_3^{1,0,k}\left(\tfrac{s}{u}\right)^x (u^{k-1}r - \left(\tfrac{s}{u}\right)\tfrac{qu^k}{r}).$$

If you choose λ so that $u = qr$, $qu^k r = 1$ and $\left(\frac{s}{u}\right) = u^{k-1}r$, then the above cocycle is cohomologous to $\psi_2^{1,0,k}$. As a consequence, we can assume that the secondary term is either of the form $\psi_2^{1,0,k}$ or $\psi_3^{1,0,l}$ for some k or l.

7.5.1. *Extensions of d_* with secondary term $\psi_2^{1,0,k}$.* Let

$$d_e = \psi_3^{1,1,0} + \psi_2^{1,0,k}.$$

Recall that for any generalized automorphism of d_e, its linear part λ preserves d_* and $\lambda^*(\psi_2^{1,0,k})$, given by equation (15), is D_*-cohomologous to $\psi_2^{1,0,k}$. It follows that $u = qr$, $qu^k = r$ and $s = 0$. But this implies that $\lambda^*(d_e) = d_e$. This fact greatly simplifies the study of extensions of d_e.

The D_e-coboundaries of the D_*-cohomology classes are given by

$$\begin{aligned} D_e(\varphi_n) &= -\psi_2^{1,0,k+n-1}(k+n) \\ D_e(\varphi'_n) &= -\psi_2^{1,n-1,k}(k-n) + \psi_3^{1,n-2,k+1}(n-1). \end{aligned}$$

Thus $\psi_2^{1,0,m}$ is always a D_e-coboundary if $m \geq k$. The case with φ'_n is more complicated. Note that in the first two reduction formulas, since $b = 0$, the middle upper index on the right hand side always drops by 2, and the right hand side of the first reduction formula vanishes for $\psi_3^{1,1,v}$. Thus we conclude that any term of the form $\psi_3^{1,2m+1,v}$ is a D_e-coboundary. Moreover, any term of the form $\psi_2^{1,2m,v}$ reduces to a multiple of $\psi_2^{1,0,v+m(k+1)}$, so is also a D_e-coboundary. When $n = 2m+1$ is odd,

$$D_e(\varphi'_n) = \psi_2^{1,2m,k}(2m+1-k) + \psi_3^{1,2m-1,k+1}(2m),$$

which is a D_e-coboundary, so φ'_{2m+1} extends to a D_e-cocycle.

Notice that for d_e, the reduction formulas simplify to

$$\begin{aligned}\psi_3^{1,p+1,v} &\sim -\,\psi_3^{1,p-1,v+k+1}\tfrac{p}{v+k+1}\\ \psi_2^{1,p+2,v} &\sim -\,\psi_2^{1,p,v+k+1}\tfrac{p+2}{v+1}\\ \psi_2^{1,1,v} &\sim \psi_3^{1,0,v+1}\tfrac{1}{v+1}\end{aligned}$$

When $n = 2m$, applying the reduction formulas to the coboundary of φ'_n yields a coboundary of the form $\varphi_3^{1,0,m(k+1)}$, multiplied by a nonzero constant. Since

$$D_e(\varphi'_{2m}) = \psi_2^{1,2(m-1)+1,k}(2m-k) + \psi_3^{1,2(m-1),k+1}(2m-1),$$

we note that after applying the recursion formulas, both terms reduce to multiples of $\psi_3^{1,0,m(k+1)}$. Adding the coefficient arising in reducing the first term to the one arising in reducing the second term gives an overall nonzero coefficient

$$\frac{(-1)^{m-1}\prod_{i=1}^{m}(2i+1)}{(k+1)^m m!} \tag{25}$$

for the leading cochain $\psi_3^{1,0,m(k+1)}$ in the reduced form of $D_e(\varphi'_{2m})$.

Thus, φ'_{2m} gives rise to a nontrivial D_e-coboundary, and it does not extend to a D_e-cocycle. The even part of the cohomology of d_e has a basis given by the extensions of φ'_{2m+1} to cocycles, while the odd part of the cohomology has a basis given by $\psi_3^{1,0,l}$ for those l which are not multiples of $k+1$. As a consequence, in an extension of d_e of the form $d_{k,l} + \mathrm{ho}$, with $k < l$, if l is a multiple of $k+1$, we can add a D_e-coboundary to eliminate this term. Thus, in classifying extensions of d_e, we only need consider those for which l is not a multiple of $k+1$. In particular, we do not need to consider the case when $l = k+1$. In section 7.5.3 we will consider extensions of d_e of the form

$$d_{k,l} = \psi_3^{1,1,0} + \psi_2^{1,0,k} + \psi_3^{1,0,l}, \qquad l \neq m(k+1).$$

The reduction formulas of the previous section will enable us to determine the leading terms of $D_{k,l}$-coboundaries. We will also have occasion to consider extensions of $d_{k,l}$ of the form

$$d_{k,l,m}=\psi_3^{1,1,0}+\psi_2^{1,0,k}+\psi_3^{1,0,l}+\psi_3^{1,0,m}b, \tag{26}$$

and will want recursion and reduction formulas for this codifferential as well. The modified recursion formulae are

$$D_{k,l,m}(\varphi_2^{0,0,v+1})= \psi_2^{1,1,v}(v+1)-\psi_3^{1,0,v+1}+\psi_2^{1,0,v+l}(v+1)+\psi_2^{1,0,v+m}b(v+1) \tag{27}$$

$$D_{k,l,m}(\varphi_3^{0,p,v+1}-\varphi_1^{1,p,v}k)=\psi_3^{1,p+1,v}(v+k+1)+\psi_3^{1,p-1,v+k+1}p + \psi_3^{1,p,v+l}(v+k+1-l)+\psi_3^{1,p,v+m}b(v+k+1-m) \tag{28}$$

