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A SURVEY ON T-JOINS, T-CUTS, AND CONSERVATIVE WEIGHTINGS

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The book of L. Lovász and M. Plummer on matching theory includes a good overview of the topic indicated in the title. In the present paper we exhibit the main developments of the area in the last decade. After summarizing the basic concepts and their simple properties, we discuss several min-max theorems on packing T-cuts. Here the central result is a theorem of A. Sebő on distances in a graph endowed with a conservative weighting. We show how it gives rise to a simple proof of a theorem of P. Seymour on packing T-cuts in grafts not T-contractible to K_4. We also provide a new proof of a recent result of E. Korach. An extension of a theorem of E. Korach and M. Penn is presented along with its relationship to the planar edge-disjoint path problem.

1. INTRODUCTION

Matching theory, a fundamental part of graph theory and combinatorial optimization, is mainly concerned with subgraphs of a graph satisfying upper and lower bound constraints on the degrees.

In many applications, however, only the parity of the degrees that really matters. In this survey-type paper we attempt to outline the theory of T-joins and T-cuts along with the most interesting applications. An earlier account on the topic may be found in the book of L. Lovász and M. Plummer [15]. Though our main goal is to overview the developments of the area in the last decade, in order to provide an easy access to the material, we will

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also cover earlier results. Not only the most important theorems will be discussed but in most cases we provide proofs as well.

After presenting the required terminology in Section 1 we summarize some elementary properties of T-joins and T-cuts along with their relationship to conservative weightings. Section 2 exhibits how a minimum weight T-join may be determined with the help of a weighted matching algorithm. It also includes some applications of the minimum T-join problem, such as the Chinese Postman problem. In Section 3 we briefly describe the T-cut packing problem and its relation to planar edge-disjoint paths. Section 4 is devoted to the basic min-max results and their relationship to conservative weightings. The fundamental theorem of A. Sebő on the structure of distances is proved in Section 5. A consequence, due to E. Korach and M. Penn [13], and its recent extension by A. Frank and Z. Szigeti [9] will then be derived from Sebő's theorem. Section 6 includes two theorems concerning the special role the complete graph K_4 plays in the theory. The first one is due to E. Korach while the second one is Seymour's theorem on max-flow min-cut matroids specialized to T-joins. In Section 7 we apply the theory to the planar edge-disjoint paths problem.

Let $G = (V, E)$ be an undirected graph and $T \subseteq V$ a subset of nodes of even cardinality. The pair (G,T) is said to be a *graft*. We call a subset X of nodes *T-odd* (*T-even*) if $|X \cap T|$ is odd (even). The elements of T are called *T-nodes*. For $Z \subseteq V$ let $q_T(Z)$ denote the number of T-odd components of $G - Z$. When $T = V$ we use the abbreviation $q(Z) := q_V(Z)$, that is, $q(Z)$ is the number of components of $G - Z$ with odd cardinality.

A subset J of edges is called a *T-join* if $d_J(v)$ is odd precisely when $v \in T$. Here $d_J(v)$ denotes the number of elements of J incident to v. The minimum cardinality of a T-join is denoted by $\tau(G,T)$. More generally, for an arbitrary weight-function $w : E \to \mathbb{R}$, $\tau(G,T,w)$ denotes the minimum weight of a T-join.

Given a graph $G = (V, E)$, by a *cut* $\delta(X) := [X, V - X]$ we mean the set of edges connecting X and $V - X$. If $|X| = 1$ we speak of a *star-cut*. A cut $\delta(X)$ of G is called a *T-cut* if X is T-odd. We will use the abbreviation *odd cut* for a cut of odd cardinality. $E(X)$ denotes the set of edges induced by X.

Given a partition $\mathcal{P} = \{V_1, V_2, \ldots, V_k\}$ of V, by a *multicut* $B = B(\mathcal{P})$ we mean the set of edges connecting different parts of $\mathcal{P}$. If each V_i is T-odd and induces a connected subgraph, B is called a *T-border*. Then clearly k is even, hence $\mathrm{val}(B) := k/2$ is an integer, called the *value* of the T-border. When $k = 2$, a T-border forms a T-cut. Note that the value of a T-cut is one. By a *packing* of T-borders (T-cuts) we mean a family of pairwise disjoint T-borders (T-cuts). The maximum cardinality of a packing of T-cuts is denoted by $\nu(G,T)$.

More generally, for a non-negative integer-valued weight-function $w : E \rightarrow \mathbb{Z}_+$ we say that a family of cuts is a *w-packing* if every edge e belongs to at most $w(e)$ members of the family. $\nu(G,T,w)$ denotes the maximum cardinality of a w-packing of T-cuts. When $w \equiv 1$ a w-packing reduces to a packing and $\nu(G,T,w) = \nu(G,T)$. When $w \equiv 2$ we call a w-packing a *2-packing* and $\nu(G,T,w)$ is denoted by $\nu_2(G,T)$.

Observe that if B is a T-border of value $k/2$ defined by a partition $\{V_1,\ldots,V_k\}$, then the family $\{\delta(V_1),\ldots,\delta(V_k)\}$ is a 2-packing of T-cuts and hence any packing of T-borders of total value K determines a 2-packing of T-cuts of cardinality K.

A weighting $w : E \rightarrow \mathbb{R}$ is called *conservative* if $w(C) \geq 0$ for every circuit C of G, that is, there is no negative circuit. (In directed graphs the existence of negative directed circuits is excluded.)

Let $w : E \rightarrow \mathbb{R}$ be an arbitrary weight-function and F a subset of edges. By $w' := w[F]$ we mean the following weight-function: $w'(e) = w(e)$ if $e \notin F$ and $w'(e) = -w(e)$ if $f \in F$. Let N_w denote the set of negative edges and $T_w := \{v : d_{N_w}(v) \text{ is odd}\}$, that is, T_w is the set of nodes which are incident to an odd number of negative edges.

For a subset J of edges let $\kappa_J : E \rightarrow \{+1,-1\}$ be defined by $\kappa_J(e) := -1$ if $e \in J$ and $\kappa_J(e) := +1$ if $e \in E - J$. We call J a *join* if κ_J is conservative. Let $T_J := \{v \in V : d_J(v) \text{ is odd})$. Note that any subset J of edges is a T_J-join.

In the second half of this introduction we summarize some elementary properties of T-joins and T-cuts.

Proposition 1.1. *A subset J of edges is a T-join if and only if J is the union of edge-disjoint circuits and $|T|/2$ paths connecting disjoint pairs of T-nodes.*

Proposition 1.2. *The intersection of a T-join J and a cut $B = [X, V - X]$ is of odd cardinality if and only if B is a T-cut.*

Proposition 1.3. *A graph G has a T-join if and only if every component of G is T-even.*

Proposition 1.4. *A set F of edges intersecting each T-join includes a T-cut.*

In what follows, one of our main concerns will be to compute and describe the structure of minimum weight T-joins. The following lemma establishes a relationship between minimum weight T-joins and conservative weightings.

Lemma 1.5. *For an arbitrary weight-function w, a T-join J is of minimum w-weight if and only if $w[J]$ is conservative.*

Proof. If $w[J]$ is not conservative, there is a circuit C for which $w[J](C) < 0$, that is, $w(C \cap J) > w(C - J)$. Then $J' = C \oplus J$ is a T-join for which $w(J') = w(J) - w(C \cap J) + w(C - J) < w(J)$.

Conversely, let J be a T-join for which $w[J]$ is conservative and let J' be an arbitrary T-join. Since $J \oplus J'$ is a cycle, it decomposes into circuits. If, indirectly, $w(J') < w(J)$, then for at least one of these circuits we have $w(C \cap J') < w(C \cap J)$, that is, $w[J](C) < 0$, a contradiction. ■

When $w \equiv 1$ the above proposition is specialized to the following:

Guan's Lemma 1.5a. [11] *A T-join J is of minimum cardinality if and only if*

$$|C \cap J| \leq |C - J| \tag{1.1}$$

holds for every circuit C, or equivalently, the ± 1 weighting κ_J is conservative. ■

Condition (1.1) is called the *circuit condition.* This simple lemma often (though not always; see Theorem 5.2) enables us to reformulate theorems concerning minimum T-joins in terms of ± 1-weightings and vice versa.

Proposition 1.6. *Let T_1 and T_2 be even-cardinality subsets of V and J_i a T_i-join $(i = 1, 2)$. Then $J := J_1 \oplus J_2$ is a T-join where $T := T_1 \oplus T_2$. In particular, the symmetric difference of two T-joins is a cycle and the symmetric difference of a T-join and a cycle is a T-join.* ■

2. Minimum weight T-joins and their applications

Given a graft (G, T) and an arbitrary weight-function w defined on the edge-set of G, we are concerned with the following problem.

Problem (A) Determine a T-join of minimum weight.

This will be done by a reduction to the weighted matching problem but before doing so we exhibit some applications.

2.1. Conservative weightings *Given $G = (V, E)$ and a weight-function $w : E \to \mathbb{R}$, decide if w is conservative or not.*

Let $T := \emptyset$. Then the T-joins are precisely the cycles (union of edge-disjoint circuits) of G. Since the empty set is a T-join of 0 weight, w is conservative if and only if the minimum weight of a T-join is 0.

2.2. Max cycle and max cut problem *Given again G and w, determine a maximum weight cycle.*

This problem is a maximum weight T-join problem for $T = \emptyset$. By negating the weight-function we are at Problem (A).

For planar graphs, through planar dualization, the maximum weight cycle problem is equivalent to the maximum weight cut problem. This, in turn, is equivalent to finding a minimum weight subset of edges whose removal leaves a bipartite graph.

2.3. Shortest paths *Let $G = (V, E)$ be a graph with two specified nodes s and t and $w : E \to \mathbb{R}$ a weight-function. Find a minimum weight simple path connecting s and t.*

If there is no restriction on w, this problem is NP-complete for both directed and undirected graphs as the (directed) Hamiltonian path problem can be formulated as a special case. For directed graphs a relatively simple solution is available if w is conservative.

Therefore it is natural to consider the shortest paths problem for undirected graphs when w is conservative. If w happens to be non-negative, the problem can easily be reduced to one for directed graphs by replacing each undirected edge by two oppositely directed edges. This reduction, however, does not work for general conservative weightings since for a negative edge we would introduce a negative (two-element) circuit.

An answer to Problem (A) solves the shortest paths problem for conservative weight-functions. Namely, let $T := \{s, t\}$. Then any T-join consists of a path P connecting s and t and some edge-disjoint circuits. If w is conservative, there is a minimum w-weight T-join consisting of one path connecting s and t.

Observe that in this case T-cuts are the cuts separating s and t.

