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Minimal Edge-Coverings of Pairs of Sets

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We derive a new min–max formula for the minimum number of new edges to be added to a given directed graph to make it k-node-connected. This gives rise to a polynomial time algorithm (via the ellipsoid method) to compute the augmenting edge set of minimum cardinality. (Such an algorithm or formula was previously known only for $k = 1$). Our main result is actually a new min–max theorem concerning "bisupermodular" functions on pairs of sets. This implies the node-connectivity augmentation theorem mentioned above as well as a generalization of an earlier result of the first author on the minimum number of new directed edges whose addition makes a digraph k-edge-connected. As further special cases of the main theorem, we derive an extension of (Lubiw's extension of) Győri's theorem on intervals, Mader's theorem on splitting off edges in directed graphs, and Edmonds' theorem on matroid partitions.

1. INTRODUCTION

Given a directed graph $D = (V, A)$, add a minimum number of new edges to D to obtain a digraph with some specified connectivity properties. This is the general form of the connectivity augmentation problem we are considering in the present paper. Depending on the connectivity property required for the augmented graphs, which might be called the target connectivity, one may formulate various augmentation problems.

Eswaran and Tarjan [4] proved a min–max theorem (and provided a linear time algorithm) for the minimum number of new edges whose addition to D leaves a strongly connected digraph.

A natural generalization, the edge-connectivity augmentation problem, consists of finding a minimum number of new edges whose addition to D

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leaves a k-edge-connected digraph. A min–max formula for this minimum and a polynomial time algorithm was described by Frank [5].

Another natural generalization, the node connectivity augmentation problem, consists of finding a minimum number of new edges whose addition to D leaves a k-node-connected digraph. This problem was previously open even in the special case $k=2$. One of our main purposes is to solve the node-connectivity augmentation problem for arbitrary k. We prove a min–max formula for the minimum in question and describe a polynomial time algorithm to compute the minimum augmentation. (The algorithm makes use of the formula, rather than proving it, and invokes the ellipsoid method.)

The solution method in [5] to the edge-connectivity augmentation problem heavily uses submodular functions. This naturally gives rise to the idea of trying to apply some general frameworks concerning sub-and supermodular functions. In the past two decades a large number of such models have been developed such as polymatroids, submodular flows, lattice polyhedra, linking systems, polymatroidal flows, independent flows, kernel systems, and Δ-matroids. A general account on these models and their relationship can be found in a paper of Schrijver [22]. There is a single most important feature of these models which is in common: each of them is described by a totally dual integral (TDI) linear system [3]. This implies, loosely speaking, that the corresponding primal and dual linear programming problems have integer-valued optima for any integer-valued primal cost function.

This central property of TDI-ness is an explanation why (apart from results including parity considerations) a great part of min–max theorems in graph theory and combinatorial optimization, especially those involving sub- or supermodular functions, are implied by the models above.

But one runs into a serious obstacle when trying to apply any model with a TDI description system to connectivity augmentation problems. Already Eswaran and Tarjan [4] have pointed out that, though the minimum cardinality augmentation problem to make a digraph strongly connected is tractable, the minimum cost version is NP-complete as it includes the directed Hamiltonian circuit problem. Naturally, the same difficulty occurs in the general k-edge connectivity and k-node-connectivity augmentation.

Therefore we cannot expect that a TDI system may be useful for solving these augmentation problems. From this point of view TDI-ness is too strong a property and this led us to consider general frameworks where integral results hold only for a restricted class of cost functions while the problem for general costs may well include NP-complete problems.

The main purpose of the present paper is to introduce such a model. We are going to consider bi-supermodular functions and to prove a min–max theorem concerning minimum "coverings" of these functions by edges. (Note that bi-submodular functions have been investigated by Schrijver earlier in a different context [21].)

This implies a min–max theorem concerning the node-connectivity augmentation problem mentioned above which was our initial prime interest. It implies a generalization of the edge-connectivity augmentation theorem of Frank [5]. A nice theorem of Mader [19] on splitting off edges in directed graphs is also a consequence, as well as Edmonds' theorem [2] on matroid partitions.

Finally, we show that an extension of (Lubiw's extension [17] of) S. Győri's [11] difficult min–max theorem on intervals is also an easy special case of our main result. This beautiful theorem has so far notoriously resisted every attempt to relate it to other well-cultivated parts of combinatorial optimization. Lubiw writes in a paper [17] generalizing Győri's theorem: "*In* [Gy] *Győri proved a min–max equality for intervals which is remarkable for the difficulty of the proof ... and for the lack of similarity to previous min–max results in combinatorial optimization.*"

Our model does provide the missing link and also a simple proof. Actually, we are going to extend Lubiw's generalization in two senses as will be explained in Section 6.

To conclude this introductory section we remark that the proof of the main theorem is short and based on the standard uncrossing technique. (A price we must pay for this simplicity is that the proof is not algorithmic. The only algorithm we have at present does use the main theorem and some ideas from its proof and relies on the ellipsoid method.)

The reader may ponder on this phenomenon how can a theorem, with a rather routine proof, have these far from trivial consequences? An explanation may be based on the new view we take: we do not insist on TDI-ness and require integrality only for a restricted class of objective functions. We hope that this new view may also be successful in other areas and would like to encourage the readers (and ourselves, as well) to work out other general models in this vein.

The organization of the paper is as follows. In Section 2 we state and prove the main results. Section 3 includes degree-constrained and minimum cost variations as well as the description of a relationship of our model to contra-polymatroids. Node-connectivity and edge-connectivity augmentation problems of directed graphs are discussed in Sections 4 and 5, respectively. Section 6 deals with Győri's theorem on intervals and its extensions. Algorithmic aspects are the topic of the last section.

Hopefully, it will not cause ambiguity that we often do not distinguish between a one-element set $\{s\}$ and its element s. Let V be a set. Given two elements x, y of V we say that a subset X of V is an *$x\bar{y}$-set* if $x \in X$, $y \notin X$. A family of disjoint subsets of V is called a *sub-partition*. Two subsets X, Y

of V are *co-disjoint* if $V-X$ and $V-Y$ are disjoint. X, Y are *intersecting* if none of $X-Y$, $Y-X$, $X\cap Y$ is empty. If, in addition, $V-(X\cup Y)\neq\emptyset$, X and Y are *crossing*. A family of subsets of V is called *cross-free* if it contains no two crossing sets.

We extend these notions to ordered pairs of sets. Let S and T be two non-empty subsets of V and let $A^*:=A(S,T)$ denote the set of all directed edges st with $s\in S$, $t\in T$. Let $\mathscr{A}^*:=\mathscr{A}(S,T)$ denote the set of all pairs (X,Y) with $\emptyset\neq X\subseteq S$, $\emptyset\neq Y\subseteq T$. The first member X (respectively, the second member Y) of pair (X,Y) is called the *tail* (the *head*) of the pair.

For an edge $e=st\in A^*$ and a pair $(X,Y)\in\mathscr{A}^*$ we say that e *covers* (X,Y) if $s\in X$, $t\in Y$, that is, if $e\in A(X,Y)$. For a vector $z\colon A^*\to\mathbf{R}$ we use the notation $z(X,Y):=\sum(z(xy)\colon x\in X, y\in Y)$. For a non-negative function w on $\mathscr{A}^*$ let $c_w\colon A^*\to\mathbf{Z}_+$ be a function defined by $c_w(e):=\sum(w(X,Y)\colon e$ covers $(X,Y))$.

Two pairs (A,B), (A',B') are *tail-disjoint* (respectively, *head-disjoint*) if their tails (resp., heads) are disjoint. If at least one of these two holds, the pairs are called *half-disjoint* (or sometimes *independent*). The pairs are *comparable* if $A\subseteq A'$, $B'\subseteq B$ or $A'\subseteq A$, $B\subseteq B'$. The pairs (A,B), (A',B') are called *non-crossing* if they are half-disjoint or comparable. Otherwise (A,B), (A',B') are said to *cross* or to be *crossing*.

A family $\mathscr{F}\subseteq\mathscr{A}^*$ of pairs of sets is *cross-free* if $\mathscr{F}$ contains no two crossing members. $\mathscr{F}$ is called *crossing* if both $(A\cap A', B\cup B')$ and $(A\cup A', B\cap B')$ belong to $\mathscr{F}$ for any two crossing members (A,B), (A',B') of $\mathscr{F}$.

We define a partial order $P:=(\mathscr{A}^*,\leqslant)$ on $\mathscr{A}^*$ as follows. For $(X,Y),(X',Y')\in\mathscr{A}^*$ let $(X,Y)\leqslant(X',Y')$ if $X\subseteq X'$ and $Y\supseteq Y'$. Note that two members of $\mathscr{A}^*$ are comparable if they are comparable in P. The restriction of P to a subset $\mathscr{F}$ of $\mathscr{A}^*$ will be denoted by $P(\mathscr{F})$.

In a directed graph $D=(V,A)$ for subsets $X,Y\subseteq V$ let $\delta_D(X,Y):=\delta_A(X,Y):=\delta(X,Y)$ denote the number of edges xy of D for which $x\in X$, $y\in Y$. If $X=V-Y$, then let $\rho_A(Y):=\rho_D(Y):=\rho(Y):=\delta_A(X):=\delta_D(X):=\delta(X):=\delta_D(X,Y)$, that is, $\rho(Y)$ (respectively, $\delta(X)$) is the number of edges entering Y (leaving X). For a function $x\colon A\to\mathbf{R}$, let $\rho_x(X):=\sum(x(e)\colon e\in A$, e enters $X)$ and let $\delta_x(X):=\sum(x(e)\colon e\in A$, e leaves $X)$.

Let $\mathscr{F}$ be a sub-family of $\mathscr{A}(S,T)$. For a positive integer h we say that $\mathscr{F}$ is *h-independent* if every edge $e\in A(S,T)$ covers at most h members of $\mathscr{F}$. For $h=1$ we simply say *independent*. This is equivalent to requiring that the members of $\mathscr{F}$ are pairwise half-disjoint. More generally, we say that a non-negative function w on $\mathscr{A}^*$ is *h-independent* if $c_w(e)\leqslant h$ holds for every $e\in A^*$.

Throughout the paper we adopt the convention that if a function f is defined explicitly only on some elements of $\mathscr{A}^*$, then we mean f to be 0 on all other elements of $\mathscr{A}^*$. For a function f on the elements of $\mathscr{A}^*$ and for $\mathscr{F}\subseteq\mathscr{A}^*$, we use the notation $f(\mathscr{F}):=\sum(f(X,Y)\colon(X,Y)\in\mathscr{F})$.

We call a set function $q\colon 2^V\to\mathbf{R}$ *fully supermodular* (in short, supermodular) if

$$q(X)+q(Y)\leqslant q(X\cap Y)+q(X\cup Y) \tag{1.1}$$

holds for every pair of subsets $X,Y\subseteq V$. If, in addition, q is non-negative and monotone increasing (that is, $q(X)\geqslant q(Y)$ whenever $Y\subseteq X$), then we speak of a *contra-polymatroid function*. For such a q, a polyhedron $Q:=\{x\in\mathbf{R}^V_+\colon x(A)\geqslant q(A)$ for all $A\subseteq V\}$ is called a *contra-polymatroid*. It is known that Q uniqely determines q. If (1.1) is required only for crossing sets X,Y, we say that q is *crossing supermodular*. In the next section we are going to extend these notions to functions defined on pairs of sets.

For any number x, let $x^+:=x$ if $x\geqslant 0$ and $x^+:=0$ if $x<0$. For a real-valued function f, f^+ denotes a function defined by $f^+(x):=f(x)^+$.

2. Main Results

Let S and T be two non-empty subsets of a ground-set V. Let $(S,T;A^*)$ denote a directed graph with node-set $S\cup T$ and edge-set $A^*=A(S,T)$. Let $\mathscr{A}^*:=\mathscr{A}(S,T)$ and p be a non-negative integer-valued function on $\mathscr{A}^*$. We call p *crossing bi-supermodular* if

$$p(X,Y)+p(X',Y')\leqslant p(X\cap X', Y\cup Y')+p(X\cup X', Y\cap Y') \tag{2.1}$$

holds whenever $X\subseteq S$, $Y\subseteq T$, $X\cap X'$, $Y\cap Y'\neq\emptyset$, and $p(X,Y)$, $p(X',Y')>0$.

Throughout this section p denotes such a function. If the reverse inequality holds in (2.1) we speak of *crossing bi-submodular functions*. Bi-submodular functions were extensively studied by Schrijver [21].

With the help of an edge $e=st\in A^*$ we define a 0–1 function b_e on $\mathscr{A}^*$ by $b_e(A,B):=1$ if e covers (A,B) and 0 otherwise. For a non-negative vector $x\colon A^*\to\mathbf{R}_+$ let us define a function $b_x\colon\mathscr{A}\to\mathbf{R}_+$ by $b_x(A,B):=x(A,B)$. The first part of the following claim is trivial, the second one easily follows from the first.

Claim 2.1. *b_e is bi-submodular. b_x is bi-submodular.*

We introduce two operations concerning a crossing bi-supermodular function p. For any edge $e=xy\in A^*$ define $p^e:=(p-b_e)^+$; that is, $p^e(X,Y):=p(X,Y)-1$ if $p(X,Y)>0$ and $x\in X$, $y\in Y$ and $p^e(X,Y):=p(X,Y)$ otherwise. Since b_e is bi-submodular, $p-b_e$ is crossing bi-supermodular and hence so is p^e. We call p^e a *reduction* of p (along e).

Let $S' \subseteq S$, $T' \subseteq T$ be two non-empty subsets. A *projection* p' of p onto $\mathscr{A}(S', T')$ is defined as follows. For $(X', Y') \in \mathscr{A}(S', T')$ let $p'(X', Y') := \max(p(X, Y): (X, Y) \in \mathscr{A}(S, T),\ X \cap S' = X',\ Y \cap T' = Y')$. It is easy to check that p' is crossing bi-supermodular.