$$\begin{aligned} D_{k,l,m}(\varphi_2^{0,p+1,v+1}-\varphi_1^{1,p,v+1}+\varphi_2^{0,p,v+l+1}+\varphi_2^{0,p,v+m+1}b)= \\ \psi_2^{1,p+2,v}(v+1)+\psi_2^{1,p+1,v+l}(l+2+2v)+\psi_2^{1,p+1,v+m}b(m+2+2v) \\ +\psi_2^{1,p,v+k+1}(p+2)++\psi_2^{1,p,v+2l}(l+1+v)+\psi_2^{1,p,v+2m,}b^2(m+1+v) \\ +\psi_2^{1,p-1,v+k+1+l}p+\psi_2^{1,p-1,v+k+1+m}bp \end{aligned} \tag{29}$$

7.5.2. *Extensions of d_* with secondary term $\psi_3^{1,0,l}$.* Let

$$d_e=\psi_3^{1,1,0}+\psi_3^{1,0,l}.$$

The linear part λ of any generalized automorphism of d_e preserves d_*, and so its action on $\delta=\psi_3^{1,0,l}$ is given by equation (14). Since we must have $\lambda^*\delta\sim\delta$, it follows that $u^{l-1}q=1$, and this is the only condition necessary. Of course, when $\lambda^*(\delta)\neq\delta$, we must follow λ^* with some formal automorphism g^* such that $g^*(\lambda(\delta))=\delta+\mathrm{ho}$, in order to obtain a generalized automorphism λg of d_e. In fact, if we choose

$$\alpha=-\sum_{x=0}^{l}\varphi_3^{0,x-1,l-x+1}\tfrac{1}{l-x+1}\left(\tfrac{s}{u}\right)^x,$$

then $\exp(-\operatorname{ad}\alpha)(\lambda(\delta))=\delta+\mathrm{ho}$.

Because $D_e(\varphi_n)=-\psi_3^{1,0,l+n-1}l$, we see that $\psi_3^{1,0,m}$ occurs as the leading term of a D_e-coboundary for any $m\geq l$, so in any extension of d_e, we can assume that no such terms occur. Thus, up to equivalence, any nontrivial extension of d_e is of the standard form

$$d_{k,l,e}=\psi_3^{1,1,0}+\psi_3^{1,0,l}+\psi_2^{1,0,k}a+\mathrm{ho},$$

where all higher order terms are of the form $\psi_2^{1,0,m}a_m$, and $a\neq 0$. Since the linear part λ of a generalized automorphism λg of $d_{k,l,e}$ is the linear part of a generalized automorphism of d_e, we know that $u^{l-1}q=1$, and $u=qr$. Note that if $2l\neq k+1$, then from equation (15), we can find a diagonal automorphism λ such that the coefficient of $\psi_2^{1,0,k}$ in $\lambda^*(d_{k,l})$ is 1. Moreover, in that case, then any generalized automorphism of $d_{k,l}$ will satisfy $u^{k+1-2l} = 1$, which limits the diagonal part of λ to just a few possibilities.

Let us consider a formal automorphism $g = \exp(-\alpha_2)\cdots$ such that λg is a generalized automorphism of $d_{k,l}$. Then $Q = g^*\lambda^*(d_{k,l})$ is of the form $d_{k,l} + \text{ho}$. We will show that for a certain value of m the coefficient of $\psi_2^{1,0,m}$ appearing in Q is exactly a_m plus a nonzero multiple of s. Thus, for an appropriate value of s, the coefficient becomes zero. As a consequence, if we only consider extensions of $d_{k,l}$ such that the coefficient of $\psi_2^{1,0,m}$ is zero, we do not lose any of the equivalence classes. Moreover, if an automorphism of $d_{k,l,e}$ preserves this property, then the coefficient s in its linear part must vanish. This observation will allow us to restrict our consideration to automorphisms whose linear part is diagonal. In most cases, there are only a few diagonal matrices satisfying our requirements, and they always preserve $d_{k,l}$. Because of this, we will be able to use our main conjecture to classify the extensions of $d_{k,l}$.

If $n < l$, then α_n must be a D_*-cocycle; otherwise terms of degree lower than l would appear in Q. Moreover, no term of type $\varphi_2^{0,p,n-p}$ can appear in α_n for any $n < k$, because otherwise Q would contain a term of type $\psi_2^{1,p+1,n-p-1}$. If $n < l$, the only combination of $\varphi_3^{0,0,n}$ and $\varphi_1^{1,0,n-1}$ which can appear in α_n is a multiple of φ_n, since this is the only D_*-cocycle combination of these terms. However, since $d_e(\varphi_n) = -\psi_3^{1,0,l+n-1}l$, and there are no terms like this in Q if $1 < n < k-l+1$, the coefficient of φ_n must vanish unless $n \geq k-l+1$.

If $l \leq n < k-l+1$ and terms of type $\varphi_3^{0,0,n}$ and $\psi_3^{1,0,n-1}$ occur in α_n, then since $d_e(\varphi_3^{0,0,n})$ and $d_e(\psi_3^{1,0,n-1})$ each contain a term of type $\psi_3^{1,0,l+n-1}$, then they appear as a multiple of the cochain η_n^0, where

$$\eta_n^p = \varphi_1^{1,p,n-p-1}\left(\tfrac{n-p-l}{l}\right) + \varphi_3^{0,p,n-p}\left(\tfrac{1}{l}\right).$$

Since η_n^0 and φ_n span the two dimensional subspace spanned by $\varphi_3^{0,0,n}$ and $\varphi_1^{1,0,n-1}$, when $n \geq k-l+1$ we can discuss their contributions to α_n separately.