2.4. The Chinese postman problem or how to make a graph Eulerian *A postman is supposed to pass every street of a certain district* *so that he starts from the central post office and must return there. Find a tour for him so that the total distance he must cover is minimum.*

Translating this problem to the language of graphs we arrive at the following problem. Given a (connected) graph $G = (V, E)$, call a closed walk a ***postman tour*** if it uses every edge at least once. The problem is to find a postman tour of minimum total length. By the ***total length*** of a walk W we mean $\sum x(e)$ (or more generally, $\sum x(e)w(e)$, where $w : E \to \mathbb{R}_+$ is a weight function) where $x(e)$ denotes the number of times W uses e.

Clearly, if there is a postman tour W using every edge precisely once, then W is optimal. Such a walk exists if and only if G is Eulerian. That is, for Eulerian graphs there is nothing to be optimized: any Eulerian walk serves as an optimal postman tour.

The problem becomes interesting when G is not Eulerian. The key observation is that there is a one-to-one correspondence between the postman tours and the Eulerian supergraphs of G. Namely, an Eulerian walk of an Eulerian supergraph of G determines a postman tour of G. Conversely, let W be an arbitrary postman tour. Replace each edge e by $x(e)$ parallel edges. By definition, the resulting graph $G' = (V, E')$ is Eulerian and the total length of W is $|E'|$.

Therefore our problem is equivalent to making a graph G Eulerian by adding a minimum number of new edges parallel to existing ones. Notice that in an optimal solution at most one parallel edge is added for every old edge. Indeed, for otherwise, discarding two new edges parallel to each other, we would have a smaller solution. That is, our problem is finding a minimum subset J of edges so that $(*)$ doubling parallelly each element of J makes G Eulerian.

By observing that a certain J satisfies $(*)$ if and only if J is a T-join, where T denotes the set of nodes of odd degree, we can conclude that the Chinese Postman problem is equivalent to finding a minimum T-join.

In this case T-cuts are precisely the odd-cardinality cuts.

2.5. Perfect matchings *Determine if a graph includes a perfect matching.*

Let us define $T = V$. Then every T-join has at least $|V|/2$ elements and a T-join J has precisely that many elements if and only if J is a perfect matching of G.

In other words G has a perfect matching if and only if the minimum cardinality of a T-join is $|V|/2$. Hence an answer to Problem (A) automatically provides a characterization for the existence of a perfect matching.

In this sense perfect matchings are special cases of T-joins. What is even more striking is that, conversely, T-joins can also be reduced to matchings. Specifically, we show now how an algorithm for Problem (A) can be obtained from a weighted matching algorithm. The approach is due to Edmonds and Johnson [3].

Let us first assume that $w \geq 0$. For $x, y \in T$ let $\lambda(x, y)$ denote the minimum w-weight of a path connecting x and y. Let K_T denote the complete graph on node-set T and let $M := \{x_1y_1, x_2y_2, \ldots, x_ky_k\}$ be a minimum weight perfect matching of K_T with respect to the weight-function λ. Let P_i be a minimum weight path in G connecting x_i and y_i (that is, $w(P_i) = \lambda(x_iy_i)$ $(i = 1, \ldots, k)$). Finally, let

$$J_M := E(P_1) \oplus E(P_2) \oplus \ldots \oplus E(P_k). \tag{2.1}$$

Claim 2.1. *J_M is a minimum weight T-join.*

Proof. J_M is clearly a T-join. Suppose that J is an arbitrary T-join. By Proposition 1.1 J partitions into circuits and paths R_i $(i = 1, \ldots, k)$. Let s_i and t_i be the two end-nodes of R_i and $M' := \{s_1t_1, s_2t_2, \ldots, s_kt_k\}$. Then we have $w(J) \geq \sum w(R_i) \geq \sum \lambda(s_i, t_i) = \lambda(M') \geq \lambda(M) = \sum w(P_i) \geq w(J_M)$, as required. (Here the first and the last inequalities hold since w is non-negative.) ∎

Therefore, in case of non-negative weight-functions the minimum weight T-join problem can be solved, as follows. First, compute the w-distance of each pair of nodes in T. Second, determine a minimum weight matching M of the complete graph on T. Finally take the paths P_i in G realizing the distance of the ends-points of each matching edge s_it_i. J_M, as defined in (2.1), is a minimum weight T-join.

Let us assume now that in Problem (A) w is an arbitrary weight-function. We are going to define an equivalent problem where the weight-function is non-negative.

Let $|w|$ denote the weight-function for which $|w|(e) := |w(e)|$ for every $e \in E$. Recall the definition of N_w and T_w and the fact that N_w is a T_w-join. Define a function $\varphi_w : 2^E \to 2^E$ by $\varphi_w(X) := X \oplus N_w$. Clearly, φ_w is idempotent, that is, $\varphi_w(\varphi_w(X)) = X$.

Proposition 2.2. *φ_w is a bijection between T-joins and $(T \oplus T_w)$-joins. Furthermore, if a T-join J and a $(T \oplus T_w)$-join J' correspond to each other, then*

$$w(J) = |w|(J') + w(N_w). \tag{2.2}$$

Proof. The first part follows from Proposition 1.6. Since $J' = J \oplus N_w$ we have $w(J) = w(J - N_w) + w(J \cap N_w) = |w|(J' - N_w) + w(J \cap N_w) = |w|(J') - |w|(N_w - J) + w(J \cap N_w) = |w|(J') + w(N_w - J) + w(J \cap N_w) = |w|(J') + w(N_w)$, and (2.2) follows. ∎

This immediately implies:

Proposition 2.3. *For an arbitrary weight-function w a subset J of edges is a minimum w-weight T-join if and only if $J' := J \oplus N_w$ is a minimum $|w|$-weight $(T \oplus T_w)$-join.* ∎

This way the minimum weight T-join problem is reduced to the case of non-negative weights. It should also be noted that the maximum weight (in particular, the maximum cardinality) T-join problem reduces to the minimization problem by simply negating the weight-function.

3. Packing T-cuts

In this section we are interested in the problem of packing T-cuts and its applications. One basic question is as follows.

Problem (B) Given a graft (G, T), what is the maximum number $\nu = \nu(G, T)$ of disjoint T-cuts?

By Proposition 1.2, a T-cut and a T-join always have an element in common, therefore the minimum cardinality τ of a T-join is always an upper bound for ν, that is, $\nu \leq \tau$. In general, we do not have equality here, as is shown by the graft $\mathbf{K}_4 := (K_4, V(K_4))$ where K_4 denotes the complete graph on 4 nodes. In this case a perfect matching is a minimum T-join and hence $\tau = 2$, furthermore there are no two disjoint cuts, therefore $\nu = 1$.

In the preceding section we showed that τ is computable in polynomial time. Can we compute ν in polynomial time, as well? A weaker problem for a graft is to decide in polynomial time whether $\nu = \tau$? (This is weaker indeed as τ is computable in polynomial time.) But even this second question is NP-complete, a result due to M. Middendorf and F. Pfeiffer [16]:

Theorem 3.1. *Determining whether $\nu(G,T) = \tau(G,T)$ is NP-complete even for planar graphs G. In particular, computing $\nu(G,T)$ is NP-complete.* ■

In spite of this negative result, there are important special classes of grafts when $\nu = \tau$ holds true. This is the case, for example, if G is bipartite, or if G is series-parallel, or if G is planar and the elements of T are in the boundary of one face. We will discuss these classes in detail in Sections 4 and 6 along with the special role played by K_4.

One of the main motivation for investigating the problem of packing T-cuts is its close relationship to the following problem.

Edge-disjoint paths problem In an undirected graph $G' = (V, E')$ we are given k pairs of vertices $(s_1, t_1), \dots, (s_k, t_k)$. Decide if there are k pairwise edge-disjoint paths $P_1, \dots, P_k$ in G' so that the end-nodes of P_i are s_i and t_i $(i = 1, \dots, k)$.

It is useful to mark each pair of terminals by a so called *demand edge* s_it_i. (The original edges in G' are sometimes called *supply* edges.) Let J denote the set of demand edges. We call the graph $H = (V, J)$ formed by the demand edges a *demand graph.* Therefore the edge-disjoint paths problem is equivalent to finding $|J|$ edge-disjoint circuits in the union graph $G = (V, E' \cup J)$ such that each circuit contains precisely one demand edge. This formulation justifies the following definition.

Let $G = (V, E)$ be a graph and J a subset of edges. We say that a circuit or a cut of G is *J-good* if it contains precisely one edge from J. We call a family of $|J|$ disjoint J-good cuts (or J-good circuits) a *complete packing.* More generally, given a non-negative, integer-valued weight-function $w : E \to \mathbb{Z}_+$, a w-packing of J-good cuts is called *complete* if every element f of J belongs to precisely $w(f)$ members. Therefore the edge-disjoint paths problem calls for finding a complete packing of J-good circuits. The following condition, called the *cut condition,* is easily seen to be necessary for the solvability of the edge-disjoint paths problem:

$$w(C \cap J) \le w(C - J) \text{ for every cut } C. \tag{3.1a}$$

When $w \equiv 1$ this specializes to

$$|C \cap J| \le |C - J| \text{ for every cut } C. \tag{3.1b}$$

This latter is equivalent to saying that every cut of G contains at least as many supply edges as demand edges. The cut condition is not sufficient in general, but there are several interesting subclasses when it is. (The most fundamental one is when J consists of k parallel edges: this is the undirected edge-disjoint version of Menger's theorem.) For a survey, see Frank [6]. Here we are going to concentrate only on the special case, when G is a planar graph, that is, the supply and the demand graph together (!) is planar. This special case will be called the *planar edge-disjoint paths problem.*

Theorem 3.1A. (M. Middendorf and F. Pfeiffer, [16]) *The planar edge-disjoint paths problem is NP-complete.* ■

We will see (Proposition 3.3) that this is just a reformulation of Theorem 3.1. The idea behind this relationship is that for planar graphs the problem of finding edge-disjoint J-good circuits is equivalent, by planar dualization, to that of finding edge-disjoint J-good cuts. This latter problem, in turn, as we immediately point out, is strongly related to the Problem (B), independent of planarity assumptions on G.

Let $G = (V, E)$ be an arbitrary undirected graph with a subset J of edges. When does there exist a complete w-packing of J-good cuts? The following *circuit condition* is clearly necessary for the existence of a complete w-packing.

$$w(C \cap J) \leq w(C - J) \text{ for every circuit } C. \tag{3.2a}$$

Again, when $w \equiv 1$ this specializes to

$$|C \cap J| \leq |C - J| \text{ for every circuit } C. \tag{3.2b}$$

Observe that (3.2b) is equivalent to saying that κ_J is conservative, which is, by Guan's Lemma, equivalent to requiring that J is a minimum cardinality T_J-join where T_J consists of those vertices of G which are incident to an odd number of elements from J.

