Discrete Mathematics 20 (1977) 11–20.

ON A CLASS OF BALANCED HYPERGRAPHS

András FRANK

Research Institute for Telecommunication, Budapest, Hungary

Received 10 October 1976

Let P be an arborescence, and let $F_u = \{U_1, \dots, U_k\}$, $F_v = \{V_1, \dots, V_\ell\}$ be two systems consisting of directed subpaths of P. Minimax theorems and algorithms are proved concerning the so called *bi-path* system $(P; F_u, F_v)$. One can define a hypergraph to every bi-path system. The class of these "*bi-path*" hypergraphs is closed under forming of dual, sub and partial hypergaph. Every *bi-path* hypergraph is balanced but not necessarily unimodular.

0.

Minimax theorems concerning finite interval system are well known since many years. For example the theorem of Gallai [1] states: in the interval system $I = \{I_1, I_2, \dots, I_m\}$ defined on a line the maximum number of pairwise disjoint intervals of i is equal to the minimum number of covering points. The dual version of Gallai's theorem also holds: the minimum number of covering intervals is equal to the maximum number of vertices from which every interval contains at most one vertex. The direct proofs of such theorems are simple but they follow from the fact that an interval hypergraph is unimodular. For the sake of exactness we shall speak about the *subpaths* of a directed path instead of an interval system, that is we consider a directed path P, as a graph, and the subpaths of P instead of the line and its intervals respectively. We can obtain a generalization of such a system if P is not a directed path but an *arborescence*.

1.

Definition 1.1. Let $P = \{x_1, x_2, \dots, x_n\}$ be a finite set of points and $G(P, E)$ a directed tree such that every vertex of G can be reached by a directed path from x_1. The graph G is called an *arborescence with root* x_1.

Remark. A directed path is an arborescence.

Definition 1.2. Let $F_u = \{U_1, U_2, \dots, U_k\}$ be a system of directed subpaths of an arborescence $G(P, E)$ such that every vertex of G is contained in some U_j. The

Definition 1.5. Let $G(P,E)$ be an arborescence with root x_1, $F_u = \{U_1, U_2, \dots, U_k\}$, $F_v = \{V_1, V_2, \dots, V_m\}$ are two path systems consisting of directed subpaths of G such that every path of one system intersects at least one path of the other. The system $(P; F_u, F_v)$ is called a *bi-path* system.

Index condition. Suppose that the indices of F_u and F_v have the property that when the initial vertex of U_i (V_i) is nearer to the root of the arborescence than that of U_j (V_j) then $i < j$.

Definition 1.6. A family F'_u of distinct paths of F_u is called *independent with respect to* F_v when $U', U'' \in F'_u$, $V \in F_v$, $U' \cap V \neq \emptyset$ imply $U'' \cap V = \emptyset$.

Definition 1.7. A family F'_v of distinct paths of F_v is called *intersecting with respect to* F_u when $U \in F_u$ implies the existence of some $V \in F'_v$ so that $U \cap V \neq \emptyset$.

Theorem 1.8. *In any bi-path system* $(P; F_u, F_v)$,

$$\max |F'_u| = \min |F'_v|,$$

where F'_u *is independent and* F'_v *is intersecting.*

Proof. The inequality $\max \leq \min$ is trivial. The following algorithm finds an independent system F'_u and an intersecting system F'_v with the same cardinalty.

At first let F'_u and F'_v be empty. Every step consists of two parts. In the first part of each step we join the path U_j of F_u to F'_u if j is the maximum index such that U_j is disjoint to every path V of F'_v. (By the *index condition* the maximality of j means that U_j is the farthest from the root among all paths of that type.)

In the second part join path V_i of F_v to F'_v when i is the minimum index such that V_i intersects that paths U_j which was joined to F'_u in the first part of the current step. Such a V_i surely exists by the definition of a *bi-path* system.

The algorithm stops when the first part of the following step can not be realized (since the desired U_j does not exist). We prove that the algorithm leads to the required systems.

(a) $|F'_u| = |F'_v|$ is obvious.