Define

$$\zeta_n^p = \varphi_1^{1,p,n-p-1}(n-p) + \varphi_3^{0,p,n-p} \quad \text{if } 0 \leq p \leq n-1.$$

Then

$$\alpha_l = -\sum_{p=0}^{l-1} \eta_l^p \left(\tfrac{s}{u}\right)^{p+1} + \zeta_n^p c_l^p,$$

where c_n^p are arbitrary constants, and $c_l^0 = 0$. In computing $g^*\lambda^*(d_{k,l,e})$, we will be computing terms of the form $(\operatorname{ad}\alpha_j)^k D^\lambda_{k,l,e}(\alpha_i)$ where $D^\lambda_{k,l}$ is the coboundary operator determined by $\lambda^*(d_{k,l,e})$. Now

$$
\begin{aligned}
D^\lambda_{k,l}(\alpha_l) = \sum_{p=0}^{l-1}\Big[&-\psi_3^{1,p+1,l-p-1}\left(\tfrac{s}{u}\right)^{p+1}\\
&+\sum_{x=0}^{l}\psi_3^{1,p+x,2l-x-p-1}\left(\tfrac{s}{u}\right)^x\left(c_l^p(x-l)-\left(\tfrac{s}{u}\right)^{p+1}\tfrac{x}{l}\right)\\
&+\sum_{v=k}^{\infty}\sum_{x=0}^{v}a_v\big[\\
&+\psi_2^{1,p+x,v+l-x-p-1}\left(\tfrac{s}{u}\right)^x\left(c_l^p(x+p-v-l)-\left(\tfrac{s}{u}\right)^{p+1}\tfrac{x+p-v}{l}\right)\\
&+\psi_3^{1,x+p-1,v+l-p-x}\left(\tfrac{s}{u}\right)^x\left(pc_l^p-\left(\tfrac{s}{u}\right)^{p+1}\tfrac{p}{l}\right)\\
&+\psi_3^{1,x+p,v+l-p-x-1}\left(\tfrac{s}{u}\right)^{x+1}\left(c_l^p(v-x)+\left(\tfrac{s}{u}\right)^{p+1}\tfrac{x+l-v}{l}\right)\big]\Big]
\end{aligned}
$$

The first set of terms have degree $l+1$ and are necessary to eliminate the terms of type $\psi_3^{1,p,l-p}$ from $\lambda^*(\delta)$, the second set have degree $2l$, while all the rest have degree $v+l$. Notice that in the coboundary above, the coefficient of the $\psi_3^{1,0,2l}$ term is zero, the coefficient of the $\psi_2^{1,0,k+l-1}$ term is $\left(\frac{s}{u}\right)\frac{k}{l}$, while the coefficient of the $\psi_3^{1,0,k+l-1}$ depends on the choice of the coefficients c_l^p.

Next, remember that in applying an exponential of a coderivation, we also obtain terms of the form $(\operatorname{ad}\alpha_l)^i(D^\lambda_{k,l}(\alpha_l)$. Let us examine the first such term. $[\alpha_l, D_{k,l}(\alpha_l)]$ will contain some terms of degree $2l$, $3l-1$, and $v+2l-1$. The terms of degree $2l$ and $3l-1$ are of the type $\psi_3^{1,p,x-p}$ where $p>0$. No term of type $\psi_3^{1,0,x}$ of or any term of type $\psi_2^{1,p,x-p}$ of degree less than $k+2l-1$ will arise in $(\operatorname{ad}\alpha_l)^i(D^\lambda_{k,l}(\alpha_l))$ for $i\ge 1$.

For $n<l$, α_n consists only of multiples of η_n^p, with $p>0$, unless $n\ge k+l-1$, in which case, note that $\varphi_n=\eta_n^0$, so we will discuss these terms when we discuss the contribution of the φ_n terms.

We have addressed how to construct α_n in order to remove all unwanted terms of type $\psi_3^{1,p,n-p}$ for $p>0$. The first place we encounter terms that must be eliminated which are not of this form is in degree $k+1$ where they come from $\lambda^*(\gamma)$. The terms in $\lambda^*(\gamma)$ which concern us are those of type $\psi_3^{1,0,k}$ and $\psi_2^{1,1,k-1}$, the latter because it gives rise to a term of the former type in the process of elimination. We have

$$D^\lambda_{k,l,e}(-\varphi_2^{0,0,k}\left(\tfrac{s}{u}\right)\tfrac{a}{k}) = -\psi_2^{1,1,k-1}\left(\tfrac{s}{u}\right)a + \psi_3^{1,0,k}\left(\tfrac{s}{u}\right)\tfrac{a}{k} + \sum_{x=0}^{k}(-\psi_2^{1,x,k+l-x-1}\left(\tfrac{s}{u}\right)^{x+1}a + \psi_3^{1,x-1,k+l-x}\left(\tfrac{s}{u}\right)^{x+1}\tfrac{ax}{k}) + \text{ho}$$

The only terms that are important to us in the expression above are the first two, and $-\psi_2^{1,0,k+l-1}\left(\frac{s}{u}\right)a$. To get rid of the the $\psi_3^{1,0,k}$ term, we use φ_{k-l+1}. We have to remember that in addition to the coefficient $-a\left(\frac{s}{u}\right)$ arising from $\lambda^*(\gamma)$, we also must remove the coefficient $\left(\frac{a}{k}\right)\left(\frac{s}{u}\right)$ which arises above. The total coefficient is $\left(\frac{s}{u}\right)\frac{a(1-k)}{k}$. We really only need to look at

$$D_{k,l}\left(\varphi_{k-l+1}\left(\tfrac{s}{u}\right)\tfrac{a(1-k)}{kl}\right) = -\psi_3^{1,0,k}\left(\tfrac{s}{u}\right)\tfrac{a(1-k)}{k} - \psi_2^{1,0,2k-l}\left(\tfrac{s}{u}\right)\tfrac{a^2(1-k)(2k-l+1)}{kl}.$$

When $k+1<2l$, it follows that $k<2k-l<k+l-1$, and the only occurrence of $\psi_2^{1,0,2k-l}$ in the calculation of the addition of coboundaries and higher order terms coming from the exponential is from the term above. Thus, by an appropriate choice of s, we can arrange that a standard form of an equivalent codifferential will have a zero value for the coefficient.