The circuit condition is not always sufficient as is shown (again) by K_4 with J being a perfect matching. We are interested in cases when the circuit condition is sufficient. A link between packing J-good cuts and T-cuts is shown by the following:

Proposition 3.2. *Suppose for a graph (G, E) and a subset J of edges that the circuit condition holds (or equivalently, κ_J is conservative). The following are equivalent:*
(a) There exists a complete packing of J-good cuts,
(b) $\nu(G, T_J) = \tau(G, T_J)$.

Proof. (a)$\to$(b). Suppose there is a family $\mathcal{B}$ of $|J|$ disjoint J-good cuts. Since κ_J is conservative, J is a minimum cardinality T_J-join, that is, $|J| = \tau(G, T_J)$. By Proposition 1.2, each J-good cut is a T_J-cut and hence $\mathcal{B}$ is a packing of $|J| = \tau(G, T_J)$ T_J-cuts.

(b)$\to$(a). If we have a packing $\mathcal{B}$ of $|\tau(G, T_J)| = |J|$ T_J-cuts, then each member of the family contains precisely one elements of J, that is $\mathcal{B}$ is a complete packing of J-good cuts. ■

This proposition shows that a theorem, stating that $\nu(G,T) = \tau(G,T)$ holds under some assumptions, might be used to derive theorems stating that, in certain circumstances, the circuit condition is sufficient for the existence of a complete packing of J-good cuts. In the sequel we will use this implication chiefly to derive results on the planar edge-disjoint paths problem. But the reverse implication is equally true, as is expressed by the following:

Proposition 3.3. *Let J be a minimum cardinality T-join in a graft (G,T). The following are equivalent:*
(a) $\nu(G,T) = \tau(G,T)$,
(b) There exists a complete packing of J-good cuts. ■

The proof is analogous to the preceding one and is left to the reader. Proposition 3.3 shows that Theorems 3.1 and 3.1A are indeed equivalent.

We conclude this section by showing that cut packings can be retaylored into non-crossing forms.

Proposition 3.4. *Let $G = (V,E)$ be a graph, (G,T) a graft, $J \subseteq E$ a subset of edges, and $w : E \to \mathbb{Z}_+$ a non-negative, integer-valued weight-function. If there is a w-packing consisting of k T-cuts (respectively, J-good cuts), there is one consisting of k non-crossing T-cuts (J-good cuts).*

Proof. We only prove the statement concerning T-cuts since the case of J-good cuts is analogous. Suppose that $\mathcal{F} = \{X_1, \ldots, X_k\}$ is a family of T-odd sets so that the family of T-cuts $\{\delta(X_i)\}$ forms a w-packing. Let s be an arbitrary node of G. By complementing, if necessary, we may assume that no X_i contains s. If $\mathcal{F}$ is a laminar family, we are done. If $\mathcal{F}$ is not laminar, it contains two intersecting sets, say X_1 and X_2. Then either (i) both $X_1' := X_1 \cap X_2$ and $X_2' := X_1 \cup X_2$ are T-odd or else (ii) both $X_1' := X_1 - X_2$ and $X_2' := X_2 - X_1$ are T-odd. Accordingly, revise $\mathcal{F}$ by replacing X_1, X_2 with X_1' and X_2'.

Apply this uncrossing technique as long as there are intersecting members of the family. We claim that the procedure is finite and hence the final family is laminar. Indeed, $\sum |X_i|$ ($\leq k|V|$) never increases and in Case (ii) it strictly decreases. Hence Case (ii) may occur at most $k|V|$ times. Moreover, if Case (i) occurs, then $\sum |X_i|^2$ ($\leq k|V|^2$) strictly increases and hence after the last occurrence of Case (ii) at most $k|V|^2$ uncrossing steps are possible. ∎

Since a non-crossing family may have at most $2n$ members Proposition 3.4 has the following corollary.

Corollary 3.5. *There is a maximum w-packing of T-cuts (a complete w-packing of J-good cuts) that can be polynomially encoded.* ∎

This result is important in order to have a polynomial time algorithm for finding a maximum w-packing whose complexity is independent of w. Even more, it actually indicates a possible way of attack for designing such an algorithm. Namely, it tells us that there always exists an element x of T so that the star-cut $B := \delta(x)$ belongs to a maximum packing. If we were somehow able to pick up such an x, we could contract B and then iterate the procedure. Although this approach cannot be expected to work in general since the 1-packing problem of J-good cuts is NP-complete, for bipartite graphs the idea works perfectly. How this really goes is the topic of the next two sections.

4. Min-max results

In this section we exhibit several min-max theorems concerning the minimum cardinality of a T-join in a graft (G, T). Historically, Tutte's theorem [23] (see below) on the existence of a perfect matching may be considered the first result of this kind because the existence of a perfect matching in a graph G is equivalent to the statement that in the graft (G, V) the minimum cardinality of a V-join is $|V|/2$.

It will be one of our purposes to extend Tutte's theorem to any graft. In Section 2 we saw that determining the distance of two nodes with respect to a conservative weighting may be reduced to a minimum T-join problem. Conversely, in order to handle problems on minimum T-joins, A. Sebő [17, 19] invented an extremely powerful technique which is based on the use of distances. At the heart of this method lies the following lemma providing a framework for inductive proofs on the structure of minimum T-joins.

Let $G = (V, E)$ be a (connected) graph, w a conservative ± 1 weighting on E. A path or a circuit will be called *negative* if its w-weight is negative. We call a path connecting nodes u and v a *uv-path*.

Let s be a node of G which is the starting-node of a negative path. Among these paths let P denote one whose w-weight is most negative, and, subject to this, P has a minimum number of edges. The other end-node and the last edge of P is denoted by t and xt, respectively.

Lemma 4.1. *(i) Edge xt is the only negative edge incident to t.*
(ii) If a circuit C of 0 w-weight uses node t, then it uses edge xt, as well.
(iii) If for some node q a minimum w-weight sq path P_q uses node t, then it uses edge xt, as well.

Proof. We call a subpath $P[y,t]$ of P an *end-segment.* By the choice of P

$$\text{each end-segment of } P \text{ has negative weight,} \tag{$*$}$$

in particular, $w(xt) < 0$.

(i) Let tz be another negative edge. If $z \in P$, then $P[z,t] + tz$ would form a negative circuit contradicting that w is conservative. If $z \notin P$, then $P' := P + tz$ would be a path with $w(P') < w(P)$ contradicting the minimal choice of P. Thus (i) follows.

(ii) Suppose C uses t but not xt. Let u and v be the two neighbours of t in C and $R := C - t$ the path connecting u and v. An arbitrary node y of R subdivides R into two segments $R[y,u]$ and $R[y,v]$. Since $w(R) < 0$, at least one of the two segments is negative.

Suppose first that P and R have a node y in common. Choose y so that $P[y,t]$ has as few edges as possible. Assume that $w(R[u,y]) < 0$, say. Property $(*)$ implies that $P[t,y] + R[y,u] + ut$ is a negative circuit in G, a contradiction.

Second, let P and R be disjoint. By (i) both edges tu and tv are positive and therefore $w(R) \leq -2$. Hence $P' := P + tu + R$ is a simple path starting at s such that $w(P') < w(P)$ contradicting the minimal choice of P.

(iii) Suppose that P_q does not use edge xt. Let y be a node belonging to P_q and $P[x,s]$ for which $P[y,t]$ has as few edges as possible. (Such a y exists since s is a node belonging to both P_q and $P[x,s]$). Then the segments $P_q[y,t]$ and $P[y,t]$ form a circuit K. Since $w(K) \geq 0$ and $w(P[y,t]) < 0$, we obtain that $w(P[y,t]) < w(P_q[y,t])$ but this contradicts the minimality of P_q: replacing the segment $P_q[y,t]$ of P_q with $P[y,t]$ we would get an sq path with w-weight smaller than that of P_q. ■

The following result of A. Sebő, which may be interesting for its own sake, is a consequence of Lemma 4.1:

Let $D = (U, V; \hat{E})$ be a simple bipartite graph with at least three nodes and w a conservative ± 1-weighting on $\hat{E}$. Suppose that there is a negative path between every two nodes of the same class. Then D is a tree and w is identically -1.

The following fundamental result is due to P. Seymour [22].

Theorem 4.2. *Let $D = (U, V; \hat{E})$ be a connected bipartite graph, T an even subset of nodes and J a subset of edges.*
(A) *There is a family $\mathcal{J}$ of $|J|$ disjoint J-good cuts of D if and only if the circuit condition (3.2b) holds:*

$$|C \cap J| \leq |C - J| \text{ for every circuit } C.$$

(B)

$$\tau(D, T) = \nu(D, T). \tag{4.1}$$

By Propositions 3.2 and 3.3, the two parts of the theorem are equivalent. We do not prove here either part since the proof of a more general result (Theorem 4.4) will later be presented.

Let (G, T) be a graft where $G = (V, E)$ is an arbitrary graph. Let $D = (V, U; \hat{E})$ denote a bipartite graph arising from G by subdividing each edge by a new node. By applying Seymour's Theorem 4.2 to graft (D, T) we immediately obtain:

Corollary 4.3. *Let $G = (V, E)$ be a graph, T an even subset of nodes and J a subset of edges.*
(A) *There is a complete 2-packing of J-good cuts if and only if the circuit condition (3.2b) holds.*
(B)

$$\tau(G, T) = \nu_2(G, T)/2. \tag{4.2}$$

This theorem was formulated in a more general form by J. Edmonds and E. Johnson [3]. It was stated and proved explicitly by L. Lovász [14]. When the result is specialized to the case $V = T$ one obtains a characterization for the existence of a perfect matching in a general graph. Namely, *there is a perfect matching if and only if there is no 2-packing of more than $|V|$ odd-cardinality cuts.* This characterization can easily be derived from Tutte's theorem but it is disappointing to realize that the converse derivation does not seem to be straightforward at all. Therefore it is desirable to have a strengthening of Seymour's theorem that easily implies Tutte's theorem. The following result of Frank, Sebő and Tardos [5] not only fulfills this requirement but will also be useful to derive another fundamental theorem of P. Seymour on max-flow min-cut grafts (Section 6). The main point in this result is that an optimal family of T-cuts or J-good cuts ensured by Seymour's Theorem 4.2 may be chosen in a specially structured form. The proof we present here is due to A. Sebő [17].

Theorem 4.4A. *Let $D = (U, V; \hat{E})$ be a bipartite graph and J a subset of edges. There is a partition $\mathcal{P}$ of V so that*

$$d_J(K) \leq 1 \text{ for each } K \in \mathcal{K} \tag{4.3}$$

if and only if the circuit condition (3.2b) holds where $\mathcal{K}:= \{K : K$ a component of $D - X$ for some $X \in \mathcal{P}\}$.

Proof. Let $w := \kappa_J$. Then the circuit condition is equivalent to w being conservative.