(b) F'_u is independent with respect to F_v.

Otherwise there were $U_j, U_l \in F'_u$, $V_i \in F_v$ such that $U_j \cap V_i \neq \emptyset$ and $U_l \cap V_i \neq \emptyset$. Suppose $j > l$. This means that U_l is joined to F'_u later than U_j. Let V_h be the path joined to F'_v in the same step as U_j. By the construction $h \leq i$. Now $j > l$ means that the initial vertex of U_l is not farther to the root than that of U_j and $i \geq h$ means that the initial vertex of V_h is not farther to the root than that of V_i, moreover $U_l \cap V_h = \emptyset$, by the construction. Hence $U_l \cap V_i = \emptyset$, a contradiction.

(c) F'_v is intersecting with respect to F_u.

If $U_j \in F'_u$ the V_i being joined to F'_v in the same step as U_j, intersects U_j. $U_j \notin F'_u$

means that U_j was not joined to F'_u because there was a path V_i of F'_v such that $U_j \cap V_i \neq \emptyset$.

The flowchart of the algorithm is as given in Fig. 2.

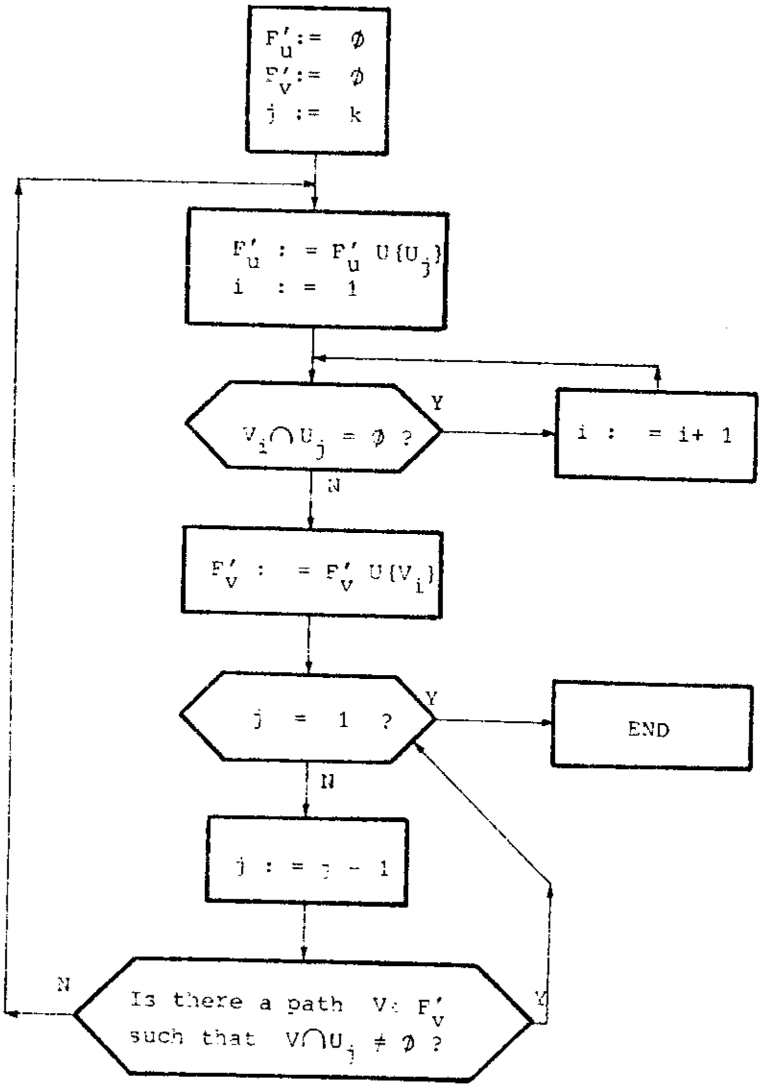

Fig. 2. Algorithm for $F_u = \{U_1, \ldots, U_k\}$, $F_v = \{V_1, \ldots, V_m\}$.