When $k+1>2l$, it follows that $k+l-1<2k-l$, and in addition to the contribution to the $\psi_2^{1,0,k+l-1}$ term from the above coboundary, we also have a contribution from the η_l^0 coboundary. The overall coefficient added is $\left(\frac{s}{u}\right)\frac{a(k-l)}{l}$. Thus we can reduce the $\psi_2^{1,0,k+l-1}$ to zero by an appropriate choice of s.

Finally, when $k+1=2l$, the terms of type $\psi_2^{1,0,2k-l}$ and $\psi_2^{1,0,k+l-1}$ are both of type $\psi_2^{1,0,2l-1}$, with total coefficient $\left(\frac{s}{u}\right)\frac{a(-1+l)(6al-2a+2l-1)}{(2l-1)l}$, which is zero precisely when $a=-\frac{2l-1}{2(3l-1)}$. Otherwise, the same idea works, and we can again assume that the λ is diagonal and that our standard form of the codifferential has a zero coefficient for the $k+l-1$ spot. We will discuss this case in more detail in the subsection devoted to the special case $k+1=2l$.

Now let us make a few remarks about d_e and its cohomology. Equation (23) reduces to

$$\psi_3^{1,0,m} = D_e(-\varphi_{m-l+1}\tfrac{1}{l}), \quad \text{if } m \geq l.$$

Since the D_e-coboundary of φ'_n can be reduced to an element of this same form, it follows that φ'_n can be extended to a D_e-cocycle. The exception is that in the case $n=1$, the term we add to φ'_1 is of the same degree, so, instead of extending to a cocycle, we see that

$$\varphi_1(1-l)+\varphi'_1 l$$

is a D_e-cocycle. Moreover, $D_e(\varphi_3^{0,1,0}) = -\psi_3^{1,1,l-1}$, which is a D_e-coboundary, so $\varphi_3^{0,1,0}$ extends to a D_e-cocycle. Thus, we obtain the following table for the cohomology of d_e.

$$H^1 = \langle \psi_2^{1,0,0}, \psi_3^{1,0,0}, \varphi_3^{0,1,0} + \mathrm{ho}, \varphi_1(1-l) + \varphi_1' l \rangle$$
$$H^n = \langle \psi_2^{1,0,n-1}, \varphi_n' + \mathrm{ho} \rangle, \quad n > 1$$

Occasionally we will need an expanded version of the reduction formulas, related to an extension of $d_{k,l}$ of the form

$$d_{k,l,m} = \psi_3^{1,1,0} + \psi_3^{1,0,l} + \psi_2^{1,0,k} a + \psi_2^{1,0,m} b. \tag{30}$$

The main difference in the nature of the recursion formulas for $d_{k,l,m}$ is that the second formula has to be applied to an infinite sum $\theta_{p,v}$ of cochains defined as follows:

$$\theta_{p,v} = \psi_3^{0,p,v+1} - \psi_1^{1,p,v} k + \sum_{i=1}^{\infty} \psi_1^{1,p,v+i(m-k)} (-1)^i (b/a)^i (m-k). \tag{31}$$

We calculate

$$\begin{aligned} D_{k,l,m}(\theta_{p,v}) = \psi_3^{1,p+1,v}(v+1+k) + \sum_{i=1}^{\infty} \psi_3^{1,p+1,v+i(m-k)} (-1)^{i+1} (b/a)^i (m-k) \\ + \psi_3^{1,p,v+l}(k+v+1-l) + \sum_{i=1}^{\infty} \psi_3^{1,p,v+l+i(m-k)} (-1)^{i+1} (b/a)^i (m-k) \\ + \psi_3^{1,p-1,v+k+1} pa + \psi_3^{1,p-1,v+m+1} pb. \end{aligned}$$

In order to obtain a recursion formula from the formula above, we need to reduce the number of occurrences of the $p+1$ index to 1. Accordingly, we define

$$\Theta_{p,v} = \theta_{p,v} + \sum_{i=1}^{\infty} \theta_{p,v+i(m-k)} (-1)^i \left(\tfrac{(b/a)^i (m-k)}{v+m+1} \right)$$

The modified recursion formulas are as follows.

$$D_{k,l,m}(\varphi_2^{0,0,v+1}) = \psi_2^{1,1,v}(v+1) - \psi_3^{1,0,v+1} + \psi_2^{1,0,v+l}(v+1). \tag{32}$$

$$\begin{aligned} D_{k,l,m}(\Theta_{p,v}) = \psi_3^{1,p+1,v}(v+1+k) \\ + \psi_3^{1,p,v+l}(k+v+1-l) + \sum_{i=1}^{\infty} \psi_3^{1,p,v+l+i(m-k)} \left(\tfrac{(-1)^{i+1}(b/a)^i (m-k) l}{v+m+1} \right) \\ + \psi_3^{1,p-1,v+k+1} pa + \psi_3^{1,p-1,v+m+1} pb \left(\tfrac{v+k+1}{v+m+1} \right). \end{aligned} \tag{33}$$

$$D_{k,l,m}(\psi_2^{0,p+1,v+1} - \psi_1^{1,p,v+1} + \psi_2^{0,p,v+l+1}) = \psi_2^{1,p+2,v}(v+1) + \psi_2^{1,p+1,v+l}(l+2+2v) + \psi_2^{1,p,v+2l}(l+v+1) + \psi_2^{1,p,v+k+1}a(p+2) + \psi_2^{1,p,v+m+1}b(p+2) + \psi_2^{1,p-1,v+k+l+1}ap + \psi_2^{1,p-1,v+m+l+1}bp. \tag{34}$$