To see the necessity of (4.3) let $\mathcal{P}$ be a partition of V satisfying the condition. Then the family $\{\delta(K) : K \in \mathcal{K}\}$ of cuts forms a partition of $\hat{E}$. Therefore, if there were a negative circuit, one of these cuts would contain at least two negative edges proving the necessity of (4.3).

Its sufficiency is proved by induction on $|J|$. Choose a node s which is incident to a negative edge (that is, to an element of J) and apply Lemma 3.1. (Recall that in the lemma t and xt denote the last node end edge, respectively). Let B denote the set of edges incident to t. B is a cut containing one negative edge. Let D' denote the graph arising from D by contracting the elements of B and w' the restriction of w on the edges of D'. Then D' is bipartite so it has the form $D' := (U', V'; \hat{E}')$. Let t' denote the contracted node.

We claim that w' is conservative. Indeed, suppose indirectly that there is a circuit C' in D' of negative w'-weight. Since w is conservative in D, the edge set in D corresponding to C' forms a negative path R connecting two distinct neighbours of t, denoted by u and v. Since D is bipartite, we have $w(R) \leq -2$. Now $C := R + ut + tv$ is a circuit of D for which $0 \leq w(C) = w(R) + w(ut) + w(vt) \leq -2 + 1 + 1 = 0$. Hence $w(ut) = w(vt) = 1$ and $w(C) = 0$. Therefore u and v are distinct from x, that is, C is a 0 circuit using t but not xt, contradicting Part (ii) of Lemma 3.1.

By induction there is a partition $\mathcal{P}'$ of V' satisfying (4.3) with respect to J'. If $t \in U$ (that is, $t' \in V'$), then $\mathcal{P}'$ determines a partition $\mathcal{P}$ of V.

If $t \in V$, then define $\mathcal{P} := \mathcal{P}' \cup \{t\}$. In both cases it is easily seen that $\mathcal{P}$ satisfies the requirements of the theorem. ∎

Note that Theorem 4.4A immediately implies Theorem 4.2A. We can reformulate Theorem 4.4A in the following form.

Theorem 4.4B. *Let $D = (U, V; \hat{E})$ be a bipartite graph. For an even subset T of nodes*

$$\tau(D,T) = \max(\sum_{X \in \mathcal{P}} q_T(X) : \mathcal{P} \text{ a partition of } V), \tag{4.4a}$$

and

$$\tau(D,T) = \max(\sum_{X \in \mathcal{P}} q_T(X) : \mathcal{P} \text{ a partition of } U). \tag{4.4b}$$

Proof. Naturally, the role of the two parts U and V is symmetric and therefore it suffices to prove only (4.4a). Let us denote the maximum in (4.4a) by $\nu' = \nu'(D,T)$. For a T-join J and a partition $\mathcal{P}$ of V we have $|J| = \sum_{X \in \mathcal{P}} d_J(X) \geq \sum_{X \in \mathcal{P}} q_T(X)$ from which $\nu' \leq \tau$ follows. To see the reverse inequality let J be a T-join of minimum cardinality. Then the circuit condition holds and hence, by the preceding theorem, there is a partition $\mathcal{P}$ of V satisfying (4.3).

A component K of $D - X$ ($X \in \mathcal{P}$) is T-odd if and only if $d_J(K) = 1$. Hence $d_J(X) = q_T(X)$ and $|J| = \sum_{X \in \mathcal{P}} q_T(X)$, as required. ∎

Theorem 4.4B also has a version concerning general graphs $G = (V, E)$. As before, let $D = (V, U; \hat{E})$ denote a bipartite graph arising from G by subdividing each edge by a new node. Here sets E and U are in a one-to-one correspondence and we will not distinguish between their corresponding elements. In particular, a subset of U will be considered as a subset of E and vice versa. We are also given an even subset T of V. While applying Theorem 4.4B to (D,T) we may use (4.4a) or (4.4b). Accordingly, we will obtain two different theorems for $\tau(G,T)$. The first one is due to A. Frank, É. Tardos and A. Sebő [5] while the second to A. Sebő [18].

Theorem 4.5. *For a graft (G,T)*

$$\tau(G,T) = \frac{1}{2} \max(\sum_{X \in \mathcal{P}} q_T(X) : \mathcal{P} \text{ a partition of } V). \tag{4.5}$$

Proof. Immediate from (4.4a) if we observe that $2\tau(G,T) = \tau(D,T)$ and that $q_T(X)$ is the same in G and in D. ■

Before mentioning the other characterization for $\tau(G,T)$ let us show that Theorem 4.5, satisfying our expectation, immediately implies:

Tutte's Theorem *A graph G contains no perfect matching if and only if there is a subset X of nodes such that $G - X$ includes more than $|X|$ components of odd cardinality.*

Proof. (Sufficiency.) Apply Theorem 4.4B with the choice $T := V$. Notice that in this case a set is T-odd if its cardinality is odd. If there is no perfect matching, then the minimum cardinality of a T-join is larger than $|V|/2$. By Theorem 4.5 there is a partition $\mathcal{P}$ of V so that $\sum_{X\in\mathcal{P}} q_T(X)/2 > |V|/2$, that is, $\sum_{X\in\mathcal{P}} q_T(X) > \sum_{X\in\mathcal{P}} |X|$. Therefore there must be a member X of $\mathcal{P}$ so that $q_T(X) > |X|$, that is, the number of components in $G-X$ with odd cardinality is larger than $|X|$, as required. ■

Let us recall the definition of a T-border. The *value* of a packing $\mathcal{B}$ of T-borders is defined by *val*$(\mathcal{B}) := \sum_{B\in\mathcal{B}}$*val*$(B)$. The next result was proved by A. Sebő [18]. It will be invoked in Section 6 to prove Theorem 6.5 of P. Seymour. The present proof is taken from Frank and Szigeti [8].

Theorem 4.6. *For a graft (G,T)*

$$\tau(G,T) = \max(\ \textit{val}\ (\mathcal{B}) : \mathcal{B} \text{ a packing of } T\text{-borders}). \tag{4.6}$$

Proof. Since $|B \cap J| \geq$*val*(B) for any T-border B and T-join J, we get $\tau(G,T) \geq \max$ *val*$(\mathcal{B})$. In order to show the reverse inequality we are going to prove that there is a T-join J of G and a packing $\mathcal{B}$ of T-borders of G so that

$$|J| = \text{val}\ (\mathcal{B}). \tag{4.7}$$

By Theorem 4.4B (formula (4.4b)) there is a partition $\mathcal{U}$ of U and a T-join J' of D for which

$$|J'| = \sum(q_T(X) : X \in \mathcal{U}). \tag{4.8}$$

Assume that $l := |\mathcal{U}|$ is as large as possible and let Z be an arbitrary member of $\mathcal{U}$ with $q_T(Z) > 0$. Let $K_1, K_2, \ldots, K_h$ be the components of $D - Z$, $V_i := V \cap K_i$ and $\mathcal{P} := \{V_1, \ldots, V_h\}$.

Clearly, $Z \supseteq B(\mathcal{P})$ and, actually, here we have equality since if an edge e induced by V_i belonged to Z, then $|Z| \geq 2$ and in $\mathcal{U}$ we could replace

Z by two sets $Z - e$ and $\{e\}$ without destroying (4.8), contradicting the maximality of l. We also claim that each V_i is T-odd for otherwise $|Z| \geq 2$ and for an edge $e \in Z$ leaving V_i we could replace Z by $Z - e$ and $\{e\}$ without destroying (4.8), contradicting again the maximality of l.

Let $\mathcal{B} := \{Z \in \mathcal{U} : q_T(Z) > 0\}$. We have seen that each member Z of $\mathcal{B}$ is a T-border of G with $val(Z) = q_T(Z)/2$. Hence (4.7) and the theorem follows by noticing that J' corresponds to a T-join J of G with $|J| = |J'|/2$. ■

Seymour's Theorem 4.2A shows that the circuit condition is sufficient for the existence of a complete packings of J-good cuts if the graph is bipartite. This is not true for non-bipartite graphs in general, but the following easy theorem shows that it is true if J forms a tree.

Theorem 4.7. *Let $G = (V, E)$ be a graph and $H = (V(J), J)$ a sub-tree of G. There is a complete packing of J-good cuts if and only if the circuit condition holds.* ■

Proof. Let s be an arbitrary node of H and let $e = ux$ be any edge of H such that u is closer to s than x.

Let V_x consist of the nodes in $V - V(J)$ plus the nodes in the component of $H - e$ containing x. Let $S_x := \{v \in V_x :$ there is path in $G(V_x)$ from x to v whose κ_J-length is non-positive.$\}$ Then it is not difficult to check that $\mathcal{J}_s := \{S_x : x \in V(J) - s\}$ is laminar and the family $\mathcal{B}_s := \{\delta(S_x) : x \in V(J) - s\}$ is a complete packing of J-good cuts. ■

In Theorems 4.2 and 4.4 we have seen that in bipartite graphs the packing problem of J-good cuts is equivalent to the problem of deciding for a graft if $\tau(D, T) = \nu(D, T)$. The two problems, however, are not necessarily equivalent if some extra restrictions are imposed. For example, if we want to reformulate Theorem 4.7 in terms of T-joins, we run into the following:

Open problem 4.5. Given a graft (G, T), decide if there is a T-join of minimum cardinality which is, in addition, connected. More generally, find a T-join of minimum cardinality that consists of a minimum number of components.

We conclude this section by stating a result of W. Pulleyblank. *The problem of finding a connected T-join is* NP-*complete.* The proof relies on the fact that the Hamiltonian circuit problem is NP-complete even in 3-regular graphs.

5. Conservative Weightings and Distances

In the preceding section we have seen the benefit of making use of distances to obtain short and simple proofs of theorems concerning minimum cardinality T-joins and maximum packing of T-cuts. In this section we go one step further and show that distances serve not only as a powerful proof technique but they are also the subject of a fundamental theorem due to A. Sebő [19]. His result implies Theorem 4.4 and has some other important theoretical and algorithmical consequences, as well, to be discussed below.

Let $D = (U, V; \hat{E})$ be a bipartite graph and $w : \hat{E} \to \{+1, -1\}$ a conservative weighting on the edges. Let J denote the subset of negative edges. The minimum w-weight of a uv-path is called the *distance* of u and v. It is denoted by $d(u, v) = d_w(u, v)$.

Let s be a specified node of D and for every node u let $\lambda(u) := \lambda_w(u) := d_w(s, u)$. Clearly, $\lambda(s) = 0$ and it is also evident that $|\lambda(x) - \lambda(y)| = 1$ for every edge xy. Let m and M denote the smallest and the largest distance from s, respectively. For every integer i, $m \leq i \leq M$, define a *level set* $L_i := \{x : \lambda(x) = i\}$ and a *down-set* $D_i := \{x : \lambda(x) \leq i\}$. Let $\mathcal{D}_i$ denote a family of those subsets of nodes which form a connected component of the subgraph induced by a down-set D_i. The family $\mathcal{D} := \cup \mathcal{D}_i$ is laminar (i.e., any two members are either disjoint or one includes the other) and the cuts determined by its members form a partition of the edge set $\hat{E}$.