Let w be a non-negative, integer-valued function on $\mathscr{A}(S, T)$. The *support* $\mathscr{S}_w$ of w is the family of pairs (X, Y) for which $w(X, Y)$ is positive. We say that w is *cross-free* if its support is cross-free. Let $p(w) := \sum (w(X, Y)\, p(X, Y): (X, Y) \in \mathscr{A}(S, T))$. Define $c_w: A^* \to \mathbf{Z}_+$ by $c_w(e) := \sum (w(X, Y): (X, Y) \in \mathscr{A}(S, T),\ e$ covers $(X, Y))$ where $e \in A^*$.

Lemma 2.2 *Given a function $w: \mathscr{A}(S, T) \to \mathbf{Z}_+$, there exists a cross-free function $\bar{w}: \mathscr{A}(S, T) \to \mathbf{Z}_+$ for which $p(\bar{w}) \geqslant p(w)$ and $c_{\bar{w}} \leqslant c_w$.*

Proof. Let us choose a function $\bar{w}: \mathscr{A}(S, T) \to \mathbf{Z}_+$ such that $p(\bar{w}) \geqslant p(w)$, $c_{\bar{w}} \leqslant c_w$, and such that $s(\bar{w}) := \sum (\bar{w}(X, Y)\, f(X, Y): (X, Y) \in \mathscr{A}^*)$ is as large as possible where $f(X, Y) := (|X| - |Y|)^2$.

One can easily check that $f(X, Y) + f(X', Y') = f(X \cap X', Y \cup Y') + f(X \cup X', Y \cap Y') - 2(|X - X'| + |Y' - Y|)(|X' - X| + |Y - Y'|)$. Hence f is bi-supermodular and $f(X, Y) + f(X', Y') = f(X \cap X', Y \cup Y') + f(X \cup X', Y \cap Y')$ holds if and only if (X, Y) and (X', Y') are comparable.

We claim that $\bar{w}$ is cross-free. Suppose to the contrary that $\mathscr{S}_{\bar{w}}$ has two crossing members (X, Y) and (X', Y'). Revise $\bar{w}$ by decreasing its value on (X, Y) and (X', Y') by 1 and increasing it on $(X \cap X', Y \cup Y')$ and $(X \cup X', Y \cap Y')$ by 1. By Claim 2.1, $c_{w'} \leqslant c_{\bar{w}}$. Since p is crossing bi-supermodular, $p(w') \geqslant p(\bar{w})$. Since $s(w') > s(\bar{w})$ we are in a contradiction with the choice of $\bar{w}$. ∎

Let p be a non-negative, integer-valued function on $\mathscr{A}(S, T)$. We say that a non-negative vector z on A^* *covers* p or that z is a *covering* of p if $b_z \geqslant p$, or equivalently $z(X, Y) \geqslant p(X, Y)$ for every $(X, Y) \in \mathscr{A}(S, T)$. We will be interested in integer-valued coverings minimizing cz for certain linear objective functions c. Our basic result concerns the case $c \equiv 1$. An extension concerning more general cost-functions will be derived in the next section.

Theorem 2.3. *For an integer-valued crossing bi-supermodular function $p \geqslant 0$ the following min–max equality holds. $\tau_p := \min(z(A^*)$: $z \geqslant 0$ an integer-valued covering of $p) = \nu_p := \max(p(\mathscr{F})$: $\mathscr{F}$ independent$)$.*

Proof. First we prove that $\nu_p \leqslant \tau_p$. Let z be a covering of p and $\mathscr{F}$ an independent family of pairs. Since no edge of A^* can cover more than one member of $\mathscr{F}$, we have $z(A^*) \geqslant \sum (z(X, Y): (X, Y) \in \mathscr{F}) \geqslant \sum (p(X, Y): (X, Y) \in \mathscr{F}) = p(\mathscr{F})$ and hence $\nu_p \leqslant \tau_p$.

We prove now the reverse inequality. For an edge $e \in A^*$ let p^e be the reduction of p along e. Since $p^e \leqslant p$, $\nu_p - 1 \leqslant \nu_{p^e} \leqslant \nu_p$. We call an edge e *reducing* if $\nu_{p^e} < \nu_p$. The next lemma is the key in the proof.

Lemma 2.4. *If $p(A, B) > 0$ for a pair $(A, B) \in \mathscr{A}^*$, then there is a reducing edge $e = xy$ with $x \in A$, $y \in B$.*

Proof. Suppose indirectly that $\nu_{p^e} = \nu_p$ for every edge $e = xy \in A^*$ covering (A, B), that is, there is an independent family $\mathscr{F}_e$ for which $p(\mathscr{F}_e) = \nu_p$ and e does not cover any member of $\mathscr{F}_e$.

For each $e \in A^*$ covering (A, B) let w_e be a 0–1 function on $\mathscr{A}^*$ so that w_e is 1 on the members of $\mathscr{F}_e$ and 0 otherwise. Let w_0 be 1 on (A, B) and 0 otherwise. Define $w := w_0 + \sum (w_e$: e covers $(A, B))$ and let $h := |A|\,|B|$. Then $p(w) = \nu_p h + p(A, B)$ and hence

$$p(w) \geqslant \nu_p h + 1 \tag{2.2a}$$

and

$$w \text{ is } h\text{-independent.} \tag{2.2b}$$

By Lemma 2.2 we may assume that w is cross-free. Let $\mathscr{F}' := \mathscr{S}_w$ be the support of w. Recall the partial order $P' := P(\mathscr{F}')$ defined in the Introduction. Because $\mathscr{F}'$ is cross-free, if two members of $\mathscr{F}'$ are non-comparable in P', then they are half-disjoint. By (2.2) there is no chain of P' with length greater than h. By a weighted version of the polar Dilworth theorem (*given a non-negative integer weighting w of a partially ordered set P', the maximum weight of a chain is equal to the minimum number of anti-chains covering all elements $v \in P'$ by $w(v)$ times*), it follows that there is an anti-chain of weight at least $\nu_p + 1$, contradicting the definition of ν_p. This contradiction proves the lemma. ∎

To prove that $\nu_p \geqslant \tau_p$, we use induction on $\sum p(X, Y)$. If this sum is zero, then $\nu_p = 0 = \tau_p$. Let now $p(A, B) > 0$ for a pair $(A, B) \in \mathscr{A}^*$. By Lemma 2.4 there is a reducing edge $e = st \in A^*$ with $s \in S$, $t \in T$. If z' is a covering of p^e, then increasing its value on edge e by one, we obtain a covering of p and hence $\tau_p \leqslant \tau_{p^e} + 1$. Since $\sum p^e(X, Y) < \sum p(X, Y)$, we can apply the inductive hypothesis, from which we obtain $\tau_p - 1 \leqslant \tau_{p^e} = \nu_{p^e} \leqslant \nu_p - 1$, and hence $\tau_p \leqslant \nu_p$, as required. ∎

The following corollary will be used in Section 6.

Theorem 2.5. *Given a crossing family $\mathscr{L}$ of pairs of non-empty subsets of V, the maximum cardinality of an independent sub-family of $\mathscr{L}$ is equal to the minimum number of directed edges covering all members of $\mathscr{L}$.*

Proof. Let p be a 0–1-valued function on the pairs of non-empty subsets of V defined to be 1 on a pair (X, Y) precisely when $(X, Y) \in \mathscr{L}$. Clearly, p is crossing bi-supermodular and, by choosing $S := T := V$, Theorem 2.3 implies the result. ∎

We can use the above min–max theorem to derive feasibility results concerning coverings. Let S, T, p be the same as in Theorem 2.3. Let $m_{\text{out}}: S \to \mathbf{Z}_+$ and $m_{\text{in}}: T \to \mathbf{Z}_+$ be two integer-valued functions for which $\gamma := m_{\text{out}}(S) = m_{\text{in}}(T)$.

THEOREM 2.6. *Let p be a non-negative, integer-valued crossing bi-supermodular function on $\mathscr{A} := \mathscr{A}(S, T)$. There exists an integer-valued covering $z: A(S, T) \to \mathbf{Z}_+$ of p for which*

$$\delta_z(v) = m_{\text{out}}(v) \qquad \textit{for every} \quad v \in S \tag{2.3a}$$

and

$$\rho_z(v) = m_{\text{in}}(v) \qquad \textit{for every} \quad v \in T \tag{2.3b}$$

if and only if

$$m_{\text{out}}(Z) \geqslant p(\mathscr{F}) \tag{2.4a}$$

holds for every $Z \subseteq S$ and for every independent family $\mathscr{F} \subseteq \mathscr{A}(Z, T)$, and

$$m_{\text{in}}(Z) \geqslant p(\mathscr{F}) \tag{2.4b}$$

holds for every $Z \subseteq T$ and for every independent family $\mathscr{F} \subseteq \mathscr{A}(S, Z)$.

In particular, if there is a covering satisfying (2.3a) *and there is a covering satisfying* (2.3b), *then there is one satisfying both* (2.3a) *and* (2.3b).

Proof. We prove only the necessity of (2.4a) since (2.4b) is analogous. Let z be a covering of p satisfying (2.3a), Z a subset of S and $\mathscr{F} \subseteq \mathscr{A}(Z, T)$ an independent family. Then, clearly, $\gamma = \sum (\delta_z(s): s \in S) = \sum (\delta_z(s): s \in S - Z) + \sum (\delta_z(s): s \in Z) \geqslant m_{\text{out}}(S - Z) + p(\mathscr{F}) = \gamma - m_{\text{out}}(Z) + p(\mathscr{F})$ and hence $m_{\text{out}}(Z) \geqslant p(\mathscr{F})$, as required for (2.4a).

Sufficiency. Define p' by modifying p as follows. Let $p'(v, T) := m_{\text{out}}(v)$ for every $v \in S$ and $p'(S, v) := m_{\text{in}}(v)$ for every $v \in T$, while $p'(X, Y) := p(X, Y)$ whenever $|X|, |Y| \geqslant 2$.

Since a family consisting of one pair is independent, (2.4) implies that $p(v, T) \leqslant m_{\text{out}}(v)$ for every $v \in S$. Analogously, $p(S, v) \leqslant m_{\text{in}}(v)$ for every $v \in T$. Therefore $p' \geqslant p$. It can easily be seen that p' is crossing bi-supermodular. Apply Theorem 2.3 to p' and let z be a minimum covering of p'. Clearly, z is a covering of p, as well, and $\gamma' := z(A^*) \geqslant \gamma$.

If $\gamma' = \gamma$, then we claim that z is a covering of p satisfying (2.3). Indeed, we have $\gamma' = \sum (\delta_z(v): v \in S) \geqslant \sum (m_{\text{out}}(v) \in S) = \gamma$ and hence $\delta_z(v) = m_{\text{out}}(v)$

follows for every $v \in S$, that is, (2.3a) holds. Equation (2.3b) is seen analogously.

Suppose now that $\gamma' > \gamma$. By Theorem 2.3 there exists an independent family $\mathscr{F}'$ of members of $\mathscr{A}(S, T)$ for which $p'(\mathscr{F}') = \gamma'$. Since $\mathscr{F}'$ is independent, it cannot contain pairs of both forms (s, T) and (S, t) where $s \in S$ and $t \in T$. So suppose that $\mathscr{F}'$ does not contain pairs of form, say, (s, T). Let Z' denote the subset of T consisting of those elements t for which (S, t) belongs to $\mathscr{F}'$ and let $Z := T - Z'$. (Z' may be empty.) Let $\mathscr{F}_1 := \{(S, t): t \in Z'\}$ and $\mathscr{F} := \mathscr{F}' - \mathscr{F}_1$. We have $p(\mathscr{F}) = p'(\mathscr{F}) = \gamma' - p'(\mathscr{F}_1) = \gamma' - m_{\text{in}}(Z') > \gamma - m_{\text{in}}(Z') = m_{\text{in}}(Z)$. Since $\mathscr{F}'$ is independent, so is $\mathscr{F} \subseteq \mathscr{A}(S, Z)$ and hence the inequality $p(\mathscr{F}) > m_{\text{in}}(Z)$ contradicts (2.4). ∎

One may be interested in the case when a prescription is given only on the out-degrees of elements of S while the in-degrees in T are not specified.

THEOREM 2.7. *Let $m_{\text{out}}: S \to \mathbf{Z}_+$ be a function on S. There is a covering z of p satisfying* (2.3a) *for every $v \in S$ if and only if* (2.4a) *holds.*

Proof. We have already seen the necessity of (2.4a). To see the sufficiency, let z be a covering of p minimizing $z(A^*)$ and let $m_{\text{in}}(v) := \rho_z(v)$ for every $v \in T$. Now $m_{\text{in}}(T) = z(A^*) = \tau_p = \nu_p \leqslant m_{\text{out}}(S)$ where the inequality follows from (2.4a). If $m_{\text{in}}(T) < m_{\text{out}}(S)$, then increase $m_{\text{in}}(t)$ by $m_{\text{out}}(S) - m_{\text{in}}(T)$ for an arbitrarily chosen element t of T. Let m_{in} denote the resulting function. Now (2.4a) holds by hypothesis and (2.4b) holds due to the construction of m_{in}. We can apply Theorem 2.6 from which the result follows. ∎

In some important special cases ν_p can be expressed in considerably simplified form. Let $A^* := A(V, V)$ and $\mathscr{A}^* := \mathscr{A}(V, V)$. Suppose that $p(X, Y)$ is a crossing bi-supermodular function which may be positive only if $\{X, Y\}$ is a bipartition of V. Such a function can be identified with a crossing supermodular function p'' defined on the subsets of V (namely, $p''(Y) := p(V - Y, Y)$) and we will formulate the theorems to concern p''. The following observation is the basis for the simplification.