7.5.3. *General Extensions of d_*, with $2l \neq k+1$.* We consider the codifferential

$$d_{k,l} = \psi_3^{1,1,0} + \psi_2^{1,0,k} + \psi_3^{1,0,l}$$

When $k < l$, this codifferential arises as an extension of the codifferential considered in (7.5.1), so we can assume that l is not a multiple of $k+1$. When $l < k$, this codifferential arises as an extension of the codifferential considered in (7.5.2), so will will only consider extensions of $d_{k,l}$ in which the coefficient of $\psi_2^{1,0,k+l-1}$ vanishes.

No matter whether this is an extension of the d_e in (7.5.1) or the one in (7.5.2), the condition that $\lambda^*(d_{k,l}) - d_{k,l}$ be the leading term of a d_e-coboundary is satisfied precisely when $u = qr$, $r = qu^k$ and $u^{l-1}q = 1$, so we can solve $r = u^l$, $q = u^{1-l}$, $u^{2l-(k+1)} = 1$. Moreover, it is easily checked that $\lambda^*(d_{k,l}) = d_{k,l}$. There are only at most $|2l-(k+1)|$ solutions for λ, and λ applied to a cochain of fixed degree simply multiplies it by a power of u.

The group $U_{k,l}$ of $|2l-(k+1)|$ roots of unity acts in an obvious way on the set of extensions of $d_{k,l}$. Thus, in studying extensions of $d_{k,l}$, we can determine equivalence by studying coboundaries. Our main goal in the following will be to add terms to $d_{k,l}$ until all of the even cocycles have been killed off. Once we have arrived at an extension d_{fin} of $d_{k,l}$ for which no even cohomology classes remain, the set of equivalence classes of d_{fin} is given by the set of extensions of d_{fin} such that no term of degree larger than the maximal degree in d_{fin} is a coboundary with respect to d_{fin}, modulo the action of the group $U_{k,l}$.

The formula for the coboundary of φ_n allows us to convert cocycles of the form $\psi_2^{1,0,m}$ to the form $\psi_3^{1,0,m+l-k}$ and vice versa, up to a coboundary term. Explicitly, we have the conversion formulas

$$\psi_2^{1,0,v} \sim -\psi_3^{1,0,v+l-k}\tfrac{l}{v+1}$$
$$\psi_3^{1,0,v} \sim -\psi_2^{1,0,v+k-l}\tfrac{v+k+1-l}{l}$$

Studying the reduction formulas, observe that whenever a middle upper index reduces by 2 in a reduction formula, the right upper index increases by either $k+1$ or $2l$. Whenever the middle upper index reduces by 1 in a reduction formula, the right upper index increases by l, with the exception that for $\psi_2^{1,1,v}$, the right upper index in the first term only increases by 1, but in this case, we also convert the lower index from a 2 to a 3, so if you convert the term with a lower index 3 back into a term with lower index 2, the net effect is that the right upper index increased by $k+1-l$. At the last step, we also can convert terms with lower index 2 to terms with lower index 3, and then the upper right hand coefficient increases by $l-k$.

Looking at the first term in $D_{k,l}(\varphi'_n)$, and noting that its upper right index is k, we see that in reductions to terms of middle upper index 0 and lower index 3, at the last step, the k is replaced by an l. Putting all these facts together we obtain that the reduced form of $D_{k,l}$ contains only terms of the type $\psi_3^{1,0,i(k+1)+jl}$, where $2i+j=n$, and every such term arises in the reduction process, possibly with a net zero coefficient. Moreover, if $2i+j=2i'+j'$, then $i(k+1)+jl\neq i'(k+1)+j'l$, since $2l\neq k+1$.

If $k+1<2l$, then if $n=2m$, the smallest upper right index is $m(k+1)$, while if $n=2m+1$, then the smallest such index is $m(k+1)+l$. If $k+1>2l$, then the smallest upper right index is always just nl. Notice that when $k+1=2l$ all coefficients have the same index. Thus the three cases are best treated separately. Note that in all three cases, the smallest upper right index appearing in the expression is different for different values of n, so $\varphi'(n)$ and $\varphi'(n')$ do not reduce to terms with the same order, if the coefficient of the smallest degree term is nonzero.

Case 1: $k+1<2l$

The analysis depends somewhat on whether n is even or odd, so we treat these cases separately. If n is even, say $n=2m$, then the coefficient of the $\psi_3^{1,0,m(k+1)}$ term is the the same coefficient given in equation (25). In particular, it is nonzero. When n is odd, the situation is more complicated.

If $n=2m+1$, the coefficient of the $\psi_3^{1,0,m(k+1)+l}$ term is

$$\frac{(-1)^m(-2l+(2m+1)(k+1))\prod_{i=1}^m(2l+(2i+1)(k+1))}{(k+1)^{m+1}\prod_{i=1}^m(l+i(k+1))},$$

which vanishes only when $2l=n(k+1)$. Thus if $k+1<2l$, we can only have a zero coefficient for at most one value of n, and for most values of k and l this doesn't occur. Since there are some differences in these cases, we treat them separately.