Theorem 5.1. (A. Sebő, [19]) *Every member K of $\mathcal{D}$ is entered by no negative edge or by one negative edge according to K contains s or not. Equivalently, the cuts determined by the members of $\mathcal{D}$ not containing s form a complete packing of J-good cuts.*

Proof. Sebő, [20] Induction on the number of nodes. We will contract a certain cut B of D resulting in a bipartite graph $D' = (U', V'; \hat{E}')$. In general, we use the notational convention that for any part X of D (such as node, level set, distance, family of subsets etc.) the corresponding part of D' is denoted by X'. For example, $\lambda'(v)$ denote the distance of node v from s' in D'. Also, w' stands for the restriction of w on $\hat{E}'$. Let $\Gamma(x)$ denote the set of neighbours of a node x.

We distinguish two cases.

Case 1. $m = 0$. Then no negative path of D ends at s and hence there is no zero-circuit containing s. Therefore if we contract the set B of edges incident to s, no negative circuit arises, i.e. w' is conservative. Clearly,

$\lambda'(v) = \lambda(v) - 1$ for every node of D'. Therefore the level sets for D and for D' correspond to each other, as follows. $L'_{-1} = L_0 - s$, $L'_0 = L_1 - \Gamma(s) + s'$, and for $i \geq 1$, $L'_i = L_{i+1}$.

Let $K_1 := \{s\}$ and $K_2 := \Gamma(s) + s$. Then $K_1, K_2 \in \mathcal{D}$ and we have $\mathcal{D}' = \mathcal{D} - \{K_1, K_2\} + \{s'\}$. We are done by induction if we notice that the preimage of s' is just K_2 and remember that no negative edge is incident to s.

Case 2. $m < 0$. Let P be a path starting at s with length m (that is, P is a most negative path) and assume that P has a minimum number of edges.

We can apply Lemma 3.1 (and use the notation P, t, xt appearing there). The cut B consisting of the edges incident to t contains one negative edge xt (the last edge of P). By Part (ii) of Lemma 3.1 w' is easily seen to be conservative (as was already shown in detail in the proof of Theorem 4.4A).

Claim. *$\lambda'(t') = m + 1$ and $\lambda'(z) = \lambda(z)$ for every node $z \neq t'$ of D'.*

Proof. Since w' is conservative, no negative path in $D - t$ connects two neighbours of t. Hence if R is a path of D between s and a node $z \neq t$, then for the corresponding contracted path R' we have $w'(R') \leq w(R)$. It follows that $\lambda'(t') \leq m + 1$ and $\lambda'(z) \leq \lambda(z)$ for every node $z \neq t'$ of D'.

We cannot have strict inequality in the first case, for if there were a path R' in D' from s to t' with $w'(R') \leq m$, then there would exist a path R in D between s and a neighbour of t for which $w(R) \leq m$, contradicting the fact that each neighbour of t is in level set L_{m+1}.

To see the equality in the second case, suppose indirectly that there is an sz-path R' in D' with $w'(R') < \lambda_w(s, z)$. Because D' is bipartite, $w'(R) \leq \lambda_w(s, z) - 2$. Now R' must contain t' and the two segments $R'[s, t']$ and $R'[t', z]$ correspond to an su-path R_1 and a vz path R_2 in D where $u, v \in \Gamma(t)$. Now $R_0 := R_1 + ut + tv + R_2$ is an sz-path in D. Neither of ut and vt can have weight -1, since then we would have $\lambda_w(s, z) \leq w(R_0) \leq w'(R') \leq \lambda_w(s, z) - 2$. If, in turn, $w(ut) = w(vt) = 1$, then $\lambda_w(s, z) \leq w(R_0) + 2 = w'(R') + 2 \leq \lambda_w(s, z)$, that is, $w(R_0) = \lambda_w(s, z)$, contradicting Lemma 3.1 (iii). ■

We can conclude that the level sets of D and of D' are essentially the same, namely, $L'_m = L_m - t$, $L'_{m+1} = L_{m+1} - \Gamma(t) + t'$, and for $i \geq m + 2$, $L'_i = L_i$.

Let $K_1 := \{t\}$ and $K_2 := \Gamma(t) + t$. Then $K_1, K_2 \in \mathcal{D}$ and we have $\mathcal{D}' = \mathcal{D} - \{K_1, K_2\} + \{t'\}$. We are done by induction since the preimage of

t' is just K_2 and by Lemma 3.1 (i) one negative edge is incident to t, that is, one negative edge enters K_1. ∎

Among the consequences of Theorem 5.1 we first derive Theorem 4.4A. Suppose that $s \in V$. For $m \leq i \leq M$ define $\mathcal{P}_i := \{P : P = L_i \cap X, \text{ for } X \in \mathcal{D}_i\}$ and let

$$\mathcal{P} := \cup(\mathcal{P}_i : i \text{ even}). \tag{5.1}$$

Clearly, $\mathcal{P}$ is a partition of V that satisfies (4.3) by Theorem 5.1. ∎

We know already how to compute a minimum cardinality T-join and also the distance of two nodes with respect to a conservative weight function. But how can we determine algorithmically a complete packing of J-good cuts or a maximum packing of T-cuts, at least when the graph is bipartite and Seymour's Theorem 4.2 is available? Or more generally, how can we compute the partition $\mathcal{P}$ described in Theorem 4.4A?

A striking consequence of Sebő's theorem is that we do not have to design any new algorithm. All we need is to determine the distance $\lambda_w(v)$ of each node v from s and to compute the components of the graphs induced by the down-sets. Once the level sets and $\mathcal{D}$ are available, we can determine a partition $\mathcal{P}$ satisfying (4.3) by (5.1).

The algorithm obtained this way is of polynomial time since it needs n (: the number of nodes) minimum weight matching-computations (to determine the distances) and some manipulation with the connected components in question. Though there are more efficient methods for computing a complete packing of J-good cuts, we stress the conceptual novelty in Sebő's approach: with the help of n primal type computations, a solution to the dual can be immediately extracted.

We mentioned in Section 3 that determining if a general graph $G = (V, E)$ has a complete packing of J-good cuts is NP-complete (where J is a join of G, that is, the circuit condition (3.1b) holds). Therefore we may be satisfied with having "near-complete" packings. The following result was proved first by E. Korach and M. Penn [13]. The present proof is due to A. Sebő [19]. For another, more direct proof, see Section 6.

Theorem 5.2. *Let J be a join in G and let the components of the subgraph $(V(J), J)$ be K_0, $J_1, J_2, \ldots, J_l$. Then it is possible to contract at most one edge from each J_i so that the resulting graph has a complete packing of J-good cuts.*

Note that this generalizes Theorem 4.7.

Proof. Let $D = (U, V; \hat{E})$ be a bipartite graph arising from G by subdividing each edge by a node. (The elements of U and E correspond to each other.) Let w be a ± 1 weighting on $\hat{E}$ so that $w(e) = -1$ precisely if e corresponds to an element of J. Since the circuit condition holds, w is conservative. Choose a node s in V that belongs to K_0 and apply Theorem 5.1.

We call a node $v \neq s$ of D *singular* if there are precisely two negative edges $u_1 v, u_2 v$ incident to v and $\lambda(v) - 1 = \lambda(u_1) = \lambda(u_2)$.

Lemma 5.3. *Each component C of J contains at most one singular node. If s belongs to C, then C contains no singular node.*

Proof. There are no two adjacent negative edges $x_1 y, x_2 y$ for which

$$\lambda(y) + 1 = \lambda(x_1) = \lambda(x_2), \tag{$*$}$$

for otherwise there would be two negative edges entering the component of $\mathcal{D}_{\lambda(y)}$ containing y, contradicting Theorem 5.1.

Suppose indirectly that the component C of J not containing s contains two singular nodes s_1 and s_2 or that a component C containing s contains a singular node s_1. In the second case let $s_2 := s$. Let y be a node of the unique path in C connecting s_1 and s_2 for which $\lambda(y)$ is minimum. Now y is distinct from s_1 and s_2 and hence the two negative edges yx_1 and yx_2 of the paths that are incident to y violate $(*)$. ■

Let us call an edge e of G *singular* if its subdividing node s_e is singular in D. By the claim each component of J_i contains at most one singular edge while K_0 contains none.

Let $\mathcal{D}' := \cup\{X \in \mathcal{D}_i, i \text{ odd}, s \notin X\}$. The cuts determined by the members of $\mathcal{D}'$ correspond to disjoint J-good cuts of G and each non-singular edge enters one of these cuts. Therefore by contracting the singular edges we obtain a graph having a complete packing of J-good cuts. ■

Korach and Penn also formulated their result in an equivalent form:

Corollary 5.4. *If there is a minimum cardinality T-join in a graft (G, T) that has at most $k \geq 1$ components, then*

$$\tau(G,T) - (k-1) \leq \nu(G,T) \leq \tau(G,T). \tag{5.2}$$

There may be cases when a nearly perfect packing of J-good cuts ensured by the previous theorem is not satisfactory. For example, when

J is a matching, Theorem 5.2 is essentially meaningless. Therefore it is a natural demand to prescribe certain (or maybe all) components of J for which no contraction is permitted. This can be achieved by requiring an appropriate strengthening of the circuit condition.

In order to generalize Theorem 5.2, assume that the components of the subgraph $(V(J), J)$ are divided into two groups: $\{K_0, K_1, \dots, K_k\}$ and $\{J_1, \dots, J_l\}$, $(k \geq 0, l \geq 0)$. For a circuit C let $k_+(C)$ denote the number of those components K_i $(i \geq 1!)$ from which C contains at least one edge. Let $w := \kappa_J$. The following result and its proof is taken from Frank and Szigeti [9].

Theorem 5.5. *If*

$$w(C) \geq k_+(C) \tag{5.3}$$

for every circuit C of G, then it is possible to contract at most one edge from each J_i so that the resulting graph has a complete packing of J-good cuts. In particular, if $l = 0$, there is a complete packing of J-good cuts.

Proof. For simplicity, here we prove only the special case $l = 0$. For a sub-tree K of a graph $\hat{G} = (V, \hat{E})$ we call a circuit C *continuous in K* if $x, y \in V(C) \cap V(K)$ implies that the unique path in K connecting x and y belongs to C. The following lemma is easy to prove.

Lemma 5.6. *If J is a sub-forest of a graph so that κ_J is not conservative, then there is a negative circuit which is continuous in each component of J.*

We define an auxiliary graph as follows. First, subdivide each edge e by a new node denoted by s_e. Second, for each node $v \in \cup(V(K_i) : i = 1, \dots, k)$ split v into two in the following sense: add a new copy v' of v to the graph along with a new edge $v'v$ and for each original edge $e = uv \notin J$ replace the edge $s_e v$ by $s_e v'$. We call the new edges $v'v$ *split* edges.