LEMMA 2.8. *Let $\mathscr{F}' \subseteq \mathscr{A}^*$ be an independent family consisting of pairs (X, Y) for which $X = V - Y$. Then the members of $\mathscr{F}'$ are pairwise head-disjoint or pairwise tail-disjoint. Equivalently, if $\mathscr{F}$ is a cross-free family of subsets of V so that no member of $\mathscr{F}$ includes another member, then $\mathscr{F}$ consists of pairwise disjoint or pairwise co-disjoint sets.*

Proof. We prove the first form. Let Y be a minimal subset V which is a tail or a head of a member of $\mathscr{F}'$. By symmetry, we may assume that Y is a head, that is, $(X, Y) \in \mathscr{F}'$ where $X = V - Y$. Since $\mathscr{F}'$ is independent, any two heads are disjoint or co-disjoint. We claim that the members of $\mathscr{F}'$ are pairwise head-disjoint. Indeed, for any member (X', Y') of $\mathscr{F}'$ distinct from (X, Y) we must have $Y \cap Y' = \varnothing$ since otherwise Y and Y' are co-disjoint and then $X' \subset Y$, contradicting the minimal choice of Y. Moreover, for a third member (X'', Y'') of $\mathscr{F}'$, since both Y' and Y'' are disjoint from Y, they cannot be co-disjoint, therefore Y' and Y'' are disjoint. ∎

THEOREM 2.9. *Let p'' be a crossing supermodular function on the subsets of V with $p''(\varnothing) = p''(V) = 0$. Then* $\min(z(A^*)$: *$z \geqslant 0$ integer-valued, $\rho_z(Y) \geqslant p''(Y)$ for every $Y \subseteq V) = \max(p''(\mathscr{F})$: $\mathscr{F}$ is a family of pairwise disjoint or pairwise co-disjoint non-empty subsets of V).*

Proof. Let $p(X, Y) := p''(Y)$ if $X = V - Y$ and zero otherwise. Clearly, p is crossing bi-supermodular. Apply Theorem 2.3 with $S = T = V$ and let $\mathscr{F}'$ denote an independent sub-family of $\mathscr{A}^*$ on which the maximum is attained. We may assume that p is positive on each member of $\mathscr{F}'$. By the assumption on p'', each member (X, Y) of $\mathscr{F}'$ forms a bipartition of V. Since $\mathscr{F}'$ is independent, Lemma 2.8 implies that $\mathscr{F}'$ consists of pairwise tail-disjoint or pairwise head-disjoint pairs. Then the family $\mathscr{F} := \{Y \subset V: (X, Y) \in \mathscr{F}'\}$ consists of pairwise disjoint or pairwise co-disjoint sets and $p''(\mathscr{F}) = p(\mathscr{F}')$. Hence the result follows from Theorem 2.3. ∎

For later purposes we formulate Theorem 2.9 in an equivalent form (and we also change the groundset from V to T):

THEOREM 2.9.A. *Let p'' be a crossing supermodular function on the subsets of T with $p''(\varnothing) = p''(T) = 0$. There exists a digraph (T, F) with at most γ edges for which $\rho_F(Y) \geqslant p''(Y)$ for every $Y \subseteq T$ if and only if*

$$\sum p''(X_i) \leqslant \gamma \tag{2.5a}$$

and

$$\sum p''(V - X_i) \leqslant \gamma \tag{2.5b}$$

holds for every sub-partition $\{X_1, ..., X_t\}$ of T.

For the feasibility problem concerning crossing supermodular functions an even bigger simplification is possible.

THEOREM 2.10. *Let $m_{\text{in}}: V \to \mathbf{Z}_+$ and $m_{\text{out}}: V \to \mathbf{Z}_+$ be two functions for which $m_{\text{in}}(V) = m_{\text{out}}(V)$ and let p'' be a non-negative, integer-valued, crossing supermodular function on the subsets of V with $p''(V) = 0$. Then there exists a directed graph (V, F) for which*

$$\delta(v) = m_{\text{out}}(v) \tag{2.6a}$$

and

$$\rho(v) = m_{\text{in}}(v) \tag{2.6b}$$

for every $v \in V$ and for which

$$\rho(X) \geqslant p''(X) \tag{2.7}$$

for every $X \subseteq V$ if and only if for every $X \subseteq V$

$$m_{\text{out}}(V - X) \geqslant p''(X) \tag{2.8a}$$

and

$$m_{\text{in}}(X) \geqslant p''(X). \tag{2.8b}$$

In particular, if there is a digraph satisfying (2.7) *and* (2.6a) *and if there is a digraph satisfying* (2.7) *and* (2.6b), *then there is one satisfying* (2.7) *and both* (2.6a) *and* (2.6b).

Proof. The necessity of (2.8) is straightforward. To see the sufficiency, let $p(X, Y) := p''(Y)$ if $X = V - Y$ and $:= 0$ otherwise. Clearly, p is crossing bi-supermodular. Choose $S := T := V$.

We claim that (2.4a) holds. Let Z be a subset of V and $\mathscr{F}' \subseteq \mathscr{A}(Z, V)$ an independent family so that p is positive on each member of $\mathscr{F}'$. Then $X = V - Y$ for each member (X, Y) of $\mathscr{F}'$ and, by Lemma 2.8, the members of $\mathscr{F}'$ are pairwise tail-disjoint or pairwise head-disjoint. In the former case, by (2.8a), we have $m_{\text{out}}(Z) \geqslant \sum (m_{\text{out}}(X): (X, V - X) \in \mathscr{F}') \geqslant \sum (p''(V - X): (X, V - X) \in \mathscr{F}') = \sum (p(X, V - X): (X, V - X) \in \mathscr{F}') = p(\mathscr{F}')$, as required for (2.4a). If the members of $\mathscr{F}'$ are not pairwise tail-disjoint, then they are pairwise head-disjoint, and the tails then are pairwise co-disjoint. Hence Z must be V and, by (2.8b), we have $m_{\text{out}}(Z) = m_{\text{out}}(V) = m_{\text{in}}(V) \geqslant \sum (m_{\text{in}}(X): (V - X, X) \in \mathscr{F}') \geqslant \sum (p''(X): (V - X, X) \in \mathscr{F}') = \sum (p(V - X, X): (V - X, X) \in \mathscr{F}') = p(\mathscr{F}')$, that is, (2.4a) is satisfied in this case, as well.

Inequality (2.4b) can be proved analogously. From Theorem 2.6 it follows that there is an integer-valued covering z of p satisfying (2.3), which is equivalent to saying that there is a digraph satisfying (2.6) and (2.7). ∎

Remarks. It is useful to compare Theorems 2.6 and 2.10 and observe that condition (2.8) in Theorems 2.10 is much simpler than (2.4). Namely, in (2.4) an inequality is required for every subset Z and family $\mathscr{F}$, while (2.8) concerns only subsets X. We also note that in [6] a different, more direct proof of Theorem 2.10 is described. We will see in Section 5 that Theorem 2.10 may be considered as an abstract extension of a theorem of Mader on splitting off edges in directed graphs.

If only out-degree specifications are given for the digraph, the following holds.

THEOREM 2.11. *Let $m_{\text{out}}: V \to \mathbf{Z}_+$ be a function and let p'' be a non-negative, integer-valued, crossing supermodular function on the subsets of V with $p''(V) = 0$. Then there exists a directed graph (V, F) satisfying (2.6a) and (2.7) if and only if (2.8a) holds for every $X \subset V$ and*

$$m_{\text{out}}(V) \geq \sum_i (p(X_i)) \tag{2.9}$$

holds for every sub-partition $\{X_i\}$ of V.

Proof. The necessity of (2.8a) was already shown (and straightforward anyway). Suppose there is a digraph (V, F) satisfying the requirements and let $\{X_i\}$ be a sub-partition of V. Then $m_{\text{out}}(V) = |F| \geq \sum_i \rho_F(X_i) \geq \sum_i p(X_i)$, and (2.9) follows.

To see the sufficiency, suppose that (2.8a) and (2.9) hold. Let $\gamma := m_{\text{out}}(V)$. Now (2.5a) is the same as (2.9) and (2.5b) follows from (2.8a) since $\sum_i p''(V - X_i) \leq \sum_i m_{\text{out}}(X_i) \leq \gamma$. By Theorem 2.9.A there is a digraph (V, F') of γ edges satisfying (2.7). Define $m_{\text{in}}(v) := \rho_{F'}(v)$ for every $v \in V$. Now (2.8b) is satisfied and therefore Theorem 2.10 applies and implies the existence of a digraph satisfying (2.6a) and (2.7). ∎

In Section 5 (after Theorem 5.2) we show an example to demonstrate that (2.8a) without (2.9) is not sufficient in Theorem 2.11. This is related to the fact that, in the proof of Theorem 2.10, while deriving (2.4a) we made use of not only (2.8a) but (2.8b), as well (at least for $X = V$).

3. Degree Constraints and Node Costs

In the second part of the previous section we proved feasibility type theorems concerning coverings of p, that is, we were interested in coverings of p satisfying out-degree and in-degree specifications on the nodes of S and T respectively. Relying on these results, we study now the extension when the degrees of a covering are required to satisfy lower and upper bound constraints. The ground for the generalization is that the set of degree-vectors of coverings, as we will prove it, spans a "contra-polymatroid." This fact will enable us to handle not only degree-constrained coverings but the problem of minimum-cost coverings, as well, provided that the cost-function on A^* is induced by a cost-function on $S \cup T$.

Let V, S, T, A^*, $\mathscr{A}^*$, and p be the same as in Theorem 2.3 and recall the notation $c_w(e)$ given in the Introduction. Let q be an (integer-valued) contra-polymatroid function defined on the subsets of S. A *contra-polymatroid* in $\mathbf{R}^S_+$ is a polyhedron

$$C(q) := \{x\colon x(X) \geq q(X) \text{ for every } X \subseteq S\}. \tag{3.0}$$

It is well-known that a contra-polymatroid uniquely determines its defining contra-polymatroid function q (namely, $q(X) = \min(x(X)\colon x \in C(q))$. (A more general class of polyhedra, g-polymatroids, was studied by Frank and Tardos in [8]. For a relationship of contra-polymatroids and edge-connectivity augmentations see [5].)

Define a set-function p^v_{out} on the power set of S, as follows. For a subset Z of S, let $p^v_{\text{out}}(Z)$ denote $\max(p(\mathscr{F})\colon \mathscr{F}$ an independent sub-family of $\mathscr{A}(Z, T))$. The notation is justified by the following result.

THEOREM 3.1. *For every $Z \subseteq S$,*

$$p^v_{\text{out}}(Z) = \min\left(\sum (\delta_z(v)\colon v \in Z)\colon z \geq 0 \textit{ an integer-valued covering of } p\right). \tag{3.1}$$

Proof. For a covering z of p and for an independent sub-family $\mathscr{F}$ of $\mathscr{A}(Z, T)$ we have $\sum (\delta_z(v)\colon v \in Z) = z(Z, T) \geq \sum (z(X, Y)\colon (X, Y) \in \mathscr{F}) \geq \sum (p(X, Y)\colon (X, Y) \in \mathscr{F}) = p(\mathscr{F})$ from which the "$\leq$" direction follows in (3.1).

Conversely, since the restriction p' of p to $\mathscr{A}(Z, T)$ is crossing bi-supermodular, we may apply Theorem 2.3 and conclude that there is an integer-valued covering z' (defined on $A(Z, T)$) of p' for which $\sum (\delta_{z'}(v)\colon v \in Z) = z'(Z, T) = v_{p'} = p^v_{\text{out}}(Z)$. Define $z(xy) := z'(xy)$ if $x \in Z$, $y \in T$ and $z(xy) := M$ if $x \in S - Z$, $y \in T$ where M is a big enough number (say, M is the maximum of p). Clearly z is a covering of p for which $\sum (\delta_z(v)\colon v \in Z) = \sum (\delta_{z'}(v)\colon v \in Z) = p^v_{\text{out}}(Z)$, which proves (3.1). ∎

Perhaps the most important feature of function $p^v_{\text{out}}(Z)$ is expressed by the following result.

THEOREM 3.2. *p^v_{out} is a contra-polymatroid function.*

Proof. Clearly, p^v_{out} is non-negative, monotone increasing, and zero on the empty set. The main content of the theorem is that p^v_{out} is fully supermodular.

For a subset Z of S let $\mathscr{F}_Z$ denote an independent sub-family of $\mathscr{A}(Z, T)$ for which $p(\mathscr{F}_Z) = p^v_{\text{out}}(Z)$. Let $w_Z\colon \mathscr{A}(Z, T) \to \mathbf{Z}_+$ denote the characteristic vector of $\mathscr{F}_Z$, that is, $w_Z(X, Y) := 1$ if $(X, Y) \in \mathscr{F}_Z$ and $:= 0$ otherwise. Let

X and Y be two subsets of S and let $w := w_X + w_Y$. By definition, $p(w) = p(w_X) + p(w_Y)$. For any edge $e = st$ ($s \in S$, $t \in T$) (i) $c_w(e) \leqslant 2$ when $s \in X \cap Y$, (ii) $c_w(e) \leqslant 1$ when $s \in (X \cup Y) - (X \cap Y)$, and (iii) $c_w(e) = 0$ when $s \in S - (X \cup Y)$.

By Lemma 2.2 there is a cross-free $\bar{w}$ satisfying (i), (ii), and (iii) such that $p(\bar{w}) \geqslant p(w)$. (Note that the range of w is $\{0, 1, 2\}$). The cross-freeness of $\bar{w}$ means, by definition, that its support family $\mathscr{F}$ is cross-free.