Subcase 1: $n=2m+1$ *and* $2l=n(k+1)$

Note that this case only occurs when $k<l$, since $k+1<2l$. Note that if $x\geq 0$, then

$$i(k+1)+(2x+1)l=(i+nx)(k+1)+l. \tag{35}$$

If $2i+j=n$ and $j>1$, then $j=2x+1$ for some $x>0$. But then the term of type $\psi_3^{1,0,i(k+1)+jl}$ in $D_{k,l}(\varphi'_n)$ is the same type as the leading term of $D_{k,l}(\varphi'_{2(i+nx)+1})$. Since the coefficient of the term $\psi_3^{1,0,m(k+1)+l}$ vanishes, this shows that $D_{k,l}\varphi'_n$ is a sum of coboundaries of different $\varphi's$,

Moreover, every upper right index in any nonleading term in the coboundary of any of the φ' terms occurs as the upper right index of a leading term of some other such coboundary. As a consequence, every term in the coboundary of any φ', with the exception of its leading order term, occurs as a leading order term of the coboundary of some $\varphi'_{n'}$, with $n\neq n'$ Thus, by subtracting appropriate coboundaries of higher order φ's, one arrives at the conclusion that $\varphi_3^{1,0,m'(k+1)}$ and $\varphi_3^{1,0,m'(k+1)+l}$ are actual coboundaries, not just leading order terms of coboundaries, with the exception of $\varphi_3^{1,0,m(k+1)+l}$.

Moreover, φ'_{2m+1} extends to a $D_{k,l}$-cocycle. Thus exactly one higher order even cohomology class remains and many odd cohomology classes remain, those whose upper right index is not a multiple of $(k+1)$ or of the form $m'(k+1)+l$ for some $m'\neq m$. In order to complete the classification of our extensions, we need to go one step further, because we have not yet killed off all the even cohomology. Consider an extension

$$d_{k,l,x}=\psi_3^{1,1,0}+\psi_2^{1,0,k}+\psi_3^{1,0,l}+\psi_3^{1,0,x}b \tag{36}$$

of $d_{k,l}$, where we assume that x is not a multiple of $k+1$, nor is x of the form $x=y(k+1)+l$ for any $y\neq m$. (Otherwise, the extension is equivalent to $d_{k,l}$.) Using the extended recursion formulas (27,28,29) from section 7.5.1, we can compute that the coefficient of the term of type $\psi_3^{1,0,m(k+1)+x}$ term in the reduced form of $D_{k,l,m}(\varphi'_n)$ is

$$\frac{(-1)^m(-2x+(2m+1)(k+1))\prod_{i=1}^m(2x+(2i+1)(k+1))}{(k+1)^{m+1}\prod_{i=1}^m(x+i(k+1))}, \tag{37}$$

which does not vanish. Moreover, our condition on x guarantees that the index $m(k+1)+x$ is not of the form $v(k+1)+l$. Since the upper right index in the leading term in the reduced form of $D_{k,l,m}(\varphi'_{n'})$ is of the form $v(k+1)+l$ when $n'\neq n$, we know that $m(k+1)+x$ is not the upper right index of a $D_{k,l}$-coboundary. Of course, there are terms in the reduced form of $D_{k,l,m}(\varphi'_n)$ which are of the form $i(k+1)+jl$, where $2i+j=n$, which may have smaller degree than $m(k+1)+x$. However, as before, they occur as leading terms of $D_{k,l,m}$-coboundaries of $\varphi's$ of higher degree, which have reduced forms with indices with $k+1$ and l terms (which can be expressed in terms of even higher degree $\varphi's$), and terms with indices involving $k+1$ and l and x of degree higher than $m(k+1)+x$. Thus, we are able to conclude that $\psi_3^{1,0,m(k+1)+x}$ is the leading term of the reduced form of the coboundary of φ'_n. Thus we have finally killed off all the even cohomology. Any two extensions of $d_{k,l,x}$ of the form

$$d_{k,l,x,e} = \psi_3^{1,1,0} + \psi_2^{1,0,k} + \psi_3^{1,0,l} + \psi_3^{1,0,x} b + \sum_{y=x+1}^{\infty} \psi_3^{1,0,y} b_y, \tag{38}$$

where $b_y = 0$ if y is a multiple of $k+1$, is of the form $y = z(k+1)+l$ for some $z \neq m$, or is equal to $m(k+1)+x$, are equivalent precisely when they are equivalent under the action of the group $U_{k,l}$ of $2l-(k+1)$ roots of unity.

Subcase 2: $n = 2m+1$ *and* $2l \neq n(k+1)$

Note that when $k \leq l$, the leading terms of the coboundaries associated to φ_n and φ'_n are $\psi_2^{1,0,k+n-1}$ and $\psi_3^{1,0,m(k+1)}$ or $\psi_3^{1,0,m(k+1)+l}$ depending on whether n is even or odd. Thus, when $k \leq l$ we obtain that any extension of $d_{k,l}$ is equivalent to one of the form

$$d_{k,l,x} = \psi_3^{1,1,0} + \psi_2^{1,0,k} + \psi_3^{1,0,l} + \sum_{x=l+1}^{\infty} \psi_3^{1,0,x} b_x, \tag{39}$$

where $b_x = 0$ if x is a multiple of $k+1$, or l plus a multiple of $k+1$.

When $l < k+1 < 2l$, then the terms we should be adding should be converted to the opposite kind. Thus the leading terms of coboundaries are $\psi_3^{1,0,l+n-1}$ and $\psi_2^{1,0,(m+1)(k+1)-l-1}$ or $\psi_2^{1,0,(m+1)(k+1)-1}$, depending on whether n is even or odd. Thus any extension of $d_{k,l}$ is equivalent to one of the form

$$d_{k,l,e} = \psi_3^{1,1,0} + \psi_3^{1,0,l} + \psi_2^{1,0,k} + \psi_2^{1,0,x} a_x,$$

where $a_x = 0$ if $x = (m+1)(k+1)-l-1$ or $x = (m+1)(k+1)-1$, and this condition classifies these extensions up to the action of the group $U_{k,l}$.