It is immediately seen that the resulting graph D is bipartite. In D let J' consist of the split edges plus the set of edges arisen by the subdivision of the elements of J. Let s be an original node from $V(K_0)$.

From the hypothesis of Theorem 5.5 and from the construction of D, Lemma 5.6 implies (when applied to J') that J' is a join and thus we can apply Theorem 5.1.

Let us consider the level sets $L_i := \{x \in V(D) : \lambda(s, x) = i\}$. The original nodes of K_0 lie in even levels while the original nodes of $K_1, \dots, K_k$ lie in odd levels.

(**) The subdividing node of an original edge e is in an even level precisely if e is in $K_1 \cup \ldots \cup K_k$.

Let $\mathcal{D}' := \cup\{X \in \mathcal{D}_i : i \text{ odd}, X \notin s\}$ and let $\mathcal{B}'$ be the family of cuts in the original graph determined by the members of $\mathcal{D}'$ not containing split edges. For each singular edge e belonging to some K_i $(i \geq 1)$, two members of $\mathcal{B}'$ contain e. Revise $\mathcal{B}'$ by leaving out one of them and let $\mathcal{B}$ denote the resulting family of cuts.

By Sebő's theorem and by property (**), $\mathcal{B}$ consists of edge-disjoint J-good cuts covering each edge of J. Therefore $\mathcal{B}$ is a complete packing of J-good cuts. ■

We close this section by citing another type of result on conservative ± 1 weightings of the edge-set of a 2-edge-connected undirected graph $G = (V, E)$. We are interested in such a weighting for which the number of negative edges is as large as possible. This is equivalent to the problem of finding a conservative ± 1 weighting w for which the total weight $w(E)$ of the edge-set is minimum.

Before answering this question let us mention an apparently quite different problem. It is well-known that if G is 2-edge-connected, then it has an ear-decomposition, that is, G can be obtained from a vertex by adding successively paths so that the two (possibly identical) end-nodes of the currently added path P belong to the subgraph constructed so far while the inner nodes (if there is any) are new. The number of paths in an ear-decomposition is $|E| - |V| + 1$. A less trivial result, due to L. Lovász, asserts that each ear can be chosen of odd length if and only if G is factor-critical (that is, $G - v$ has a perfect matching for every node v of G.) For general graphs one may be interested in ear-decompositions using as many odd ears as possible. The following result shows a surprising relationship between the two problems.

Theorem 5.7. (Frank, [7]) *Given a 2-edge-connected graph $G = (V, E)$, the minimum of $w(E)$ over all conservative ± 1 weightings w of G is equal to the maximum number of odd ears in an ear-decomposition of G.*

6. The role of K_4

We have mentioned that the example K_4 prevents us to extend Seymour's Theorem 4.2 to arbitrary graphs. One may expect that the chances for having a complete packing of J-good cuts are better if the graph does not include K_4 in some sense. In this section we exhibit two theorems, due to E. Korach and to P. Seymour, showing that this is indeed the case. But the "sense" in which K_4 is excluded is quite different in the two theorems.

While proving min-max theorems concerning grafts on bipartite graphs, Lemma 4.1 provided a useful reduction technique. Roughly, it implied that, given a conservative ± 1 weighting, a cut (namely the star of node t) can be contracted without creating a negative circuit. This is not true for non-bipartite graphs, in general, but something can be said and that will be enough for our purposes.

Lemma 6.1. *Let $G = (V, E)$ be a graph and $H = (V(J), J)$ a sub-forest of G for which the circuit condition (3.6b) holds. If every component of H, except possibly one, has at least two edges, then there is an end-node t of H (i.e., $d_J(t) = 1$) such that (i) no member of J is induced by the neighbours of t in G and (ii) contracting the star-cut $B := \delta(t)$ does not destroy the circuit condition (that is, (3.6b) holds in $G' := G/B$ with respect to $J' := J/B$).*

Proof. Let $H_i = (V(J_i), J_i)$ $(i = 1, \ldots, k)$ denote the components of H (where each $J_i \neq \emptyset$) so that if there is an exceptional tree (:having one edge), it is H_1. Let $w := \kappa_J$ and $w' := \kappa_{J'}$. Recall that the circuit condition with respect to G and J is equivalent to saying that w is conservative.

Let s be an arbitrary node of $V(J_1)$ and P a path starting at s so that its weight $m := w(P)$ is minimum and, in addition, P has as few edges as possible. Let t denote the other end-node of P and xt the last edge of P. We call a subpath $P[y, t]$ of P an *end-segment*. Clearly $m < 0$ by the choice of s and

$$\text{any end-segment of } P \text{ has negative weight,} \qquad (*)$$

in particular, $w(xt) < 0$.

We claim that $d_J(t) = 1$. Assume to the contrary that beside xt there is another edge tz in J. But this is impossible since if $z \in P$, then $P[z, t] + tz$ would form a negative circuit, and if $z \notin P$, then $P' := P + tz$ would be a path with $w(P') < w(P)$. The following claim shows that node t satisfies both (i) and (ii).

Claim. *In $G - t$ there is no negative path R connecting two neighbours u, v of t.*

Proof. Let R be such a path for which $w(R)$ is minimum and suppose for a contradiction that $w(R) < 0$. Clearly u and v are distinct from x since otherwise $R + ut + tv$ would form a negative circuit in G.

An arbitrary node y of R subdivides R into two segments $R[y,u]$ and $R[y,v]$. Since $w(R) < 0$, at least one of the two segments has negative weight. We use this fact throughout the proof.

Suppose first that P and R have a node y in common. Choose y so that $P[y,t]$ has as few edges as possible. Assume that $w(R[u,y]) < 0$. Using $(*)$ we get that $P[t,y] + R[y,u] + ut$ is a negative circuit in G, a contradiction.

Let now P and R be disjoint and let y be an arbitrary node of R. Assume that $w(R[u,y]) < 0$ and let zy denote the last edge of $R[u,y]$. Now $w(zy) = -1$ for otherwise $w(R[u,z]) \leq -2$ and hence $P' := P + tu + R[u,z]$ would be a path starting at s with $w(P') < m$. Moreover, we claim that zy is the only negative edge incident to y. To the contrary, let yz' be another one. By the minimal choice of R, yz' cannot be a chord of R. Let $R' := tu + R[u,y] + yz'$. This is a simple path with $w(R') < 0$. But now if $z' \in P$, then $(*)$ implies that $P[z',t] + R'$ is a negative circuit, if $z' \notin P$, then $P' := P + R'$ is a path starting at s with $w(P') < m$, and both cases are impossible.

We can conclude that any node y of R is incident to precisely one negative edge. This implies that a negative edge of R forms a component of H contradicting the hypothesis. ■

We briefly mention that the lemma above immediately implies the theorem of Korach and Penn that was proved in Section 5 via Sebő's Theorem 5.1.

Proof of Theorem 5.2. Let us call a packing satisfying the requirements in Theorem 5.2 *near-complete*. We use induction on $|J|$. The statement is trivial for $|J| = 1$, so we have $|J| > 1$. We claim that every J_i $(i = 2, \ldots, k)$ has at least 2 elements. Indeed, if J_2, say, consists only of one edge, then contracting this edge we are done by induction.

Therefore we can apply Lemma 6.1. By the inductive hypothesis there is a near-complete packing $\mathcal{B}'$ of J'-good cuts in G'. Adding the star-cut $\delta(t)$, provided by Lemma 6.1, to $\mathcal{B}'$ we obtain a near-complete packing of J-good cuts in G. ■

Theorem 4.7 shows that the circuit condition is sufficient for the existence of a complete packing of J-good cuts if J is connected. In what follows we consider the next special case when H consists of two disjoint trees $H_i = (V(J_i), J_i)$ (i=1,2). In this case the circuit condition is not sufficient in general. Our goal now is to find a further condition to ensure the existence of a complete packing.

Let W be a graph arising from the complete graph K_4 by subdividing its edges with some nodes. The four nodes of W corresponding to the nodes of K_4 are called the *principal* nodes while the others are the *inner* nodes. *Principal* paths of W are those connecting principal nodes. A circuit C of W is a *4-circuit* (resp., *3-circuit*) if C contains 4 (resp., 3) principal nodes. Recall that the circuit condition is equivalent to saying that $w := \kappa_J$ is conservative. We say that W is a *bad K_4-graph* (with respect to J or w) (in short, a *bad-K_4*) if it has two 4-circuits of zero weight and W is non-bipartite. Notice that the union of two 4-circuits of W is W. If a graph G includes a bad K_4-graph as a subgraph, we say that G has a bad-K_4 or that a bad-K_4 is in G. One can easily show that each 3-circuit of a bad-K_4 is of odd length.

The name "bad" is justified by the observation that in a bad K_4-graph W no complete packing $\mathcal{B}$ of J-good cuts may exist where J denotes the set of negative edges. Indeed, if such a $\mathcal{B}$ exists, then every edge of a circuit of zero weight and therefore every edge of W must belong to a member of $\mathcal{B}$. That is, W is the union of disjoint cuts and therefore it is bipartite. It also follows that, given a graph G and a subset J of its edges, there is no complete packing of J-good cuts if G has a bad-K_4.

Lemma 6.2. *Let C_1 and C_2 be two 4-circuits of zero length in a bad K_4-graph W. (i) The two (disjoint) principal paths shared by C_1 and C_2 have negative weight while the other four principal paths have non-negative. (ii) Any two nodes u and v of W is connected by a non-positive path.*

Proof. (i) Let s_i $(i = 1,2,3,4)$ denote the principal nodes of W, let P_{ij} $(1 \leq i < j \leq 4)$ denote the six principal paths and let $w_{ij} := w(P_{ij})$. Assume that C_1 is the union of $\{P_{12}, P_{23}, P_{34}, P_{14}\}$ and C_2 is the union of $\{P_{12}, P_{24}, P_{34}, P_{13}\}$.

$w(C_1) = 0$ implies that one of the two segments of C_1 connecting s_1 and s_3 is non-positive. Since P_{13} forms a circuit with both segments and since w is conservative, we have $w_{13} \geq 0$. Analogously, we obtain $w_{24} \geq 0$, $w_{23} \geq 0$ and $w_{14} \geq 0$. Among these four values let w_{13} be a smallest one. Since each 3-circuit is of odd length we have $0 < w_{13} + w_{12} + w_{23} = w_{13} - w_{14} - w_{34}$. Hence $w_{14} \geq w_{13} > w_{14} + w_{34}$ and therefore $w_{34} < 0$. $w_{12} < 0$ follows analogously.