Let $\mathscr{F}_1$ consist of the minimal elements of the partial order $P' := P(\mathscr{F} \cap \mathscr{A}(X \cap Y, T))$. Due to property (i), every pair (A, B) with $\bar{w}(A, B) = 2$ belongs $\mathscr{F}_1$. Let $\mathscr{F}_2 := \{(A, B) \colon \bar{w}(A, B) = 2\} \cap (\mathscr{F} - \mathscr{F}_1)$. By this definition we have

$$p(\mathscr{F}_1) + p(\mathscr{F}_2) = p(\bar{w}) \geqslant p^v_{\text{out}}(X) + p^v_{\text{out}}(Y). \tag{3.2}$$

The family $\mathscr{F}_1$ is independent since if it has two comparable elements, then the larger one is not minimal in P', contradicting the definition of $\mathscr{F}_1$. It follows that $p^v_{\text{out}}(X \cap Y) \geqslant p(\mathscr{F}_1)$.

We claim that $\mathscr{F}_2$ is independent, as well. Suppose indirectly that, for two members of $\mathscr{F}_2$, $(A, B) > (A', B')$. By property (i), $\bar{w}(A, B) = \bar{w}(A', B') = 1$ and hence both pairs belong to $\mathscr{F} - \mathscr{F}_1$. Since $A' \subseteq A$ and $B \subseteq B'$, (ii) implies that $A' \subseteq X \cap Y$. Therefore (A', B') is in $\mathscr{A}(X \cap Y, T)$ but not in $\mathscr{F}_1$. Hence there is a member (A'', B'') of $\mathscr{F}_1$ for which $(A'', B'') < (A', B')$. But then the existence of these three pairs contradicts (i).

It follows from (iii) that $\mathscr{F}_2$ is an independent sub-family of $\mathscr{A}(X \cup Y, T)$ and therefore $p^v_{\text{out}}(X \cup Y) \geqslant p(\mathscr{F}_2)$. By (3.2) we obtain $p^v_{\text{out}}(X \cap Y) + p^v_{\text{out}}(X \cup Y) \geqslant p(\mathscr{F}_1) + p(\mathscr{F}_2) \geqslant p^v_{\text{out}}(X) + p^v_{\text{out}}(Y)$, as required. ∎

Quite analogously, one can define a set-function p^v_{in} on the power set of T by $p^v_{\text{in}}(Z) := \max(p(\mathscr{F}) \colon \mathscr{F}$ an independent sub-family of $\mathscr{A}(S, Z))$ where $Z \subseteq T$. Clearly, each theorem concerning p^v_{out} may be formulated to concern p^v_{in}. Let C_S (respectively, C_T) denote the contra-polymatroid defined by p^v_{out} (resp., p^v_{in}).

Our next purpose is to show a relationship between coverings of p and contra-polymatroids.

THEOREM 3.3. (A) *An integer-valued vector $m_{\text{out}} \colon S \to \mathbf{Z}_+$ belongs to C_S if and only if there is an (integer-valued) covering z of p for which $m_{\text{out}}(s) = \delta_z(s)$ for every $s \in S$.*

(B) *For any integer-valued vectors $m_{\text{out}} \in C_S$ and $m_{\text{in}} \in C_T$ for which $m_{\text{out}}(S) = m_{\text{in}}(T)$ there is an (integer-valued) covering z of p for which $\delta_z(v) = m_{\text{out}}(v)$ for every $v \in S$ and $\rho_z(v) = m_{\text{in}}(v)$ for every $v \in T$.*

Proof. Part (A) is a re-formulation of Theorem 2.7. Part (B) is equivalent to Theorem 2.6. ∎

Let q_1 be a contra-polymatroid function on the subsets of a set V_1 and $C_1 := C(q_1)$ the contra-polymatroid associated with q_1. We are given two non-negative integer-valued functions $f_1 \colon V_1 \to \mathbf{Z}_+$ and $g_1 \colon V_1 \to \mathbf{Z}_+ \cup \{\infty\}$ for which $f_1 \leqslant g_1$. The following result occurs in a more general form in [8] and more concretely in [5, Proposition 6.9].

LEMMA 3.4. *C_1 has an integer-valued element m_1 for which $f_1 \leqslant m_1 \leqslant g_1$ and $m_1(V_1) = \gamma$ if and only if $\gamma \leqslant g_1(V_1)$ and $q_1(X) \leqslant \min(\gamma - f_1(V_1 - X), g_1(X))$ holds for every $X \subseteq V_1$.*

By combining these results, it is possible to handle degree-constrained and minimum-cost versions of the covering problem. First, let us be given two non-negative integer-valued functions $f_{\text{out}} \colon S \to \mathbf{Z}_+$, $g_{\text{out}} \colon S \to \mathbf{Z}_+ \cup \{\infty\}$ for which $f_{\text{out}} \leqslant g_{\text{out}}$ and two non-negative, integer-valued functions $f_{\text{in}} \colon T \to \mathbf{Z}_+$, $g_{\text{in}} \colon T \to \mathbf{Z}_+ \cup \{\infty\}$ for which $f_{\text{in}} \leqslant g_{\text{in}}$. Let γ be a positive integer.

THEOREM 3.5. (A) *There exists a non-negative, integer-valued covering z of p with $z(A^*) = \gamma$ for which*

$$f_{\text{out}}(v) \leqslant \delta_z(v) \leqslant g_{\text{out}}(v) \tag{3.3a}$$

for every $v \in S$ and

$$f_{\text{in}}(v) \leqslant \rho_z(v) \leqslant g_{\text{in}}(v) \tag{3.3b}$$

for every $v \in T$ if and only if

$$\gamma \leqslant \min(g_{\text{out}}(S), g_{\text{in}}(T)) \tag{3.4}$$

and

$$p^v_{\text{out}}(Z) \leqslant \min(\gamma - f_{\text{out}}(S - Z), g_{\text{out}}(Z)) \quad \textit{for every} \quad Z \subseteq S, \tag{3.5a}$$

$$p^v_{\text{in}}(Z) \leqslant \min(\gamma - f_{\text{in}}(T - Z), g_{\text{in}}(Z)) \quad \textit{for every} \quad Z \subseteq T. \tag{3.5b}$$

(B) *There exists a non-negative integer-valued covering z of p satisfying (3.3) if and only if*

$$p^v_{\text{out}}(Z) \leqslant \min(\alpha - f_{\text{out}}(S - Z), g_{\text{out}}(Z)) \quad \textit{for every} \quad Z \subseteq S, \tag{3.6a}$$

and

$$p^v_{\text{in}}(Z) \leqslant \min(\alpha - f_{\text{in}}(T - Z), g_{\text{in}}(Z)) \quad \textit{for every} \quad Z \subseteq T, \tag{3.6b}$$

where $\alpha := \min(g_{\text{out}}(S), g_{\text{in}}(T))$.

Proof. (A) Necessity. Let z be a covering of p with the desired properties. $\gamma = z(A^*) = \sum(\delta_z(v): v \in S) \leqslant \sum(g_{\mathrm{out}}(v): v \in S) = g_{\mathrm{out}}(S)$. Similarly, $\gamma \leqslant g_{\mathrm{in}}(T)$, and (3.4) follows. Furthermore, $p^v_{\mathrm{out}}(Z) \leqslant \sum(\delta_z(v): v \in Z) \leqslant \sum(g_{\mathrm{out}}(v): v \in Z) = g_{\mathrm{out}}(Z)$ and $p^v_{\mathrm{out}}(Z) \leqslant \sum(\delta_z(v): v \in Z) = \gamma - \sum(\delta_z(v): v \in S - Z) \leqslant \gamma - \sum(f_{\mathrm{out}}(v): v \in S - Z) = \gamma - f_{\mathrm{out}}(S - Z)$, from which (3.5a) follows. Inequality (3.5b) is analogous.

Sufficiency. Apply Lemma 3.4 to $C_1 := C_S$, $f_1 := f_{\mathrm{out}}$, $g_1 := g_{\mathrm{out}}$, $q_1 := p^v_{\mathrm{out}}$. By (3.4) and (3.5) the conditions in Lemma 3.4 are satisfied and hence there is an integer-valued element m_{out} of C_S for which $m_{\mathrm{out}}(S) = \gamma$. Analogously, we obtain that there is an integer-valued element m_{in} of C_T for which $m_{\mathrm{out}}(S) = \gamma$. By Theorem 3.3(B) the result follows.

(B) Necessity. Let z be a covering of p satisfying (3.3) and let $\gamma := z(A^*)$. Then (3.4) and (3.5) hold and hence $p_{\mathrm{out}}(Z) \leqslant g_{\mathrm{out}}(Z)$ and $p_{\mathrm{out}}(Z) \leqslant \gamma - f_{\mathrm{out}}(S - Z) \leqslant \alpha - f_{\mathrm{out}}(S - Z)$, from which (3.6a) follows. Inequality (3.6b) is analogous.

Sufficiency. Defining $\gamma := \alpha$ we see that (3.6) implies (3.4) and (3.5) and hence part (B) of the theorem follows from part (A). ∎

We mentioned earlier that there is no hope to obtain min–max results for the general minimum cost version of the covering problem since a special case, finding a minimum cost strongly connected augmentation of a digraph, is NP-complete. However, for a special class of cost functions such a characterization exists.

Let us be given two non-negative cost functions $c_S: S \to \mathbf{R}_+$ and $c_T: T \to \mathbf{R}_+$. They induce a cost function c on the edge set $A^* := A(S, T)$ by the rule $c(st) := c_S(s) + c_T(t)$ for $st \in A^*$. It is an easy exercise to check that an integer-valued node-induced cost function on A^* can always be induced by integer-valued cost functions on S and on T. Let p be a crossing bi-supermodular function on $\mathscr{A}^* := \mathscr{A}(S, T)$.

THEOREM 3.6. *For a node-induced function $c: A^* \to \mathbf{R}_+$, the linear program*

$$\min(cz: z \text{ a covering of } p) \tag{3.7a}$$

has an integer-valued optimum. If, in addition, c is integer-valued, the dual linear program

$$\max\Big(\sum(w(X, Y)\, p(X, Y): (X, Y) \in \mathscr{A}^*): w: \mathscr{A}^* \to \mathbf{R}_+,$$
$$\sum(w(X, Y): (X, Y) \in \mathscr{A}^*, s \in X, t \in Y) \leqslant c(st) \text{ for every } st \in A^*\Big) \tag{3.7b}$$

also has an integer-valued optimum.

Proof. Recall the definition of the contra-polymatroids C_S and C_T. Let c_S and c_T be the cost functions inducing c so that both c_S and c_T are integer-valued if c is integer-valued. It is known from polymatroid theory that the system (3.0) is totally dual integral. Therefore there is a (minimizing) integer vector $m_{\mathrm{out}} \in C_S$ and there is a dual variable $y_S \in 2^S \to \mathbf{R}_+$ (integer-valued, if c_S is integer-valued) so that

$$\sum(y_S(Z): s \in Z \subseteq S) \leqslant c_S(s) \quad \text{for every } s \in S \tag{3.8a}$$

and

$$c_S m_{\mathrm{out}} = \sum(y_S(Z)\, p^v_{\mathrm{out}}(Z): Z \subseteq S). \tag{3.8b}$$

Similarly, there is a (minimizing) integer vector $m_{\mathrm{in}} \in C_T$ and there is a dual variable $y_T \in 2^T \to \mathbf{R}_+$ (integer-valued, if c_T is integer-valued) so that

$$\sum(y_T(Z): t \in Z \subseteq T) \leqslant c_T(t) \quad \text{for every } t \in T \tag{3.9a}$$

and

$$c_T m_{\mathrm{in}} = \sum(y_T(Z)\, p^v_{\mathrm{in}}(Z): Z \subseteq T). \tag{3.9b}$$

It is well-known from polymatroid theory that if an element x of contra-polymatroid $C(q)$ is minimal in the sense that no x' with $x' \leqslant x$, $x' \neq x$ belongs to $C(q)$, then $x(S) = q(S)$. From this and from the non-negativity of c_S and c_T it follows that $m_{\mathrm{out}}(S) = p^v_{\mathrm{out}}(S) = v_p = p^v_{\mathrm{in}}(T) = m_{\mathrm{in}}(T)$.

By Theorem 3.3 there is a covering z_0 of p for which $\delta_{z_0}(v) = m_{\mathrm{out}}(v)$ for every $v \in S$ and $\rho_{z_0}(v) = m_{\mathrm{in}}(v)$ for every $v \in T$.

By definition, for each $Z \subseteq S$ there is an independent sub-family $\mathscr{S}_Z$ of $\mathscr{A}(Z, T)$ so that $p^v_{\mathrm{out}}(Z) = p(\mathscr{S}_Z)$. Similarly, for each $Z \subseteq T$ there is an independent sub-family $\mathscr{T}_Z$ of $\mathscr{A}(S, Z)$ so that $p^v_{\mathrm{in}}(Z) = p(\mathscr{T}_Z)$.

Define $w_0(A, B) := \sum(y_S(Z): (A, B) \in \mathscr{S}_Z, Z \subseteq S) + \sum(y_T(Z): (A, B) \in \mathscr{T}_Z, T \subseteq S)$. From (3.8) and (3.9) we obtain for every $s \in S$, $t \in T$ that $\sum(w_0(X, Y): (X, Y) \in \mathscr{A}^*, s \in X, t \in Y) = \sum(y_S(Z): Z \subseteq S, s \in Z) + \sum(y_Z(Z): Z \subseteq T, t \in Z) \leqslant c_S(s) + c_T(t) = c(st)$, that is, w_0 satisfies (3.7b).

Furthermore, $\sum(w_0(X, Y)\, p(X, Y): (X, Y) \in \mathscr{A}^*) = \sum(y_S(Z)\, p(\mathscr{S}_Z): Z \subseteq S) + \sum(y_T(Z)\, p(\mathscr{T}_Z): Z \subseteq T) = \sum(y_S(Z)\, p^v_{\mathrm{out}}(Z): Z \subseteq S) + \sum(y_T(Z)\, p^v_{\mathrm{in}}(Z): Z \subseteq T) = c_S m_{\mathrm{out}} + c_T m_{\mathrm{in}} = \sum(c_S(s)\, \delta_{z_0}(s): s \in S) + \sum(c_T(t)\, \rho_{z_0}(t): t \in T) = \sum(z_0(st)(c_S(s) + c_Z(t)): s \in S, t \in T) = \sum(z_0(st)\, c(st): st \in A^*) = cz_0$ showing that z_0 is a primal optimum and w_0 is a dual optimum. ∎

4. Augmenting Node-Connectivity of Digraphs

The main motivation behind the general framework we described in the preceding sections was to develop techniques to solve connectivity augmentation problems in directed graphs, in particular, the node-connectivity augmentation which was open so far. Here we exhibit these applications.