Corollary 1.9. *In a path hypergraph (P, F_u) the maximum number of pairwise disjoint paths is equal to the minimum number of covering vertices.*

Proof. Let us apply Theorem 1.8, for the *bi-path* system $(P; F_u, F_v)$ in the case when F_v consists of paths with one vertex each; every vertex of the arborescence P is a path in F_v.

Corollary 1.10. *In a path hypergraph (P, F_v) the minimum number of covering paths* *is equal to the maximum number of vertices from which every path contains at most one.*

Proof. Apply Theorem 1.8 for the bi-path system $(P; F_u, F_v)$, this time F_u consists of paths with one vertex each.

Remark. The above algorithm can be translated easily to the case of a *path* hypergraph.

Definition 1.11. Let $(P; F_u, F_v)$ be a *bi-path* system and assign a hypergraph H to this system as follows: the vertex set $\bar{F}_v = \{\bar{V}_1, \dots, \bar{V}_m\}$ of H corresponds to F_v and the edge set $\bar{F}_u = \{\bar{U}_1, \dots, \bar{U}_k\}$ of H corresponds to F_u. $\bar{V}_i \in \bar{U}_j$ if and only if $V_i \cap U_j \neq \emptyset$. The hypergraph H is called a *bi-path hypergraph.*

Proposition 1.12. *The dual. the subhypergraph and the partial hypergraph of a bi-path one is a bi-path hypergraph again.*

Obvious!

It easy to check. by the Theorem 1.8 and the definition that in a *bi-path* hypergraph H, $\nu(H) = \tau(H)$, that is the maximum number of disjoint edges is equal to the minimum number of covering points. By Proposition 1.12 every subhypergraph has this property therefore a *bi-path* hypergraph is balanced.

Observe that any *bi-path* hypergraph is balanced but not necessaritly unimodular as the example in Fig. 3 implies.

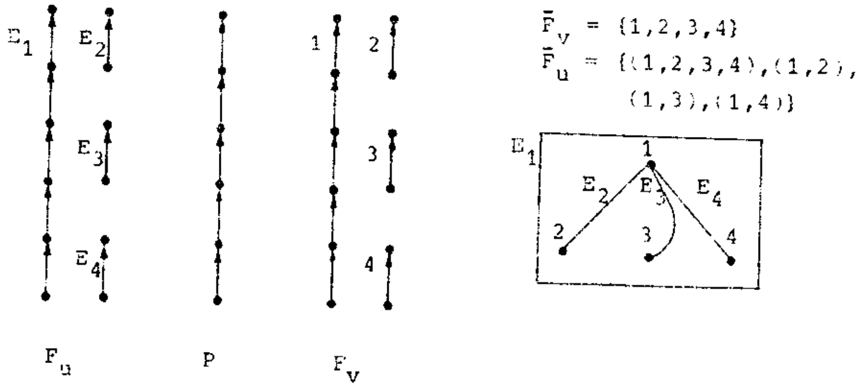

Fig. 3.

This example indicates that the concept of the bi-path hypergraphs is more general than that of the path hypergraphs. This follows also from the fact that the dual of a path hypergraph is not a path hypergraph in general (Fig. 4).

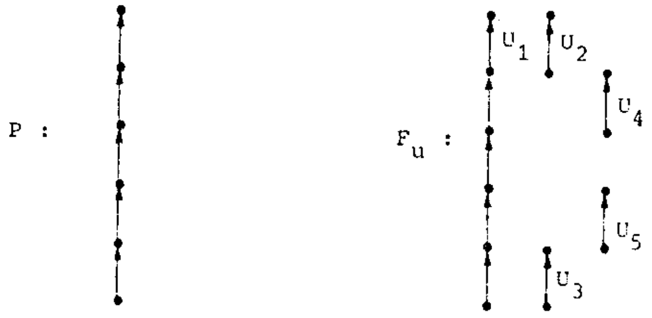

Fig. 4.

2.

In the sequel we are going to prove two theorems and algorithms concerning bi-path systems.

Assign a non-negative integer $d_j = d(U_j)$ to every path U_j for $j = 1, 2, \dots, k$ and a non-negative integer $c_i = c(V_i)$ to every path V_i for $i = 1, 2, \dots, m$.