Case 2: $k+1 > 2l$

In this case, the lowest upper right index of $D_{k,l}(\varphi'_n)$ is nl and the coefficient of this term is simply

$$\frac{(-1)^n((n+1)l-(k+1))}{(n-1)l+k+1}, \tag{40}$$

which only vanishes if $k+1 = nl$. Again, we need to treat this special case separately.

Subcase 1: $k+1 = (n+1)l$

This case is similar to the case when $2l = n(k+1)$ above, in that $\psi_3^{1,0,ml}$ is a coboundary, not just the leading term of a coboundary, when $m \neq n$. Moreover φ'_n extends to a cocycle. Since $k > l$, the terms $\psi_3^{1,0,xl}$, which arise from our application of the reduction formulas to φ'_x are not of the type that we use to extend $d_{k,l}$, because $k > l$, and we should convert these terms to ones with lower index 2. Thus terms of the form $\psi_3^{1,0,l+x-1}$ are leading terms of coboundaries, and those of the form $\psi_2^{1,0,k+(x-1)l}$ are coboundaries when $x \neq n$. Thus, any extension of $d_{k,l}$ is of the form

$$d_{k,l,m} = \psi_3^{1,1,0} + \psi_3^{1,0,l} + \psi_2^{1,0,k} + \psi_2^{1,0,m}a,$$

where xm is not of the form $(x-1)l + k$ if $x \neq n$. Using the modified recursion formulas (32,33,34) from section 7.5.2, and using only the first term in the infinite series in equation (33), since the others will clearly lead to larger upper right indices, we find after reduction that there are terms with upper right indices of the form $i(k+1)+jl+z(m+1)+k-l$ and $i(k+1)+jl+z(m+1)+m-l$ for any triple (i,j,z) of integers such that $2i + j + 2z = n$ as long as $i \geq 1-n$, $0 \leq j \leq n$, $z \geq 0$. The ones where $z = 0$ correspond to the terms from the unmodified reduction formulas. These terms are handled by the same type of reasoning as in the previous subcase 1. The next lowest degree term is of type $\psi_2^{1,0,m+(n-1)l}$, and its coefficient can be shown to equal

$$\frac{-b((n-1)l+1)((n+1)l-(m+1))}{nl}. \tag{41}$$

This coefficient would vanish only if $m+1 = (n+1)l$, which is impossible, since $m > k$. Thus our extended codifferential picks up an additional leading coboundary term $\psi_2^{1,0,m+(n-1)l}$, and in extensions of $d_{k,l,m}$ we avoid adding terms of that type, as well as those we excluded before.

Subcase 2: $k+1$ is not a multiple of l

In this case, all the even cocycles have been eliminated and $\psi_3^{1,0,ml}$ is the leading term of a coboundary for all $m > 0$. Thus, up to equivalence, and the action of the group $U_{k,l}$, extensions of $d_{k,l}$ are of the form

$$d_{k,l,e} = \psi_3^{1,1,0} + \psi_3^{1,0,l} + \psi_2^{1,0,k} + \sum_{v=k+1}^{\infty} \psi_2^{1,0,v} a_v, \tag{42}$$

where $v \neq k + xl$ for any positive integer x.

7.5.4. *Extensions of d_* when $k+1 = 2l$.* Let

$$d_e = \psi_3^{1,1,0} + \psi_3^{1,0,l}. \tag{43}$$

Recall from section 7.5.2 that $\varphi_1(1-l)+\varphi_1'l$ is the only non trivial D_e-cohomology class of order 1. In general, its $D_{k,l}$-coboundary is $-\psi_2^{1,0,k}a(k+1-2l)$, so that usually $\psi_2^{1,0,k}$ represents a $D_{k,l}$. But in the present case, we see that $\varphi_1(1-l)+\varphi_1'l$ is a $D_{k,l}$-cocycle, and $\psi_2^{1,0,k}$ is not a coboundary. This is consistent with the observation that we cannot assume that the constant a appearing in $d_{k,l}$ is 1. Thus we begin with an extension of d_e of the form

$$d_{k,l}=\psi_3^{1,1,0}+\psi_3^{1,0,l}+\psi_2^{1,0,2l-1}a. \tag{44}$$

It is easy to see that every term which occurs in the reduced form of the coboundary of φ_n' has type $\psi_3^{1,0,nl}$. The coefficient occurring in the reduced form of $D_{k,l}(\varphi_n')$, for $n>1$ is

$$\frac{C(n)(a(n+1)^2+nl)\prod_{2p+1<n}(a(n-2p-1)^2-p(n-p-1)l)}{l^{[n/2]}}, \tag{45}$$

where $C(n)$ is a nonzero constant, depending only on n. As a consequence, except when $n=1$, the coefficient of this term does not vanish for generic values of a. In particular, when $a=1$, this coefficient does not vanish for $n>1$. On the other hand, for any value of a which can be expressed in the form $a=\frac{p(n-p-1)l}{(n-2p-1)^2}$, the coefficient vanishes, and moreover, if one of these coefficients vanishes, then an infinite number of them do. In fact, if p is the smallest value for which there is some n such that $a=\frac{p(n-p-1)l}{(n-2p-1)^2}$, then $a=\frac{p'(n'-p'-1)l}{(n'-2p'-1)^2}$, iff $p|p'$ and $(n-1)|(n'-1)$. Therefore, for some very special values of a, we obtain a more complicated phenomena than usual; especially, we see that there are an infinite number of values of n for which φ_n' extends to a nontrivial cocycle, so we have some nonzero even cohomology classes.