Let $D=(V, A)$ be a digraph with possible parallel edges. The *local node-connectivity* from x to y, denoted by $\kappa(x, y; D)$, is the maximum number of pairwise openly node-disjoint paths from x to y. We say that D is *k-node-connected* (or, in short, *k-connected*) if $\kappa(x, y; D) \geqslant k$ holds for every ordered pair of nodes (x, y) of D. When $k=1$, the term *strongly connected* is used. We will consider k-node-connected digraphs only for $k<|V|-1$ since for larger k their structure is uninteresting: for every possible pair $\{x, y\}$ of nodes at least $k-|V|+2$ parallel edges from x to y must belong to the digraph. Also, for node-connectivity augmentation problems we may assume that the starting digraph is simple.

The connectivity augmentation problem for digraphs consists of finding a minimum set of new edges whose addition to a given directed graph leaves a k-connected digraph. When $k=1$, the problem was solved by Eswaran and Tarjan [4]. For larger k only very little was known. Masuzawa *et al.* [20] solved the special case when the starting digraph D is an arborescence (that is, a directed tree in which very node is reachable from a root.) Jordán [16] described a (combinatorial) polymial time approximation algorithm to augment the node connectivity of a digraph from $k-1$ to k and proved that the augmentation of his algorithm uses at most $k-1$ more edges than the optimum.

One of the major open question of the area is to decide if the node-connectivity augmentation problem for undirected graphs belongs to co-NP or if it is NP-complete. The problem is polynomially solvable for $k=1$ (trivial), for $k=2$ [4], for $k=3$ [24, 13] and, providing that the starting graph is 3-connected, for $k=4$ [12]. For higher k, the NP-completeness status is not known even for the special case when we want to increase the node-connectivity only by one. For that problem Jordán [14, 15] developed an approximation algorithm that provides an augmenting set of edges whose cardinality is at most $k-2$ larger than the optimum. We only mention these developments to provide a more general picture: the present paper has little to say concerning node-connectivity augmentations in undirected graphs (see Corollary 4.8).

Assume that $D=(V, A)$ is a simple directed graph (that is, there are no loops and no parallel edges from x to y for each $x, y \in V$). We assume that the target connectivity $k<|V|-1$. For a pair (X, Y) of disjoint non-empty subsets of V, let $h(X, Y):=|V-(X \cup Y)|$. We call such a pair (X, Y) a

one-way pair if there is no edge from X to Y, that is, $\delta(X, Y)=0$. (It is easy to see that the family of one-way pairs is crossing.)

Claim 4.1. *A digraph $D^+=(V, A^+)$ is k-connected if and only if*

$$h(X, Y) \geqslant k \tag{4.1}$$

holds for every one-way pair (X, Y).

Proof. If D^+ is k connected, there are k openly disjoint paths from a node $x \in X$ to $y \in Y$. Since there is no edge from X to Y, each of the k paths must contain a node in $V-(X \cup Y)$ from which the necessity of (4.1) follows.

Conversely, assume that (4.1) holds. Suppose, indirectly, that there are no k openly disjoint paths from a node x to another node y. If there is no edge from x to y, then, by the directed node version of Menger's theorem, there is a subset $C \subseteq V-\{x, y\}$ of less than k nodes covering all paths from x to y. Now the set X of nodes reachable from x in D^+-C and $Y:=V-X-C$ violates (4.1). If there is an edge $e=xy$, then, by applying the same argument to D^+-e, we conclude that there is a pair (X, Y') of disjoint subsets of V such that $x \in X$, $y \in Y'$, $h(X, Y') \leqslant k-2$, and e is the only edge of D^+ from X to Y. It follows from (4.1) that the in-degree of any node is at least k. Hence we cannot have $Y'=\{y\}$. But then $Y:=Y'-y$ is non-empty and (X, Y) is a one-way pair violating (4.1), a contradiction. ∎

The *deficiency* of a pair (X, Y) of disjoint non-empty subsets is defined to be $p_{\mathrm{def}}(X, Y):=(k-h(X, Y))^+$ if (X, Y) is a one-way pair and $:=0$ otherwise. Clearly, $p_{\mathrm{def}}(X, Y)=(k-k\delta_A(X, Y)-h(X, Y))^+$ for every (X, Y). This latter form shows that p_{def} is crossing bi-supermodular because both $\delta_A(X, Y)$ (by Claim 2.1) and $h(X, Y)$ are crossing bi-supermodular.

The following statement easily follows from Claim 4.1:

Claim 4.2. *The addition of a set F of new edges to a digraph $D=(V, A)$ leaves a k-connected digraph $D^+:=(V, A+F)$ if and only if*

$$\delta_F(X, Y) \geqslant p_{\mathrm{def}}(X, Y) \tag{4.2}$$

holds for every one-way pair (X, Y).

The main result of this section provides an answer to the directed node-connectivity augmentation problem, the basic motivation of the whole paper. Recall that a family $\mathcal{F}$ of pairs of disjoint subsets of V was called independent if every two members (X, Y), (X', Y') of $\mathcal{F}$ were half-disjoint, that is, at least one of $X \cap X'$ and $Y \cap Y'$ is empty.

THEOREM 4.3. *A digraph $D=(V,A)$ can be made k-node-connected by adding at most γ new edges if and only if*

$$\sum (p_{\mathrm{def}}(X,Y): (X,Y)\in\mathscr{F}) \leqslant \gamma \tag{4.3}$$

holds for every choice of independent families $\mathscr{F}$ of one-way pairs.

Proof. The theorem immediately follows from Claim 4.2 and Theorem 2.3 when $S=T=V$ and $p:=p_{\mathrm{def}}$. ∎

In Section 7 we briefly indicate a polynomial-time algorithm, based on the ellipsoid method, to compute the optimal augmentation in Theorem 4.3. No polynomial-time combinatorial algorithm is known even if we want to increase the connectivity by 1. In [7] we developed such an algorithm when the starting digraph is strongly connected and the target connectivity is 2.

With the same approach, one can easily derive the following extension. We say that a digraph $D^+=(V,A^+)$ is *k-connected from S to T*, where $S,T\subseteq V$, if there are k openly disjoint paths from every element of S to every element of T. When $S=T=V$ we are back at k-connectivity. By Menger's theorem, D^+ is k-connected from S to T if and only if

$$\delta_{A^+}(X,Y)\geqslant k-h(X,Y) \tag{4.4}$$

holds for each pair (X,Y) of disjoint non-empty subsets of V for which

$$X\cap S\neq\varnothing \quad\text{and}\quad Y\cap T\neq\varnothing. \tag{4.5}$$

We say that two pairs (X,Y) and (X',Y') of pairs of non-empty disjoint subsets of V are *(S,T)-independent* if at least one of $X\cap X'\cap S$ and $Y\cap Y'\cap T$ is empty. A family $\mathscr{F}$ is *(S,T)-independent* if it consists of pairwise (S,T)-independent pairs satisfying (4.5).

For a pair (X,Y) of disjoint non-empty subsets of V satisfying (4.5), define $p_{\mathrm{D}}(X,Y):=(k-\delta_A(X,Y)-h(X,Y))^+$. By Claim 2.1, p_{D} is crossing bi-supermodular.

THEOREM 4.4. *A digraph $D=(V,A)$ can be made k-node-connected from S to T by adding at most γ new edges with tails in S and heads in T if and only if*

$$\sum (p_{\mathrm{D}}(X,Y): (X,Y)\in\mathscr{F}) \leqslant \gamma \tag{4.6}$$

holds for every choice of (S,T)-independent families $\mathscr{F}$.

Proof. Let F be a set of new edges satisfying the requirements of the theorem. By (4.4) $\delta_A(X,Y)+\delta_F(X,Y)\geqslant k-h(X,Y)$ and hence $\delta_F(X,Y)\geqslant p_{\mathrm{D}}(X,Y)$ must hold for every pair (X,Y) satisfying (4.5). Since an edge

from S to T can cover at most one member of an (S,T)-independent family $\mathscr{F}$, we have $\gamma\geqslant|F|\geqslant\sum(\delta_F(X,Y):(X,Y)\in\mathscr{F})\geqslant\sum(p_{\mathrm{D}}(X,Y):(X,Y)\in\mathscr{F})$ from which the necessity of (4.6) follows.

To see the sufficiency, let p' denote the projection of p_{D} onto $\mathscr{A}(S,T)$. It follows from (4.6) that $p'(\mathscr{F}')\leqslant\gamma$ holds for every independent sub-family $\mathscr{F}'$ of $\mathscr{A}(S,T)$. By applying Theorem 2.3 to p' we obtain that there is an integer-valued covering $z: A(S,T)\to\mathbf{Z}^+$ of p' for which $z(A(S,T))\leqslant\gamma$. Define F by taking $z(e)$ parallel copies of each edge $e\in A(S,T)$. Then F satisfies the requirements of the theorem. ∎

Note that other results of Sections 2 and 3 can also be applied to the special bi-supermodular functions p_{def} or p_{D}. Therefore the degree-constrained and minimum-node-cost connectivity augmentation problems are also tractable. The detailed formulations of these results are left to the reader; here we mention only one consequence. Let $\kappa(D)$ denote the connectivity of a digraph D. Recall the notation $h(X,Y)=|V-(X\cup Y)|$.

THEOREM 4.5. *For a given a digraph $D=(V,A)$ let γ denote the minimum cardinality of a set F of new edges whose addition to $D=(V,A)$ leaves a k-connected digraph ($k<|V|-1$). Then F may be chosen in such a way that*

$$\rho_F(v)\leqslant k-\kappa(D) \quad\text{and}\quad \delta_F(v)\leqslant k-\kappa(D) \tag{4.7}$$

for every node v of D.

Proof. In the proof let $p:=p_{\mathrm{def}}$. Let $Z\subseteq V$ be a non-empty subset and $\mathscr{F}\subseteq\mathscr{A}(Z,V)$ an independent family of one-way pairs for which $p(\mathscr{F})$ is as large as possible. That is, $p^v_{\mathrm{out}}(Z)=p(\mathscr{F})$.

LEMMA 4.6. *$\mathscr{F}$ has at most $|Z|$ members.*

Proof. Let $n:=|V|$ and $t:=|\mathscr{F}|$. First observe that $\sum(k-\delta_D(v): v\in Z)\leqslant\sum((k-\delta_D(v))^+: v\in Z)\leqslant p^v_{\mathrm{out}}(Z)=p(\mathscr{F})$. Second, let $L(Z)$ denote the set of edges of D whose tail is in Z. Since $\mathscr{F}$ consists of independent one-way pairs belonging to $\mathscr{A}(Z,V)$, the edge-sets $A(X,Y)$ ($(X,Y)\in\mathscr{F}$) are disjoint from each other and form $L(Z)$, as well. Since $|A(X,Y)|=|X|\,|Y|\geqslant n-1-h(X,Y)$ holds for any one-way pair (X,Y) and the number of non-loops in $A(Z,V)$ is $|Z|\,(n-1)$, we have $|Z|\,(n-1)\geqslant|L(Z)|+\sum(|X|\,|Y|: (X,Y)\in\mathscr{F})$. Combining these, we have $|Z|\,(n-1-k)+p(\mathscr{F})\geqslant|Z|\,(n-1-k)+\sum(k-\delta_D(v): v\in Z)=|Z|\,(n-1-k)+|Z|\,k-\sum(\delta_D(v): v\in Z)=|Z|\,(n-1)-\sum(\delta_D(v): v\in Z)=|Z|\,(n-1)-|L(Z)|\geqslant\sum(|X|\,|Y|: (X,Y)\in\mathscr{F})\geqslant\sum(n-1-h(X,Y): (X,Y)\in\mathscr{F})=t(n-1)+\sum(k-h(X,Y): (X,Y)\in\mathscr{F})-tk=t(n-1-k)+p(\mathscr{F})$.

Since $n-1-k>0$, we conclude that $t \leqslant |Z|$, as required. ∎

Because $p(X, Y) \leqslant k-\kappa(D)$, from the lemma we obtain that

$$p^v_{out}(Z) \leqslant |Z|\,(k-\kappa(D)). \tag{4.8a}$$

Since the role of p^v_{out} and p^v_{in} is symmetric, we also have

$$p^v_{in}(Z) \leqslant |Z|\,(k-\kappa(D)). \tag{4.8b}$$

Apply Theorem 3.5(A) with the choice $S:=T:=V, f_{in}:=f_{out}\equiv 0, g_{in}:=g_{out}\equiv k-\kappa(D)$. It follows from (4.8) that both (3.4) and (3.5) hold. The covering z provided by Theorem 3.5 defines a set F of γ new edges satisfying the requirements. ∎

When we want to increase the connectivity by one, Theorem 4.5 specializes to:

COROLLARY 4.7. *A κ-connected digraph $D=(V, A)$ can be optimally augmented by adding pairwise disjoint circuits and paths to obtain a $(\kappa+1)$-connected digraph. In particular, if D is κ-connected and the in-degree and out-degree of each node is κ, then there are disjoint circuits covering V whose addition to D leaves a $(\kappa+1)$-connected digraph.*

Note that, unlike edge-connectivity, node-connectivity cannot always be increased by adding a Hamiltonian circuit: choose D to be a digraph arising from $K_{3,3}$ by replacing each edge by two oppositely oriented edges. D is 3-connected, and the only way to increase its connectivity by disjoint circuits is to add two directed triangles.