Definition 2.1. A system F'_u of not necessarily distinct paths of F_u is called *c-independent* with respect to F_v if every path V of F_v intersects at most $c(V)$ paths of F'_u. F'_u is called *d-allowed* if every path U of F_u occurs in F'_u at most $d(U)$ times.

Definition 2.2. A *u-covering* system $D = D_u \cup D_v$ consists of some distinct paths of F_v $(= D_v)$, and all the paths $(= D_u)$ of F_u which are disjoint to all the paths of D_v. The *weight* $s(D)$ of a u-covering system is defined as the sum of numbers (c_i or d_j) assigned to the paths of D.

Theorem 2.3. $\max|F'_u| = \min s(D)$ where F'_u is d-allowed and c-independent with respect to F_v and D is a u-covering system.

Proof. (1) max ≤ min. Let F'_u be c-independent d-allowed and D u-covering. The number of paths of F'_u which are disjoint to every path of D_v is at most $\Sigma_{U\in D_u} d(U)$. The number of paths of F'_u which intersect some paths of D_v is at most $\Sigma_{V\in D_v} c(V)$.

(2) max = min. By the previous argument the equality $|F'_u| = s(D)$ holds when the following three conditions are true.

(A) Every path U of F'_u intersects at most one path of D_v.

(B) For every path U of D_u the number of copies of U in F'_u is exactly $d(U)$.

(C) For every path V of D_v the number of paths of F'_u which intersect V is exactly $c(V)$.

By means of the following algorithm we can construct the pair (F'_u, D) satisfying the three optimality criterion. The algorithm consists of two parts. The first one produces the system F'_u (see Fig. 5).

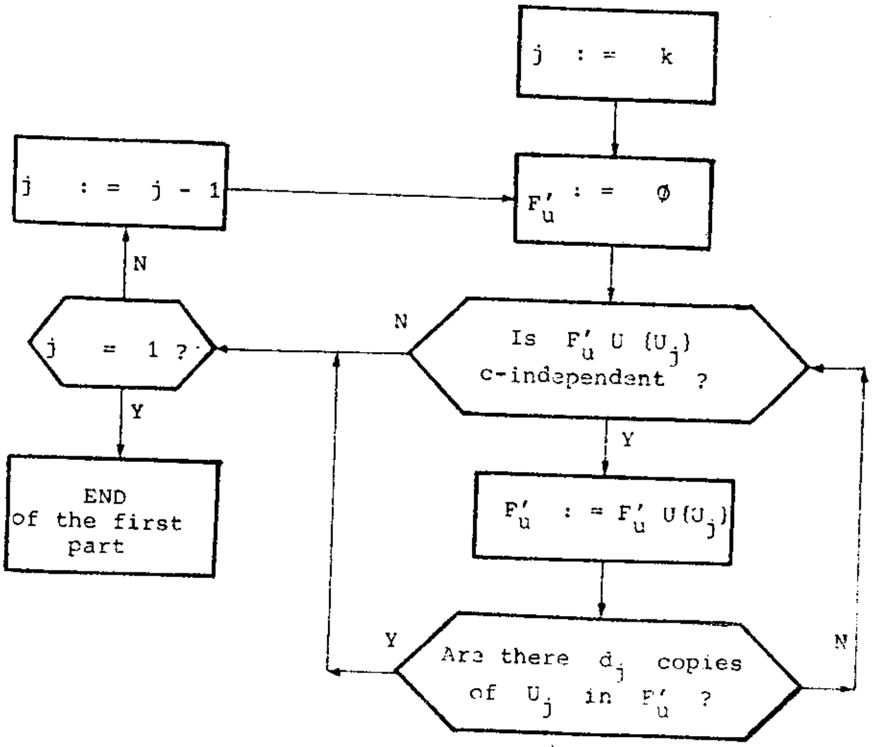

Fig. 5. The construction of F'_u.