(ii) If C_i contains both u and v, then one of the two segments of C_i connecting u and v is non-positive. Therefore we may assume that none of C_1 and C_2 contains both u and v. Then both u and v are inner nodes and, by re-numbering the indices if necessary, we may assume that u is an inner node of P_{14} and v is an inner node of P_{13}. Let P_u (respectively, P_v) denote the segment of P_{14} (resp., P_{13}) between s_1 and u (resp., s_1 and v). By symmetry we may assume that $w(P_u) \geq w(P_v)$. Now $P' := C_1 - P_u + P_v$ is a path connecting u and v and $w(P') = w(C_1) - w(P_u) + w(P_v) \leq 0$. ■

Theorem 6.3. (E. Korach, [12]) *Let $G = (V, E)$ be a graph and $H = (V(J), J)$ a sub-forest with two components so that the circuit condition holds. The following statements are equivalent.*
(a) There exists a complete packing of $|J|$-good cuts.
(b) The union of circuits with zero κ_J-weight is bipartite.
(c) G has no bad-K_4.

Proof. (The present proof follows a line different from Korach's original proof). We have already seen $(a) \to (b) \to (c)$. To prove $(c) \to (a)$ let $H_i = (V(J_i), J_i)$ $(i = 1, 2)$ denote the two components of H. We use induction on the cardinality of J. If one of J_1 and J_2 is empty, (a) follows from Theorem 4.7. Assume now that $|J_1| = |J_2| = 1$, and let $u_i v_i$ denote the only element of J_i. Since (c) holds, the four nodes u_1, v_1, u_2, v_2 do not induce a complete graph, that is, two of them, say u_1 and u_2, which are not adjacent in G. Now the two star-cuts $\delta(u_1)$ and $\delta(u_2)$ form a complete packing of J-good cuts.

Hence we may assume that both J_1 and J_2 are non-empty and $|J| \geq 3$. Then Lemma 6.1 applies and ensures the existence of an end-node t of H so that contracting the star-cut $B := \delta(t)$ does not destroy the circuit condition. The next lemma tells us that, very luckily, this contraction does not create a bad K_4-subgraph either. Let $G' := G/B$, $J' := J/B$, $J_i' := J_i/B$ $(i = 1, 2), w' := \kappa_{J'}$. Assume that $t \in V(J_1)$. Let y_1 denote the unique neighbour of t in J_1 and t' the contracted node in G'.

Lemma. *G' has no bad-K_4 (with respect to w').*

Proof. Assume, indirectly, that G' has a bad K_4-subgraph W' (with respect to J'). Let W denote the subgraph of G formed by the edges corresponding to the edges of W'. We are going to prove that G also has a bad-K_4, contradicting (c).

The contracted node t' must belong to W' for otherwise W is a bad-K_4 in G. Let us denote two neighbours of t' in W' by u_1, u_2 and if t' is principal, u_3 denotes its third neighbour. The four principal nodes of W' are denoted by $s_1, s_2, s_3, s_4 := t'$ so that u_i lies on the principal path P_{i4} $(i = 2, 3, 4)$.

Any subpath P of J' connecting two nodes of W' is in W'. For otherwise, there is a subpath P' of P connecting two nodes u and v of W' so that P' and W' are edge-disjoint. By Lemma 6.2 (ii) there is a non-positive path P'' in W' connecting u and v and hence $P' \cup P''$ would include a negative circuit.

It follows that the elements of J' belonging to W' form a forest H'' with at most two components. Actually, by Lemma 6.2 (i),

$$H'' \text{ has precisely two components.} \tag{$*$}$$

By combining these observations we can conclude that at least one of the edges $t'u_i$, say u_1t', belongs to J'. Then the edge of G corresponding to u_1t' must be u_1y_1. Let u_iy_i $(i = 2, 3)$ denote an edge of G that corresponds to an edge u_it' of W' where y_i is a neighbour of t.

Case 1. *t' is an inner node of W'.* If $y_1 = y_2$, then W is a bad-K_4 in G. If $y_1 \neq y_2$, then $W + y_1t, ty_2$ is a bad-K_4 in G (Fig. 1.).

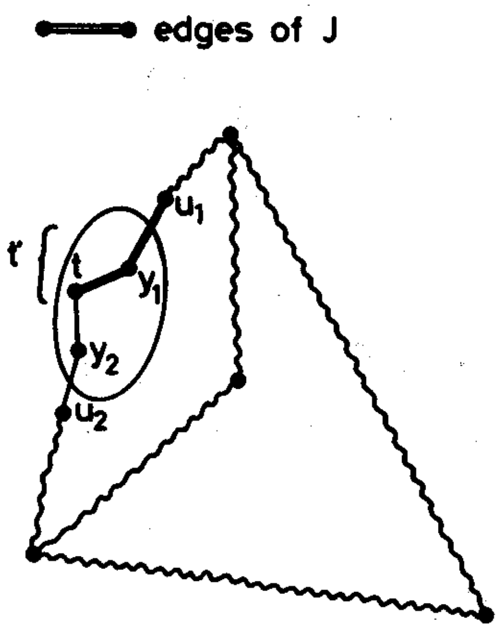

Figure 1.

Case 2. *t' is a principal node of W'.* If $y_1 = y_2 = y_3$, then W is a bad-K_4 in G (Fig. 2.). If for some permutation (i, j, k) of $(1, 2, 3)$ $y_i \neq y_j = y_k$, then $P' := (y_it, ty_j)$ is a path P' of zero length and $W + P'$ is a bad-K_4 in G with principal nodes s_1, s_2, s_3, y_j (Fig. 3.).

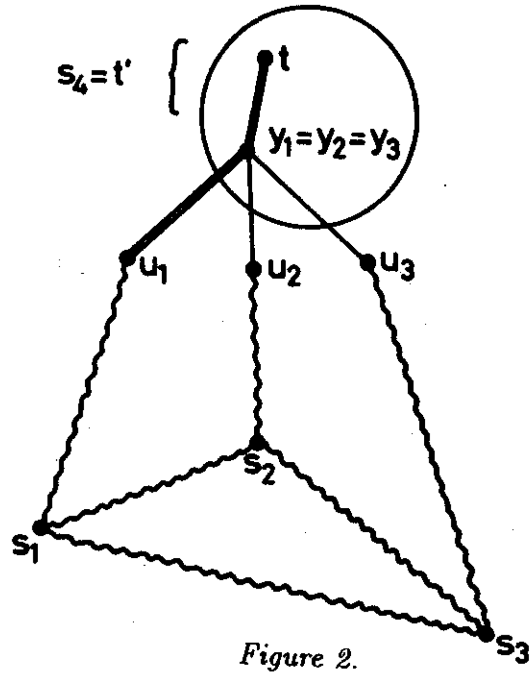

Figure 2.

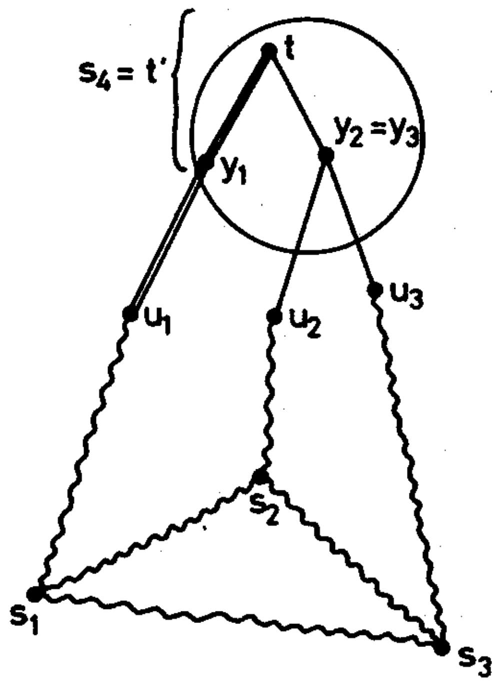

Figure 3.

Finally, let y_1, y_2, y_3 be distinct. The edges u_iy_i $(i = 2, 3)$ do not belong to J since if u_2y_2, say, belongs to J_1, then the subpath of J_1 connecting y_1 and y_2, which is in W, along with (y_1t, ty_2) would form a negative circuit in G. $(*)$ shows that u_2y_2 cannot belong to J_2.

Furthermore, $(*)$ implies that the negative principal path of W' ending at t', which is ensured by Lemma 6.2 (i), must be P_{14}. Hence both 4-circuits of W' with zero weight use P_{14}. But then $W + ty_1, +ty_2 + ty_3$ is a bad-K_4 in G (Fig. 4.). ∎

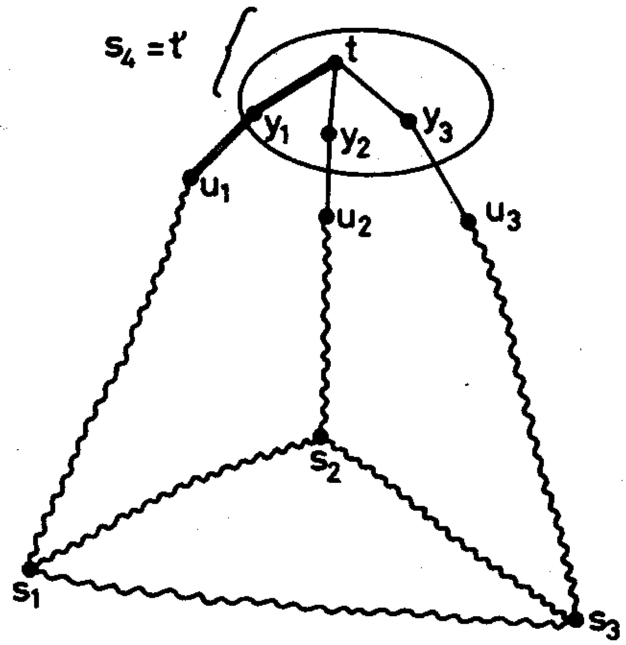

Figure 4.

Since $|J'| = |J| - 1$, the theorem, by induction, is valid for G'. It follows from the Lemma that in G' there is a complete packing $\mathcal{B}'$ of J'-good cuts. Then the cuts in G corresponding to the members of $\mathcal{B}'$ plus the star-cut $\delta(t)$ form a complete packing of J-good cuts in G. ∎

The following example (Fig. 5.) shows that Theorem 6.3 is no more true if J has more than two components. Observe that in this graph there is no circuit of zero κ_J-weight.