From the last corollary we derive the only result of this paper concerning undirected graphs.

COROLLARY 4.8. *A κ-connected undirected graph G can be made $(\kappa+1)$-connected by adding disjoint paths.*

Proof. Let $k:=\kappa+1$. Create a digraph D from G by replacing each edge by two oppositely oriented edges. Then D is κ connected and by Corollary 4.7 there is a collection of disjoint directed paths and circuits whose addition to D leaves a k-connected digraph. Using the same paths and circuits in the undirected sense we get that G can be made k connected by adding disjoint paths and circuits. Let F denote the set of new edges. We may assume that F is minimal in the sense that the addition of any proper subset of F no longer increases the connectivity of G. That is, $G+F$ is a k-connected graph but leaving out any element of F destroys k-connectivity. Mader [18] proved that *if C is a circuit of a k-connected graph G^+ so that each node of C has degree larger than k, then at least one edge of C can be deleted from G^+ without destroying k-connectivity.* We claim that F forms a forest. Indeed, if some edges of F formed a circuit C, then each node of C has degree at least $k-1$ in G and, hence, at least $k+1$ in $G+F$. By Mader's theorem this is in contradiction with the minimal property of F.

It follows that F forms a forest of disjoint paths, as required. ∎

Note that the augmentation ensured by the last corollary (unlike Corollary 4.7 for directed graphs) was not proved to be a minimum cardinality augmentation. It is tempting to conjecture that there is such an augmentation for undirected graphs, as well; that is, a $(k-1)$-connected undirected graph can be made k-connected by adding a minimum number of new edges so that the set of new edges forms a forest of disjoint paths. (This can be proved for $k \leqslant 3$.) It would also be interesting to find a more direct proof of Corollary 4.8.

5. AUGMENTING EDGE-CONNECTIVITY OF DIGRAPHS

In this section we show how some extensions of known results concerning edge-connectivity augmentation of digraphs may be deduced from the general framework. For the necessary definitions see the previous section.

For a digraph $D=(V, A)$ the *local edge-connectivity* from x to y, denoted by $\lambda(x, y; D)$, is the maximum number of pairwise edge-disjoint paths from x to y. We say that D is *k-edge-connected* if $\lambda(x, y; D) \geqslant k$ holds for every ordered pair of nodes (x, y) of D. Unlike node-connectivity augmentations, here k may be any big integer.

By the directed edge-version of Menger's theorem $\lambda(x, y) \geqslant k$ if and only if $\delta(X) \geqslant k$ holds for every subset X with $x \in X \subseteq V-y$. It follows that a digraph is k-edge-connected if and only if $\delta(X) \geqslant k$ holds for every non-empty proper subset X of V.

The feasibility form of the edge-connectivity augmentation problem consists of finding a set of new edges that satisfies in-degree and out-degree prescriptions at the nodes and whose addition to D leaves a k-edge-connected digraph. Mader [19] solved this problem by showing that a natural necessary condition is sufficient as well. (Actually, Mader's original theorem is formulated in terms of splitting off edges but his result can easily be reformulated to concern the feasibility problem; see below.)

The minimization form of the edge-connectivity augmentation problem for digraphs consists of determining the minimum number of new directed edges whose addition to D leaves a k-edge-connected graph. This problem was solved in [5] by invoking Mader's theorem and the theory of polymatroids. The solution includes a min–max theorem as well as a (combinatorial) strongly polynomial time algorithm to find the extrema in question.

As far as undirected edge-connectivity augmentation is concerned the minimization problem was solved first by Watanabe and Nakamura [23]. In [5] a generalization was solved when the desired local edge-connectivities are arbitrarily prescribed. This was even further generalized by Bang-Jensen *et al.* [1] where a generalization concerning mixed graphs was described. For a survey, see [6].

Let $H=(V, A')$ be a digraph. *Splitting off* a pair of edges $e=us$, $f=st$ of H means that we replace e and f by a new edge ut. The resulting digraph will be denoted by H^{ef}. The following important result concerning splittings is due to Mader [19]:

THEOREM 5.1. *Let $H=(V+s, A')$ be a directed graph for which $\lambda(x, y; H)\geq k$ for every $x, y\in V$ and $\rho_H(s)=\delta_H(s)$. Then for every edge $f=st$ there is an edge $e=us$ so that $\lambda(x, y, H^{ef})\geq k$ for every $x, y\in V$.*

By repeated applications one obtains

THEOREM 5.1.A [Mader, 19]. *Let $H=(V+s, A')$ be a directed graph for which $\lambda(x, y; H)\geq k$ for every $x, y\in V$ and $\rho_H(s)=\delta_H(s)$. Then the edges entering and leaving s can be partitioned into $\rho(s)$ pairs so that splitting off all these pairs leaves a k-edge-connected digraph on node-set V.*

Let $D=(V, A)$ be a digraph and $m_{\text{in}}: V\to \mathbf{Z}^+$ and $m_{\text{out}}: V\to \mathbf{Z}_+$ two functions for which $\gamma := m_{\text{in}}(V)=m_{\text{out}}(V)$. The feasibility theorem for edge-connectivity augmentation is as follows.

THEOREM 5.2. *A directed graph $D=(V, A)$ can be made k-edge-connected by adding a set F of new edges satisfying*

$$\rho_F(v)=m_{\text{in}}(v) \quad \text{and} \quad \delta_F(v)=m_{\text{out}}(v) \tag{5.1}$$

for every node $v\in V$ if and only if both

$$\delta_D(X)+m_{\text{out}}(X)\geq k \tag{5.2a}$$

and

$$\rho_D(X)+m_{\text{in}}(x)\geq k \tag{5.2b}$$

hold for every $\emptyset\neq X\subset V$.

Though Theorems 5.1.A and 5.2 sound different, they easily imply each other.

Proof of Theorem 5.2 from Theorem 5.1.A. To see that Theorem 5.2 follows from Theorem 5.1.A, extend D by a new node s and, for each $v\in V$, adjoin $m_{\text{in}}(v)$ (respectively, $m_{\text{out}}(v)$) parallel edges from s to v (from v to s). By (5.2) the resulting digraph $H=(V+s, A')$ satisfies the hypotheses of Theorem 5.1.A and hence we can split off γ pairs of edges to obtain a k-edge-connected digraph on node-set V. The resulting set of γ new edges (connecting original nodes) satisfies the requirement. ∎

Proof of Theorem 5.1.A *from Theorem* 5.2. Given $H=(V+s, A')$, for every node $v\in V$ define $m_{\text{in}}(v)$ (respectively, $m_{\text{out}}(v)$) to be the number of parallel edges of H from s to v (respectively, from v to s). Let $D:=(V, A)$ denote the digraph arising from H by deleting s.

It follows from the hypothesis of Theorem 5.1.A that $\rho_H(X)\geq k$ and $\delta_H(X)\geq k$ for every $\emptyset\neq X\subset V$. This, in turn, implies (5.2) because $\rho_D(X)+m_{\text{in}}(X)=\rho_H(X)$ and $\delta_D(X)+m_{\text{out}}(X)=\delta_H(X)$. Furthermore, $\rho_H(s)=\delta_H(s)$ is equivalent to $m_{\text{in}}(V)=m_{\text{out}}(V)$ and thus we can apply Theorem 5.2. It provides a set F of edges satisfying (5.1) whose addition to D leaves a k-edge-connected digraph. For each edge $f=uv\in F$ assign a pair of edges of H consisting of an edge from s to v and an edge from u to s. By (5.1) this can be done in such a way that the pairs assigned to distinct members of F are pairwise disjoint. By splitting off these pairs we obtain $(V, A+F)$, a k-edge-connected digraph, as required for Theorem 5.1.A. ∎

Proof of Theorem 5.2. Define $p''(X)=\max(0, k-\rho_D(X))$ if $X\neq V, \emptyset$ and $p''(V)=p''(\emptyset)=0$. This function is crossing-supermodular so Theorem 2.10 applies and it just specializes to Theorem 5.2. ∎

In a similar way we obtain from Theorem 2.11 the following:

THEOREM 5.2.A. *Given a digraph $D=(V, A)$ and a function $m_{\text{out}}: V\to \mathbf{Z}_+$, D can be made k-edge-connected by adding a set F of new edges for which $\delta_F(v)=m_{\text{out}}(v)$ for every node $v\in V$ if and only if both* (5.2a) *holds for every $\emptyset\neq X\subset V$ and*

$$m_{\text{out}}(V)\geq \sum_i (k-\rho_D(X_i)) \tag{$*$}$$

holds for every sub-partition $\{X_i\}$ of V.

Note that in Theorem 5.2.A condition (5.2a) without $(*)$ is not sufficient, as is shown by a digraph D on three nodes $\{a, b, c\}$ with edge set $\{ba, ca\}$ if $k=1$ and $m_{\text{out}}(a):=1$, $m_{\text{out}}(b):=m_{\text{out}}(c):=0$.

The following theorem, concerning the minimization form of the edge-connectivity augmentation problem, was proved in [5].

THEOREM 5.3. *Given a directed graph $D=(V,A)$, the minimum number of edges whose addition to D leaves a k-edge-connected digraph is equal to the maximum of γ_i and γ_o where*

$$\gamma_i=\max\left(\sum_j (k-\rho(X_j))\right) \quad \text{and} \quad \gamma_o=\max\left(\sum_j (k-\delta(X_j))\right)$$

where both maxima are taken over all sub-partitions $\{X_1, ..., X_t\}$ of V.

Proof. If we define p'' as before; then Theorem 2.9 specializes to Theorem 5.3. ∎

Remark. Note that in [5] the minimization problem was reduced to the feasibility problem, that is, Theorem 5.3 was proved with the help of Theorem 5.2, while in the present paper we followed an opposite approach: Theorem 2.10 (a generalization of the feasibility theorem) was derived (via Theorem 2.6) from Theorem 2.3 (the minimization theorem).

In order to generalize these results, suppose that T is a subset of nodes of a digraph $D=(V,A)$. We say that D is *k-edge-connected in T* if the local edge-connectivity $\lambda(x,y;D)$ is at least k for any two elements x, y of T. We may use Theorem 2.10 for proving a generalization of Mader's Theorem 5.1.A.

THEOREM 5.4. *Let $H=(V+s,A')$ be a directed graph and $T\subseteq V$ a subset of nodes. Suppose that H is k-edge-connected in T, $\rho_H(s)=\delta_H(s)$, and each out- and in-neighbour of s is in T. Then the edges entering and leaving s can be partitioned into $\rho(s)$ pairs so that splitting off all these pairs leaves a digraph which is k-edge-connected in T.*

The corresponding minimization problem consists of adding a minimum number of new edges to D so that the resulting digraph is k-edge-connected in T. This problem was shown in [5] to be NP-complete even for $k=1$. However, if we make a restriction on the set of possible new edges to have both end-nodes in T, then the problem is tractable. For a family $\mathscr{F}$ of subsets of V let $\mathscr{F}\,|\,T:=\{X\cap T: X\cap T\neq\emptyset, X\in\mathscr{F}\}$.

THEOREM 5.5. *Given a digraph D and a subset T of nodes, it is possible to make D k-edge-connected in T by adding at most γ new edges connecting elements of T if and only if*

$$\sum_i (k-\rho_D(X_i))\leqslant\gamma \quad \text{and} \quad \sum_i (k-\delta_D(X_i))\leqslant\gamma \tag{5.3}$$

holds for every family $\mathscr{F}=\{X_1, ..., X_t\}$ of subsets V for which $\emptyset\subset X_i\cap T\subset T$ and $\mathscr{F}\,|\,T$ is a sub-partition of T.

Proof. For every subset X of T define $p''(X):=\max((k-\rho(X\cup Z))^+: Z\subseteq V-T))$. This p'' satisfies the hypothesis of Theorem 2.9.A. Inequality (2.5) transforms to (5.3) and hence Theorem 2.9.A implies the result. ∎

More can be said if D is *di-Eulerian outside T*, that is, $\rho(v)=\delta(v)$ for every $v\in V-T$.

COROLLARY 5.6. *Let T be a subset of nodes of a digraph D so that D is di-Eulerian outside T. It is possible to make D k-edge-connected in T by adding at most γ new edges connecting elements of T if and only if (5.3) holds for every sub-partition $\mathscr{F}=\{X_1, ..., X_t\}$ of V for which $\emptyset\subset X_i\cap T\subset T$. Moreover, if D can be made k-edge connected in T by adding γ new edges, then these γ edges may be chosen to have both end-nodes in T.*

Proof. The condition is clearly necessary; we prove its sufficiency. If the condition of Theorem 5.5 is satisfied, we are done. Suppose indirectly that this is not the case, that is, there is a family $\mathscr{F}=\{X_1, ..., X_t\}$ of subsets of V for which $\mathscr{F}\,|\,T$ is a sub-partition of T and $\mathscr{F}$ violates (5.3). We may assume that $\sum|X_i|$ is minimum. Since a sub-partition of V satisfies (5.3), there are two members X, Y of $\mathscr{F}$ whose intersection is non-empty. By the hypothesis every node in $X\cap Y$ is di-Eulerian, therefore $\rho(X)+\rho(Y)\geqslant \rho(X-Y)+\rho(Y-X)$. (This inequality is a consequence of the following identity which is valid for any digraph: $\rho(X)+\rho(Y)=\rho(X-Y)+\rho(Y-X)+\bar{d}(X,Y)+(\rho(X\cap Y)-\delta(X\cap Y))$ where $\bar{d}(X,Y)$ denotes the number of edges with one end-node in $X\cap Y$ and the other end-node in $V-(X\cup Y)$. This may be proved by observing that the contribution of every edge to the two sides of the identity is the same.)

Replacing X and Y by $X-Y$ and $Y-X$ we obtain a family $\mathscr{F}'$ which also violates (5.3), contradicting the minimal choice of $\mathscr{F}$.