We go through on the paths of F_u by descending indices and join to F'_u the maximum number of copies of the current path U_j so that the number of copies of U_j is at most $d(U_j)$ and F'_u preserves the c-independence.

In the second part of the algorithm we construct D_v which already determines D_u and D. Let us consider by ascending indices the paths of F_v. We join a path V_i to D_v if V_i intersects exactly c_i paths of F', where F' denotes the set of paths of F'_u which are disjoint to the paths of the current D_v (see Fig. 6).

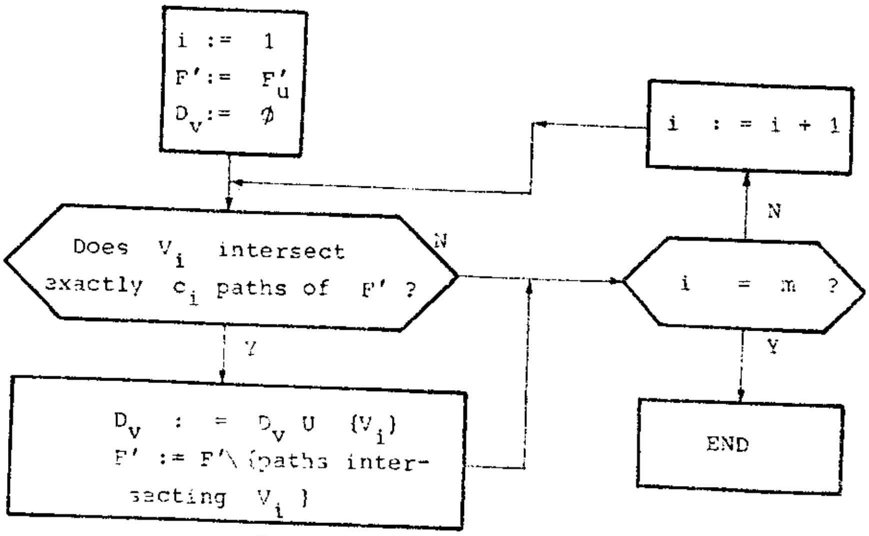

Fig. 6. The construction of D_v.

By the algorithm the optimality criterions (A) and (C) hold. We have to verify criterion (B). Suppose indirectly that there exists a path U_j of D_u with the property that the number of its copies is less than d_j. In this case there exists a path V_i of F_v which intersects exactly c_i paths of F_u' and $V_i \cap U_j \neq \emptyset$. Clearly V_i is not contained in D_v. Let V_i be of such type with maximal index. Now $U_r \in F_u'$ and $U_r \cap V_i \neq \emptyset$ imply $r > j$. Otherwise in the first part of the algorithm a further copy of U_j should be joined to F_u'. (At this point we use the maximality of i and the fact that in the first part we considered the paths of F_u by descending indices.) Hence the c_i paths of F_u' intersecting V_i do not contain any vertex lying between the initial vertex of U_j and the root of the arborescence. However in this case V_i should be joined to D_v in the second part of the algorithm, a contradiction.

Let assign a non-negative integer $d_j = d(U_j)$ to every path U_j of F_u for $j = 1, 2, \ldots, k$ and a non-negative integer $f_i = f(V_i)$ to every path of F_v for $i = 1, 2, \ldots, m$ such that for every path V_i $f(V_i) \leq \sum_{U: U \cap V_i \neq \emptyset} d(U)$.

Definition 2.4. A system F_u' of paths of F_u is called *f-intersecting* with respect to F_v if every path V_i of F_v intersects at least f_i paths of F_u'.

Definition 2.5. A *u-independent* system $D = D_u \cup D_v$ consists of some distinct paths of F_v $(= D_v)$ and all the paths of F_u $(= D_u)$ which intersect at least two paths of D_v. The *weight* $s(D)$ of a u-independent system D is

$$s(D) = \sum_{V \in D_v} f(V) + \sum_{U \in D_u} d(U) - \sum_{U \in D_u} (U, D_v)$$

where (U, D_v) denotes the number of paths of D_v intersected by U.