For generic values of a, we have only one even cohomology class, represented by $(1-l)\varphi_1+l\varphi_1'$. If we reduce the cochain $\psi_3^{1,0,nl}$ to lower index 2, we get a nonzero multiple of $\psi_2^{1,0,(n+1)l-1}$, so cocycles of the form $\psi_3^{1,0,l+n-1}$ and $\psi_2^{1,0,(n+1)l-1}$ are the leading terms of $D_{k,l}$-coboundaries. Thus, if we extend $d_{k,l}$, we would add a term of the form $\psi_2^{1,0,m}b$ to it, where $m\neq(n+1)l-1$.

Whether or not a is a very special value, if we consider any extension of the form

$$d_{k,l,m}=\psi_3^{1,1,0}+\psi_3^{1,0,l}+\psi_2^{1,0,k}a+\psi_2^{1,0,m}b, \tag{46}$$

then by applying a diagonal automorphism, we can see that the coefficient b can be taken to be 1. In fact, $D_{k,l,m}(\varphi_1(1-l)+\varphi_1'l)=-\psi_2^{1,0,m}b(m+1-2l)$, so $\psi_2^{1,0,m}$ is a coboundary, as we would expect, and there are no nontrivial 1-cocycles for the extended codifferential.

When a is not a special value, then we restrict the terms which we add so that $m \neq (n+1)l-1$ for any $n \geq 1$. When a is special for the number n, consider what happens if we let $m=(n+1)l-1$. Then for any value of n' such that $(n-1)|(n'-1)$, we have $D_{k,l}(\varphi'_{n'}) \sim 0$, and

$$(47) \quad D_{k,l,m}(\varphi'_{n'}) = \\ D_{k,l} + \psi_2^{1,n'-1,l(n+1)-1}(n'+1-l(n+1)) + \psi_3^{1,n'-2,l(n+1)}(n'-1),$$

which is reducible to a multiple of $\psi_2^{1,0,l(n+n')-1}$. Every n'' such that $(n-1)|(n''-1)$ arises exactly once in the set $n+n'-1$ as n' ranges over the integers such that $(n-1)|(n'-1)$. Thus we see that cochains of the form $\psi_2^{1,0,(n''+1)l-1}$, where $(n-1)|(n''-1)$ also appear as $D_{k,l,m}$-coboundaries of the former even $D_{k,l}$-cohomology class representatives. Thus, all the even cohomology is killed, and we restrict any extension of $d_{k,l,m}$ to one in which the additional terms are of type $\psi_2^{1,0,x}$, for x not of the form $x=(n''+1)l-1$, for any $n''>0$.

We studied the reduction of $D_{k,l,m}(\varphi'_{n'})$, and were not able to find a formula for the coefficient of the leading term of type $\psi_2^{1,0,(n+n')l-1}$. On the other hand, the terms in the reduced form are of the types $\psi_2^{1,0,p(n-1)+n'+1}$, for $p=1\cdots n'$. They are all of the form $\psi_2^{1,0,(n''+1)l-1}$ for some n'' such that $(n-1)|(n''-1)$, so that they don't appear as leading terms of $D_{k,l}$-coboundaries. The highest degree term in $D_{k,l,m}(\varphi'_{n'})$ is of type $\psi_2^{1,0,(nn'+1)l-1}$, and its coefficient is not equal to zero. Therefore, we know that $D_{k,l,m}(\varphi'_{n'})$ gives rise to a new coboundary, whose leading term is probably of type $\psi_2^{1,0,(n+n')l-1}$. If this remark is true, then we know exactly what are the types that must be excluded in the extension process, that is, those which are of the form $\psi_2^{1,0,x}$, where $x=(y+1)l-1$ for any $y>n$.

Thus, all the even cohomology is killed, and we restrict any extension of $d_{k,l,m}$ to one in which the additional terms are of type $\psi_2^{1,0,x}$, for x not of the form $x=(n''+1)l-1$, for any $n''>0$, if the leading coefficient is always nonzero, and some other pattern if the leading term does vanish.

8. Conclusions

In this paper we have succeeded in classifying all extensions of codifferentials of degree less than or equal to 2 on a 2|1-dimensional space, including classifying the moduli space of degree one and degree two codifferentials. We have not studied the moduli space of degree three or larger codifferentials. It would be interesting to know if some new phenomena arise when studying such higher degree codifferentials. Essentially, what we have been looking at is the cohomology of $\mathbb{Z}_2$-graded Lie algebras, because a degree 2 codifferential determines precisely this structure on the space. Thus the study of higher degree codifferentials is an uncharted territory.

It is clear from the complications that arise in the $2|1$-dimensional space that we are reaching the upper limit classification of L_∞ algebras by brute force construction. Our purpose was not to give an exhaustive classification of low dimensional L_∞ algebras, but to explore the ideas that arise from the study of the classification and extension problem in simple cases, to see what kind of interesting phenomena can be observed.

References

1. A. Bodin, D. Fialowski and M. Penkava, *Classifying extensions of L_∞ algebra structures*, In Preparation, 2003.
2. A. Fialowski and M. Penkava, *Deformation theory of infinity algebras*, Journal of Algebra **255** (2002), no. 1, 59–88, math.RT/0101097.
3. ———, *Examples of infinity and Lie algebras and their versal deformations*, Banach Center Publications (2002), 27–42, math.QA/0102140.
4. ———, *Classification and deformation theory of L_∞ algebras of dimension* $1|2$, preprint, 2003.
5. ———, *Versal deformations of three dimensional Lie algebras as L_∞ algebras*, preprint, 2003.
6. M. Penkava, *Infinity algebras and the homology of graph complexes*, Preprint q-alg 9601018, 1996.
7. ———, *Infinity algebras, cohomology and cyclic cohomology, and infinitesimal deformations*, Preprint math.QA/0111088, 2002.

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