The second part of the corollary follows from the first part by observing that the condition is necessary even if new edges are allowed to have end-nodes not only in T. ∎

Theorem 5.5 can even further be generalized. Let $D=(V,A)$ be a digraph with two specified non-empty subsets S, T of nodes (which may or may not be disjoint). We say that D is *k-edge-connected from S to T* if there are k edge-disjoint paths from every node of S to every node of T. (When $S=T$ we are back at k-edge-connectivity in T.) We say that a family of subsets of nodes is *(S,T)-independent* if it contains at most one $t\bar{s}$-set for every pair $s\in S$, $t\in T$.

THEOREM 5.7. *Given a digraph $D=(V, A)$ and two non-empty subsets S, T of nodes, D can be made k-edge-connected from S to T by adding at most γ new edges with tails in S and heads in T if and only if*

$$\sum_j (k-\rho_D(Y_j)) \leqslant \gamma \tag{5.4}$$

holds for every choice of (S, T)-independent family of subsets $Y_j \subset V$ where $T \cap Y_j \neq \emptyset$, $S - Y_j \neq \emptyset$ for each Y_j.

Proof. By Menger's theorem a digraph D^+ is k-edge-connected from S to T if and only if $\rho_{D^+}(Y) \geqslant k$ for every set $Y \subset V$ for which

$$Y \cap T \neq \emptyset \quad \text{and} \quad (V-Y) \cap S \neq \emptyset. \tag{$*$}$$

Therefore the addition of a set F of new edges to D leaves a digraph which is k-edge-connected from S to T if and only if

$$\rho_F(Y) \geqslant k-\rho_D(Y) \tag{5.5}$$

for every subset Y satisfying $(*)$. Since no edge from S to T can enter two (S, T)-independent sets, the number of new edges is at least $\sum_j (k-\rho_D(Y_j))$ and the necessity of (5.4) follows.

Sufficiency. Define a function $p: \mathscr{A}(V, V) \rightarrow \mathbf{Z}_+$ by $p(X, Y) := (k-\rho_D(Y))^+$ if $X=V-Y$, $\emptyset \neq Y \subset V$, and $p(X, Y) := 0$ otherwise. It follows from the submodularity of ρ_D that such a function is crossing bi-supermodular. Let p' denote the projection of p onto $\mathscr{A}(S, T)$ (for a definition, see Section 2). Then p' is also crossing bi-supermodular.

We claim that $\nu_{p'} \leqslant \gamma$. Suppose, indirectly, that there is an independent sub-family $\mathscr{F}'$ of $\mathscr{A}(S, T)$ for which $p'(\mathscr{F}') > \gamma$. With each member (X, Y) of $\mathscr{F}'$ with $p'(X, Y) > 0$ we associate a subset $\bar{Y}$ of V, as follows. By the definition of p' there exists a pair $(\bar{X}, \bar{Y}) \in \mathscr{A}(V, V)$ so that $p'(X, Y) = p(\bar{X}, \bar{Y})$, $\bar{X} \cap S = X$, and $\bar{Y} \cap T = Y$. Let this $\bar{Y}$ be associated with (X, Y). From $p(\bar{X}, \bar{Y}) > 0$ we have that $\bar{X} = V - \bar{Y}$, $\emptyset \neq Y \subset V$, and $p'(X, Y) = p(\bar{X}, \bar{Y}) = k - \rho_D(\bar{Y})$. Let $\mathscr{F}$ consist of those subsets of V which are associated with the element of $\mathscr{F}'$. The independence of $\mathscr{F}'$ is equivalent of the (S, T)-independence of $\mathscr{F}$. Since $\sum (k-\rho_D(\bar{Y}): \bar{Y} \in \mathscr{F}) = p(\mathscr{F}') > \gamma$, the family $\mathscr{F}$ violates (5.4). Therefore $\nu_{p'} \leqslant \gamma$.

From Theorem 2.3 we obtain that there is a covering $z: A^* \rightarrow \mathbf{Z}_+$ of p' (where $A^* := A(S, T)$) for which $z(A^*) \leqslant \gamma$. Define F to consist of the union of $z(s, t)$ parallel edges from s to t for each possible pair $s \in S$, $t \in T$. Then F has $z(A^*) \leqslant \gamma$ elements. We claim that F satisfies (5.5). Indeed, if there were a set $\emptyset \neq Y \subset V$ for which $\rho_F(Y) < k - \rho_D(Y)$, then $z(X \cap S, Y \cap T) = \rho_F(Y) < k - \rho_D(Y) \leqslant p'(X \cap S, Y \cap T)$ where $X := V - Y$, that is, z would not be a cover of p'. ∎

To conclude this section let us point out an important difference between the last theorem and the previous ones. In all theorems but the last one in this section sub-partitions play the main role in the characterizations in question. This enables one to find alternative proofs of these theorems using techniques developed in [5] and not using Theorem 2.3. Such an approach has the advantage that it gives rise to purely combinatorial polynomial time algorithms. In Theorem 5.7, in turn, the characterization is not sub-partition type and we do not know any constructive proof of Theorem 5.7.

(To see that this more complicated characterization is really required, consider a digraph D on four nodes $\{s_1, s_2, t_1, t_2\}$ with no edges, choose $S := \{s_1, s_2\}$ and $T := \{t_1, t_2\}$ and let $k=1$. Here the optimum augmentation F consists of the four edges from S to T and the only (S, T)-independent family of 4 members is $\{\{s_i, t_j\}: i, j \in \{1, 2\}\}$.)

6. GENERALIZING GYŐRI'S THEOREM

In 1984 Győri proved a deep min–max theorem concerning intervals of a straight line. For our purpose it is more convenient to use a terminology slightly different from Győri's and work with a system of sub-paths of a path. To be more specific, let $P = (v_0, e_1, v_1, e_2, v_2, ..., e_n, v_n)$ be a directed path or circuit where the nodes v_i's of P are distinct, except that $v_0 = v_n$ in the case P is a circuit, and each directed edge e_i of P has tail v_{i-1} and head v_i. We denote the node set of P by V. Let $\mathscr{F} := \{F_1, ..., F_k\}$ be a system of sub-paths of P. In what follows, a path will mean the set of its edges.

We say that a system $\mathscr{B}$ of subpaths of P *generates* $\mathscr{F}$ or that $\mathscr{B}$ is a *generator* of $\mathscr{F}$ if each member of $\mathscr{F}$ is the union of some members of $\mathscr{B}$. For example, $\mathscr{F}$ is a generator of itself, or the system $\{e_1, ..., e_n\}$ of one-element paths is also a generator of $\mathscr{F}$. Let $\gamma(\mathscr{F})$ denote the minimum cardinality of a generator of $\mathscr{F}$.

We call a pair (F, e) consisting of a path F and an element e of F a *represented path* and denote by $\mathscr{F}_r$ the system of all represented paths (F, e) with $F \in \mathscr{F}$.

Let $\mathscr{I} := \{I_1, ..., I_t\}$ be a family of subpaths of P and $\mathscr{R} := \{f_1, f_2, ..., f_t\}$ a system of distinct representatives of $\mathscr{I}$, that is, f_i's are distinct edges of P so that $f_i \in I_i$ for $i = 1, ..., t$. We call $\mathscr{R}$ a *strong system of representatives* if $I_i \cap I_j$ does not include a sub-path S of P with end edges f_i and f_j for i, j, $1 \leqslant i < j \leqslant t$. In this case we say that a family $\{(I_1, f_1), (I_2, f_2), ..., (I_t, f_t)\}$ of represented paths is *independent*. (When P is a path, then there is a unique subpath S with specified end edges. When P is a circuit there are two such subpaths.) $\mathscr{I} := \{I_1, ..., I_t\}$ is called *strongly representable* if it has a strong system of representatives.

It is not difficult to see, as was pointed out by Győri, that if P is a path, then $\mathscr{I}$ is strongly representable if and only if there is an ordering of the elements of $\mathscr{I}$ so that no member I of $\mathscr{I}$ is a subset of the union of the members of $\mathscr{I}$ preceding I in the given order. However, we will not use this second property since the equivalence is no longer true if P is a circuit, while Győri's theorem will turn out to hold in this case as well.

Let $\sigma(\mathscr{F})$ denote the maximum cardinality of a strongly representable sub-family of $\mathscr{F}$. It is rather straightforward to see that for any family $\mathscr{F}$ of subpaths of a path, one has $\sigma(\mathscr{F}) \leqslant \gamma(\mathscr{F})$. Győri's theorem asserts that, in fact, equality always holds:

THEOREM 6.0 [Győri, 11]. *If $\mathscr{F}$ is a family of subpaths of a path P, then $\sigma(\mathscr{F}) = \gamma(\mathscr{F})$.*

Győri used his theorem to derive a nice result in computational geometry:

> *Let R be a bounded region in the plane which is bounded by horizontal and vertical lines. Suppose that R is vertically convex in the sense that each vertical line intersects R in a (continuous) segment. Then the minimum number of rectangles (belonging to R) covering R is equal to the maximum number of points of R such that no two of them belong to a rectangle lying in R.*

Győri's original proof is a long, sophisticated argument and is not algorithmic. Later Franzblau and Kleitman [9] gave an algorithmic proof which gives rise to a polynomial-time algorithm to compute the extrema in the theorem. This proof is not short or simple either. Further extending the proof technique of Franzblau and Kleitman, Lubiw [17] was able to find a weighted generalization of Győri's theorem. Our goal here is to show that Theorem 2.5 easily implies Lubiw's result even in the more general case when P is a circuit. (Our proof is not algorithmic as it invokes Theorem 2.5 whose proof in Section 2 was not algorithmic.)

To make the exposition clearer, first we derive Győri's theorem in the more general form when P is a circuit. We then show how the same idea carries over to the weighted case. Henceforth we assume that the underlying P is a circuit.

THEOREM 6.1. *If $\mathscr{F}$ is a system of subpaths of a directed circuit P, the maximum cardinality of a strongly representable sub-family of $\mathscr{F}$ is equal to the minimum cardinality of a generator of $\mathscr{F}$, that is, $\sigma(\mathscr{F}) = \gamma(\mathscr{F})$.*

Proof. We are going to prove only the non-trivial direction $\sigma \geqslant \gamma$. Let us recall that $\mathscr{F}_r$ denotes the set of all represented paths (F, f) where $f \in F \in \mathscr{F}$. Call a member (F, f) of $\mathscr{F}_r$ *essential* if there is no member F' $(\neq F)$ of $\mathscr{F}$ for which $f \in F' \subset F$. With each essential member (F, f) of $\mathscr{F}_r$ we associate a pair (A, B) of disjoint subsets of V where (A, B) is a partition of the node set $V(F)$ of path F so that A (respectively, B) consists of the nodes of F preceding edge f (following f). Let $\mathscr{L}_{\mathscr{F}}$ denote the family of pairs obtained this way.

LEMMA 6.2. *$\mathscr{L}_{\mathscr{F}}$ is crossing.*

Proof. Let (A, B) and (A', B') be two crossing members of $\mathscr{L}_{\mathscr{F}}$. Let (F, f) and (F', f') denote the corresponding essential members of $\mathscr{F}_r$. Since (A, B) and (A', B') are crossing, $f \in F'$ and $f' \in F$.

We claim that neither A and A' nor B and B' are comparable as sets. Indeed, suppose to the contrary that, say, A includes A'. Since (F, f) is essential, B' properly includes B. But this means that (A, B) and (A', B') are comparable pairs contradicting the assumption that they are crossing.

It follows that $(A \cap A', B \cup B')$ is a pair associated with the represented path (F', f) and $(A \cup A', B \cap B')$ is a pair associated with the represented path (F, f').

In order to show that $(A \cup A', B \cap B')$ and $(A \cap A', B \cup B')$ belong to $\mathscr{L}_{\mathscr{F}}$, we have to prove that the respectively associated pairs (F', f) and (F, f') are essential. We prove this only for (F, f'); the proof is analogous for (F', f). If, indirectly, there were a member X of $\mathscr{F}$ so that $f' \in X \subset F$, then $f \notin X$ since (F, f) is essential. But then $X \subset F'$, contradicting that (F', f') is essential. ▮

Clearly, an independent sub-family of $\mathscr{L}_{\mathscr{F}}$ corresponds to a strongly representable sub-family of $\mathscr{F}$.

Moreover, let $C := \{c_1, ..., c_t\}$ be a covering of $\mathscr{L}_{\mathscr{F}}$, where $c_1, ..., c_t$ are directed edges on the ground set V. Let B_i be a sub-path of P whose first node (resp., last node) is the tail (head) of c_i and let $\mathscr{B} := \{B_1, ..., B_t\}$. We claim that $\mathscr{B}$ is a generator of $\mathscr{F}$. For otherwise there is a minimal member F in $\mathscr{F}$ that is not the union of some members of $\mathscr{B}$. Hence there is an edge f of F so that $(*)$ there is no member B of $\mathscr{B}$ for which $f \in B \subseteq F$. Because C is a covering of pairs associated with essential pairs, (F, f) cannot be essential. That is, there is a member F' of $\mathscr{F}$ so that $f \in F' \subset F$. By the minimal choice of F, F' is the union of some members of $\mathscr{B}$ contradicting $(*)$.

Now the theorem immediately follows from Theorem 2.5. ▮

Remark. Another natural possibility to generalize Győri's theorem is that we consider a family of subpaths of an arborescence. However, as Lubiw [17] showed by an example, the min–max relation is not necessarily true in this case.

Suppose that we are given a non-negative, integer-valued weight function w on the edges of P. With the help of w we define a weight function on the set of represented paths, namely, $w(F, f) := w(f)$. (It will not cause any ambiguity that the same term w is used for both functions.)