Theorem 2.6. $\min |F_u'| = \max s(D)$ *where* F_u' *is* d*-allowed* f*-intersecting,* D *is* u*-independent.*

Proof. (1) $\max \leq \min$. Let F_u' be a d-allowed f-intersecting system and D a u-independent system. We can enumerate the paths of F_u' so that we consider the sum $\sum_{V \in D_v} f(V)$. However this sum is not a lower bound for $|F_u'|$ because we have enumerated some paths several times. We get a valid lower bound for $|F_u'|$ if subtract $d(U) \cdot [(U, D_v) - 1]$ from the aforementioned sum for every path U intersecting at least one path of D_v. (We must multiply by $d(U)$ because $d(U)$ copies of U can occur in F_u'). The obtained lower bound is just $s(D)$.

(2) In order to verify the reverse inequality we have to prove that there exist two systems F_u' and D which satisfy the equality. By the previous argument the cardinality of F_u' is equal to $s(D)$ if and only if the following three condition are fulfilled:

(A) Every path of F_u' intersects at least one path of D_v.

(B) For every path U_j of D_u the number of copies of U_j occurring in F_u' is exactly d_j.

(C) Every path V_i of D_v intersects exactly f_i paths of F'_u.

By means of the following algorithm we construct the pair (F'_u, D) satisfying the three optimality criterions.

The algorithm consists of two parts. In the first one we construct the d-allowed f-intersecting system F'_u. Let us consider the paths of F_v by descending order of indices. In every step we control whether the considered path V_i of F_v is intersected at least f_i times by paths of the current F'_u. If the answer is yes we consider the next path V_{i-1} of F_v. Otherwise we choose the path U_r of F_u intersecting V_i with the least index r, provided the number of copies of U_r in the current F'_u is less than d_r (see Fig. 7).

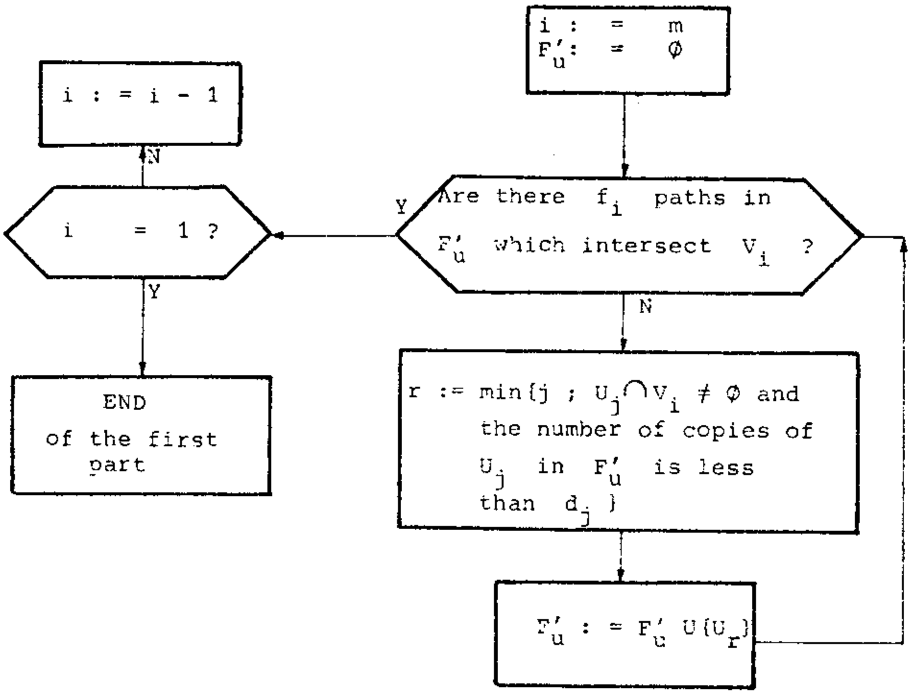

Fig. 7. The construction of F'_u.

In the second part of the algorithm we construct D_v which determines D. Let us consider the paths of F_v by ascending indices. We join a path V_i to D_v in that case when F'_u contains exactly f_i paths intersecting V_i provided the optimality criterion (B) remains true (Fig. 8).

By this algorithm the optimality criterions (B) and (C) hold. We have to verify the criterion (A). Suppose the contrary, i.e. that there exists a path U_j of F'_u which is disjoint to every path of D_v. By the first part of the algorithm we can find a path V_i of F_v for which $U_j \cap V_i \neq \emptyset$ and V_i intersects exactly f_i paths of F'_u. Let V_i be such a path with maximum index. Since $V_i \notin D_v$, there exists a path U_r in F_u with the following properties:

(i) the number of copies of U_r occurring in F'_u is less than d_r,

(ii) $U_r \cap V_i \neq \emptyset$,

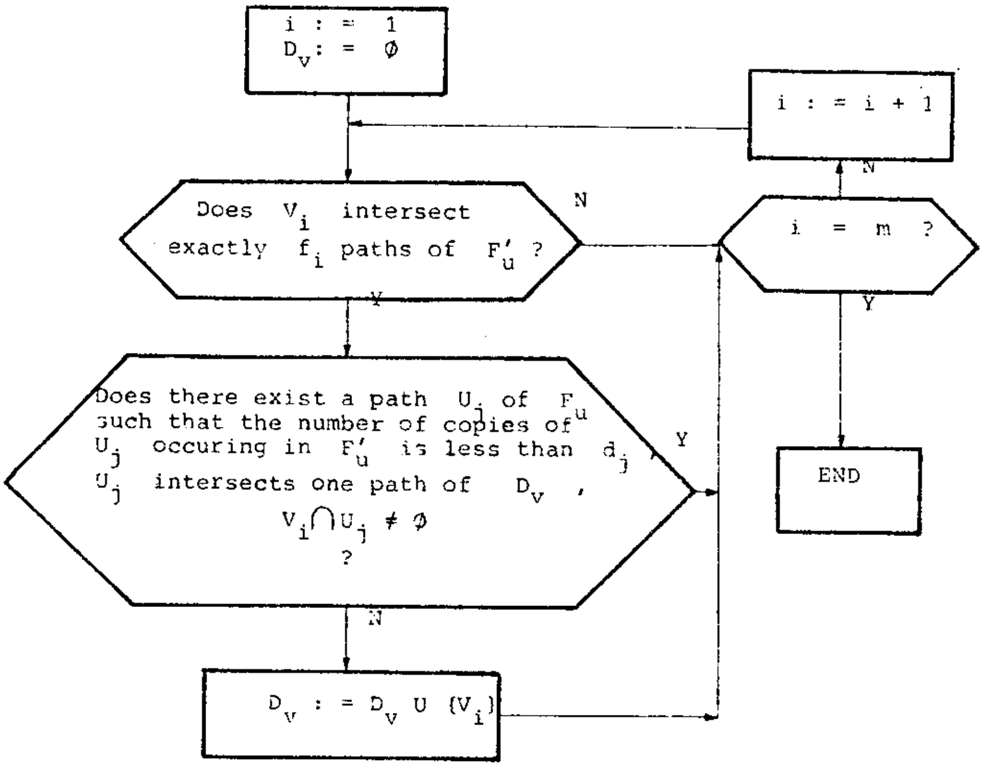

Fig. 8. The construction of D_v.

(iii) U_r intersects a path V_{i_1} of D_v with $i_1 < i$.

Since $V_{i_1} \cap U_j = \emptyset$ we have $r < j$. That means using the maximality of i that in the first part of the algorithm a further copy of U_r instead of U_j should be joined to F'_u.

Remark. Similarly to the first theorem one could formulate the 2-nd and the 3-rd ones and their algorithms concerning path hypergraphs.

References

[1] T. Gallai, Graphen mit triangulierbaren ungeraden Vielecken, Magyar Tud. Ak. Mat. Kut. Int. Közl. 7 (1962).

[2] C. Berge, Graphs and Hypergraphs (North-Holland, Amsterdam, 1973).

[3] A. Frank, Some polynomial algorithms for certain graphs and hypergraphs, Proc. Fifth British Combinatorial Conf. (1975).