We say that a family $\mathscr{B}$ of (not-necessarily distinct) sub-paths of P is a *w-generator* of $\mathscr{F}$ if for each pair (F, e) with $e \in F \in \mathscr{F}$ the family $\mathscr{B}$ contains at least $w(e)$ subpaths of F each containing e. Let $\gamma_w(\mathscr{F})$ denote the minimum cardinality of a w-generator of $\mathscr{F}$. Clearly, when $w \equiv 1$, we are back at the notion of generator. Let $\sigma_w(\mathscr{F})$ denote the maximum weight of an independent sub-family of $\mathscr{F}_r$. The following theorem was proved by Lubiw [17] in the special case when no member of $\mathscr{F}$ contains v_n as an inner node, or equivalently, the underlying P is a simple path.

THEOREM 6.3. *Given a family $\mathscr{F}$ of subpath of a directed circuit P, one has $\sigma_w(\mathscr{F}) = \gamma_w(\mathscr{F})$.*

Note that this theorem is of some interest even in the special case when w is 0–1-valued. We are back at Győri's theorem when $w \equiv 1$ and P is a path.

Proof. We define a function p' on the set of all pairs (A, B) of subsets A, B of V as follows. Let $p'(A, B) := w(f)$ if (A, B) is a member of $\mathscr{L}_{\mathscr{F}}$ associated with an essential member (F, f) of $\mathscr{F}_r$ and zero otherwise. It follows from the definition of w and from Lemma 6.2 that p' is crossing bi-supermodular.

Now the theorem follows from Theorem 2.3 in precisely the same way that Győri's theorem was derived in the preceding proof from Theorem 2.5. (That is, one observes that an independent sub-family of $\mathscr{L}_{\mathscr{F}}$ corresponds to an independent sub-family of $\mathscr{F}_r$ and hence $\sigma_w(\mathscr{F})$ is at least as big as $v_{p'}$ in Theorem 2.3. Furthermore a covering of p' corresponds to a w-generator of $\mathscr{F}_r$ and hence $\gamma_w(\mathscr{F})$ is at most $\tau_{p'}$ in Theorem 2.3.) ∎

Of course, the reduction above makes it possible to use degree constrained and/or minimum node-cost versions of Theorem 2.3 and therefore one can handle variations of Győri's theorem. For example, given two cost functions on the nodes, each possible generating path having a cost defined by the sum of the first cost of its first node and the second cost of its last node, one can derive a formula for the minimum weight of a generator of $\mathscr{F}$.

Finally we remark that the path problem in Theorem 6.3 was reduced to such a special case of the problem of covering bi-supermodular functions when the number of pairs with positive $p(X, Y)$ is bounded by $|P|^3$, a power of the size of the ground set. In Section 7 we exhibit a polynomial time algorithm for such a p. That algorithm relies on the ellipsoid method and we consider it solely as a proof of the existence of a polynomial-time algorithm.

In the special case when P is a path, Lubiw [17] designed a purely combinatorial algorithm which provides a proof of the theorem, as well. Is there an analogous combinatorial algorithm for the general case when P is a circuit?

7. ALGORITHMIC ASPECTS

How can we construct an optimal (integer-valued) covering of a bi-supermodular function? The proof of the main Theorem 2.3 includes non-constructive parts and at the present time we do not know any other proof (even for the consequence Theorem 2.5) that may give rise to a polynomial-time algorithm. (Note however that, relying on contra-polymatroids, there is a combinatorial algorithmic approach to Theorems 2.9 and 2.10.) Because of the applications we discussed in previous sections, it would be highly desirable to develop a constructive proof for Theorem 2.3.

In order to indicate the level of difficulties, here we briefly mention that Edmonds' well-known theorem on partition of matroids follows from our model.

THEOREM 7.1 [Edmonds, 2]. *Given k matroids M_i on a ground set S with rank-function r_i, S includes k pairwise disjoint bases, one from each matroid, if and only if*

$$\sum q_i(Z) \leqslant |Z| \tag{7.1}$$

holds for every subset Z of S where $q_i(Z) := r_i(S) - r_i(S - Z)$.

Proof. Note that $q_i(Z)$ may be interpreted as the minimum cardinality of the intersection of Z and a basis of matroid M_i (and this is why q_i is sometimes called the co-rank function of M_i). This interpretation shows the necessity of (7.1).

To prove the sufficiency, let $T := \{t_1, ..., t_k\}$ be a set of k new elements and define a function p: $A(S, T) \to \mathbf{Z}_+$, as follows. For $X \subseteq S$, $Y \subseteq T$, let $p(X, Y) := q_i(X)$ if $Y = \{t_i\}$ for some $i = 1, ..., k$ and $:= 0$ otherwise. This p is a crossing bi-supermodular function. Let $m_{\text{out}}(s) := 1$ for each $s \in S$.

Since q_i is monotone increasing, we see that $\mathscr{F} := \{(Z, t_i): i = 1, ..., k\}$ is an independent subfamily of $\mathscr{A}(Z, T)$ for which $p(\mathscr{F})$ is maximum and hence $p^v_{\text{out}}(Z) = p(\mathscr{F}) = \sum q_i(Z)$. Therefore (7.1) implies $p^v_{\text{out}}(Z) \leqslant |Z|$ for every $Z \subseteq S$, that is, (2.4a) is satisfied with respect to the given choice of p and m_{out}. By Theorem 2.7 there is an integer-valued covering z of p so that $\delta_z(s) = 1$ for every $s \in S$. Let $S_i := \{s \in S: z(st_i) = 1\}$. $\{S_i: i = 1, ..., k\}$ is a partition of S. Since $|S_i| = z(S, t_i) \geq p(S, t_i) = q_i(S) = r_i(S)$, each S_i includes a basis of M_i, as required. ∎

The purpose of this section is to prove that there is a (strongly) polynomial algorithm to compute a minimum covering of a crossing bi-supermodular function p. We also exhibit an algorithm, at least for a special class of bi-supermodular functions, to compute an independent family $\mathscr{F}$ for which $p(\mathscr{F})$ is maximum. These algorithms rely on the ellipsoid method and they are pretty pedestrian anyway. We do not think that this kind of algorithm might have any practical use or mathematical beauty. But their existence may serve as an encouragement to construct purely combinatorial, more efficient polynomial time algorithms for the covering problem.

Let $p, S, T, A^*, \mathscr{A}^*$ be the same as in Theorem 2.3. For simplicity we assume that S and T are disjoint. (If this were not the case, then by splitting each element of $S \cap T$ into two one can easily formulate a new covering problem which is equivalent to the original one and the corresponding sets S', T' are disjoint.) Theorem 2.3 asserts that the following pair of dual linear programs have integer-valued optima:

$$\min(z(A^*): z: A^* \to \mathbf{R}_+, z(X, Y) \geq p(X, Y) \text{ for every } (X, Y) \in \mathscr{A}^*) \quad (7.2)$$

$$\max\Big(\sum (w(X, Y)\, p(X, Y): (X, Y) \in \mathscr{A}^*): w: \mathscr{A}^* \to \mathbf{R}_+,$$
$$\sum (w(X, Y): (X, Y) \in \mathscr{A}^*, x \in X, y \in Y) \leq 1 \text{ for every } xy \in A^*\Big). \quad (7.3)$$

For a polyhedron P defined by linear inequalities a *separation algorithm* determines if a given point z belongs to P or not, and in the latter case it determines an inequality defining P which is violated by z.

LEMMA 7.2. *There is a polynomial-time separation algorithm for the covering problem* (7.2).

Proof. For an edge $e = st \in A^*$ define $p_e(X, Y) := p(X, Y)$ if e covers (X, Y) and $:= 0$ otherwise. Then p_e is crossing bi-supermodular. Clearly, z is a covering of p if and only if z is a covering of each p_e, $e \in A^*$. Therefore it suffices to solve the separation problem separately for each of the possible $|S|\,|T|$ edges.

Let us fix now $e = st$ and define a set-function q_e on the subsets of $S \cup T$ as follows. For $X \subseteq S$, $Y \subseteq T$, let $q_e(X \cup (T - Y)) := p_e(X, Y)$ if $p_e(X, Y) > 0$ and $-\infty$ otherwise. Then q_e is a fully supermodular set-function. Since δ_z is fully submodular, so is $b := \delta_z - q_e$. Now z is a covering of p_e if and only if b is non-negative. Grötschel *et al.* [10] developed a strongly polynomial algorithm to minimize a submodular function. With the help of this one can compute a subset $Z \subseteq S \cup T$ for which $b(Z)$ is minimum. If this value is non-negative, then we conclude that z is a covering of p_e. If $b(Z) < 0$, then by letting $X := S \cap Z$, $Y := T - Z$ we have $z(X, Y) = \delta_z(Z) = b(Z) + q_e(Z) = b(Z) + p_e(X, Y) < p_e(X, Y)$. ∎

Note that the algorithm of Grötschel *et al.* relies on the ellipsoid method. There are special classes of crossing bi-supermodular functions when direct combinatorial algorithms are available for the separation problem. One can show that this is the case for the special functions analyzed in the previous sections. For example, by using max-flow min-cut computations one can decide in polynomial time if a given augmentation of a digraph is k-connected. Similarly, in Győri's theorem (and in its extensions) it is easy to decide if a system of paths is indeed a generator of a specified path system.

The algorithm we propose is going to use the ellipsoid method in a second level, as well. Therefore, even in this special case when the separation is possible combinatorially, we need the ellipsoid method.

LEMMA 7.3. *There is a polynomial-time algorithm to compute fractional optimal solutions to* (7.2) *and to* (7.3) *and, in particular, to compute their optimum value* v_p *in common.*

Proof. It is well-known (Grötschel *et al.* [10, Theorem 6.4.1]) that the optimal solutions in question may be computed in polynomial time if a separation algorithm is available. This was ensured by Lemma 7.2. ∎

Algorithm for the Primal Optimum

LEMMA 7.4. *There is an algorithm to compute an optimal integer-valued solution to* (7.2) *such that the complexity of the algorithm is polynomial in* Mh *where* M *denotes the maximum value of* p *and* $h := |S|\,|T|$.

Proof. The idea behind the algorithm comes from the proof of Theorem 2.3. We consider all the edges in $A(S, T)$ in an arbitrarily specified order and compute $z(e)$ for the currently considered edge e. At the beginning $z \equiv 0$.

Choose the first edge $e = st$. Recall the definition of the projection p^e of p along e and also that e was called reducing if $(*)$ $v_p - 1 = v_{p^e}$. By Lemma 7.3 we can compute v_p and v_{p^e} and hence we can decide in polynomial time if a given edge is reducing or not.

In an *elementary step* of the procedure we increase $z(e)$ by 1 if e is reducing. Iterate this elementary step by restarting with the same edge e and with the revised function $p := p_e$ as long as e is reducing with respect to the current p. When the current edge is no more reducing, $z(e)$ is declared final and we proceed with the subsequent edge in the given ordering of edges by iterating the above procedure.

Since this algorithm is nothing but the repeated application of the elementary steps applied in the proof of Theorem 2.3, the final z will be an optimal integer-valued covering of p.

Let M denote the maximum value of p. Since any edge can be reducing at most M times, for a fixed edge we need at most M elementary steps; that is, we have to apply the algorithm ensured by Lemma 7.3 at most hM times. ∎

The above algorithm is of polynomial time only if the maximum of p is bounded by a power of h.

THEOREM 7.5. *There is a polynomial-time algorithm to compute an optimal integer-valued solution to* (7.2).

Proof. We show how the general covering problem can be reduced in polynomial time to another covering problem where the maximum value of the defining bi-supermodular function is at most h.

By Lemma 7.3 we can compute an optimal fractional solution x to (7.2). Let z_i denote the componentwise integer part of x; that is, for every $e \in A^*$, $z_i(e) := \lfloor x(e) \rfloor$ and let $x' := x - z_i$.

Define p' as follows. $p'(X, Y) := \max(0, p(X, Y) - z_i(X, Y))$. Now p' is crossing bi-supermodular and its maximum value M' is at most h. By Lemma 7.4 we can find an optimal integer-valued covering z' of p' in no more than $hM' \leqslant h^2$ elementary steps. Since x' is a fractional covering of p', Theorem 2.3 ensures that $z'(A^*) = v_{p'} \leqslant x'(A^*)$. Moreover, $z := z_i + z'$ is clearly an integer-valued covering of p and $z(A^*) \leqslant x(A^*)$. Since x is an optimal covering, $z(A^*) = x(A^*)$ and hence z is an optimal integer-valued covering of p. ∎

Algorithm for the Dual Optimum in a Special Case

We do not have any polynomial time algorithm for finding an integer-valued optimum to (7.3). In particular, we do not know how to find a family of one-way pairs violating (4.3), if there is one. There is however such an algorithm for the class of functions when the number of pairs with positive p-value is "small." Note that Győri's theorem and its extensions in Section 6 belong to this category.

LEMMA 7.6. *There is a algorithm to compute an integer-valued optimum to the dual covering problem* (7.3) *in polynomial time whenever the number f of pairs (X, Y) for which $p(X, Y) > 0$ is bounded by a power of $h := |S|\,|T|$.*

Proof. For any subset $F \subseteq A(S, T)$ we can consider a reduced problem defined by p_F where $p_F(X, Y) := p(X, Y)$ if F does not cover (X, Y) and $:= 0$, otherwise. By Claim 2.1, p_F is crossing bi-supermodular and hence with the *Primal Algorithm* we can compute v_{p_F} in polynomial time.

The idea behind the algorithm is the easy observation that a pair (X, Y) belongs to an optimal independent family $\mathscr{F}$ of pairs of subsets (in Theorem 2.3) if and only if $v_p = p(X, Y) + v_{p_F}$ where $F := A(X, Y)$.

Therefore we test this equality for each (X, Y) with positive $p(X, Y)$. Since there are f such pairs, after at most f applications of the Primal Algorithm we find a pair (X_1, Y_1) for which the equality holds. We then iterate the same procedure by starting with the reduced problem defined by p_F where $F := A(X_1, Y_1)$. Since an independent family may have at most h members, altogether we need the Primal Algorithm no more than h^2 times. ∎

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