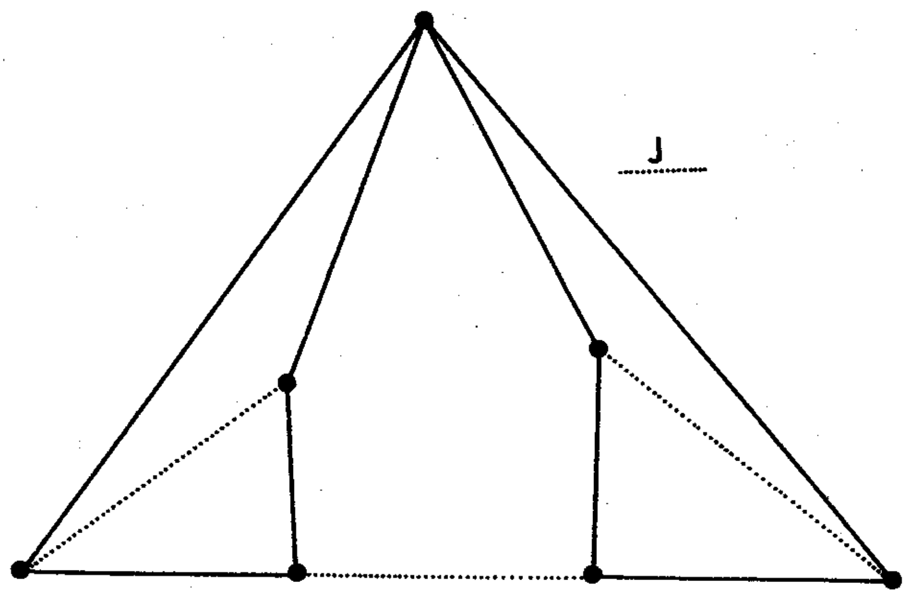

Figure 5.

In general, there is no known way to reformulate Theorem 6.3 in terms of T-joins and T-cuts since we do not know how to decide for a given graft (G,T) if there is a minimum cardinality T-join having at most 2 components. But this property is automatically guaranteed when $|T| = 4$. In this case Theorem 6.3 translates to:

Theorem 6.4. (P. Seymour, [22]) *In a graft (G,T) with $|T| = 4$*

$$\tau(G,T) - 1 \leq \nu(G,T) \leq \tau(G,T). \tag{6.1}$$

Moreover, the following are equivalent:
(a) $\tau(G,T) = \nu(G,T)$.
(b) The union of minimum T-joins forms a bipartite graph.
(c) There is no K_4-subgraph W of G with T as its principal node-set such that each of the three pairs of disjoint principal paths forms a minimum T-join and such that the 3-circuits of W have odd length.

Proof. Since $|T| = 4$, a minimum T-join consists of at most two components. Hence (6.1) follows from Corollary 5.4.

$(a) \rightarrow (b)$ By (a) there is a packing $\mathcal{B}$ of τ T-cuts. Then every edge in a minimum T-join must belong to a member of $\mathcal{B}$. Therefore the union of minimum T-joins is included in the union of some disjoint cuts which is bipartite.

$(b) \rightarrow (c)$ is trivial. To prove $(c) \rightarrow (a)$ let J be a minimum T-join (that is, $|J| = \tau$) and assume, indirectly, that $\nu < \tau$. Since $|T| = 4$, J consists of two disjoint paths. By Theorem 6.3, there exists a bad K_4-subgraph W in G. Let C_1 and C_2 denote the two 4-circuits of W with zero κ_J-weight. Then the symmetric differences $J_i := C_i \oplus J$ $(i = 1, 2)$ are minimum T-joins and hence W violates (c). ■

In Chapter 7 some applications concerning the planar edge-disjoint paths problem will be discussed.

Let us turn to the second theorem we promised where the role of K_4 is crucial, but in a different sense. The following material is taken from Frank and Szigeti [8]. We need some notation. Let $G = (V, E)$ be a graph and T an even subset of nodes. For an edge $e = uv$ we define the *elementary T-contraction* as a graft (G', T') where G' arises from G by contracting e and $T' := T - \{u, v\}$ if $|\{u, v\} \cap T|$ is even and $T' := T - \{u, v\} + x_{uv}$ if $|\{u, v\} \cap T|$ is odd where x_{uv} denotes the contracted node. The *T-contraction* of a graph means a sequence of elementary T-contractions. If $X \subseteq V$ induces a connected subgraph of G, then by T-contracting X we mean the operation of T-contracting a spanning tree of X.

Let $\mathbf{K_4}$ denote a graft $(K_4, V(K_4))$ where K_4 is a complete graph on 4 nodes. Note that a graft (G,T) can be T-contracted to $\mathbf{K_4}$ precisely if there is a partition $\{V_1, V_2, V_3, V_4\}$ of V into T-odd sets so that each V_i induces a connected subgraph and there is an edge connecting V_i and V_j whenever $1 \leq i < j \leq 4$.

A graph $G = (V, E)$ is called *bi-critical* if G contains an edge and $G - \{u, v\}$ contains a perfect matching for every pair of nodes u, v. It follows immediately from Tutte's theorem that G is bi-critical if and only if

$$q(X) \leq |X| - 2 \text{ for every subset } X \subseteq V \text{ with } |X| \geq 2 \qquad (6.2)$$

where $q(X)$ denotes the number of odd-cardinality components of $G - X$.

The *border graph* G_B of a T-border $B = B(\mathcal{P})$ is one obtained by contracting each V_i into one node. Let us call a T-border *bi-critical* if its border graph is bi-critical.

P. Seymour [21] proved a difficult and deep theorem on characterizing binary matroids with the max-flow min-cut property. A special case of his result is the following.

Theorem 6.5. *If a graft (G,T) cannot be T-contracted to $\mathbf{K_4}$, then $\tau(G,T) = \nu(G,T)$.*

Before proceeding to the proof we need two lemmas. The first one is a due to A. Sebő.

Lemma 6.6. *If in (4.5) the optimal packing $\mathcal{B}$ of T-borders is chosen in such a way that $r := |\mathcal{B}|$ is as large as possible, then each member of $\mathcal{B}$ is bi-critical.*

Proof. Suppose, indirectly, that a member $B \in \mathcal{B}$ is not bi-critical. That is, by (6.2), the border graph G_B of B includes a subset X of nodes with $|X| \geq 2$ for which $q(X) \geq |X|$. (Here $q(X)$ denotes the number of odd-cardinality components of $G_B - X$.)

For any odd component K of $G_B - X$ let us define a partition of $V(G_B)$ consisting of the elements of K as singletons and a set $V(G_B) - K$. This partition defines a T-border of G with value $(|K| + 1)/2$. For any even component L of $G_B - X$ let us define a partition of $V(G_B)$ consisting of the elements of $L - v$ as singletons and a set $V(G_B) - (L - v)$ where v is an arbitrary element of L. This partition defines a T-border of G with value

$|L|/2$. The T-borders defined this way are pairwise disjoint subsets of B and their total value is $|V(G_B)|/2$, the value of B. This contradicts the maximal choice of r. ■

The following lemma, interesting for its own right, was stated by A. Sebő [18]. He noted that it follows from Seymour's Theorem 6.5 and observed that, conversely, Theorem 6.5 is an easy consequence of Lemmas 6.6 and 6.7. Frank and Szigeti [8] contains the following simple proof.

Lemma 6.7. *The node set of an arbitrary bi-critical graph G_B on $k \geq 4$ nodes can be partitioned into four subsets V_1, V_2, V_3, V_4 of odd cardinality so that each V_i induces a connected subgraph and there is an edge connecting V_i and V_j whenever $1 \leq i < j \leq 4$. (That is, G_B can be V-contracted to $\mathbf{K}_4$.)*

Proof. Let M be a perfect matching of G_B, $uv \in M$ and $M_{uv} := M - uv$. Let $z(\neq v)$ be a neighbour of u. Since G_B is bi-critical $G_B - \{v, z\}$ contains a perfect matching M_{vz}. The symmetric difference $M_{uv} \oplus M_{vz}$ consists of node-disjoint circuits and a path P connecting z and u. Now $C := P + uz$ is an odd circuit of G_B so that, starting at u and going along C, every second edge of C belongs to M.

Let $u, u_1, \ldots, u_h$ be the nodes of C (in this order). Because of the existence of M, the component K of $G_B - V(C)$ containing v is of odd cardinality while all the other components are of even cardinality.

Let $V_1 := K$. It follows from (6.2) that G_B is 2-connected and, moreover, contains no separating set X of two elements for which $q(X) > 0$. Hence K must have at least three distinct neighbours u, u_i, u_j in C.

If there is a matching edge $xy \in M$ on C so that u, u_i, x, y, u_j reflects the order of these nodes around C (where both $u_i = x$ and $u_j = y$ are possible), then define $V_2' := \{u_1, u_2, \ldots, x\}$, $V_3' := \{y, \ldots, u_{h-1}, u_h\}$, $V_4' := \{u\}$.

If there is no such matching edge, that is, $j = i + 1$ and i is even, then define $V_2' := \{u_i\}$, $V_3' := \{u_{i+1}\}$, $V_4' := V(C) - \{u_i, u_{i+1}\}$.

In both cases $\{V_2', V_3', V_4'\}$ is a partition of $V(C)$. Let $\mathcal{L}$ denote the set of even components of $G_B - V(C)$. For each $L \in \mathcal{L}$ choose a subscript $s = s(L)(= 2, 3, 4)$ so that L is connected to a node in V_s'. For $t = 2, 3, 4$ define $V_t := V_t' \cup \cup(L \in \mathcal{L}\colon s(L) = t)$.

The partition $\{V_1, V_2, V_3, V_4\}$ constructed this way satisfies the requirements. ■

Proof of Theorem 6.5. Let $\mathcal{B}$ be an optimal packing of bi-critical T-borders provided by Lemma 6.6. We claim that each member of $\mathcal{B}$ is a

T-cut. Indeed, if $B \in \mathcal{B}$ is a T-border determined by a partition $\mathcal{P}$ of V ($|\mathcal{P}| \geq 4$) into T-odd sets, then the graft $(G_B, V(G_B))$ arises from (G,T) by T-contracting each member of $\mathcal{P}$ and then, by Lemma 6.7, (G,T) can be T-contracted to $\mathbf{K_4}$, a contradiction. ∎

Actually, Seymour formulated Theorem 6.5 in a slightly stronger form:

Theorem 6.5'. *For a graft* (G,T)

$$\tau(G,T,w) = \nu(G,T,w) \tag{6.3}$$

holds for every non-negative integer-valued weight-function w if and only if (G,T) cannot be T-contracted to $\mathbf{K_4}$.

Proof. Necessity. Assume that (G,T) can be T-contracted to K_4, that is, there exists a partition $\{V_1, V_2, V_3, V_4\}$ of V into connected T-odd sets so that there is an edge e_{ij} connecting V_i and V_j whenever $(1 \leq i < j \leq 4)$. Define w to be 1 on these six edges and 0 otherwise. Then $\tau(G,T,w) = 2$ and $\nu(G,T,w) = 1$, that is, (6.3) is not satisfied.

To see the sufficiency contract each edge e with $w(e) = 0$, subdivide every edge e by $w(e) - 1$ new nodes when $w(e) > 0$ and finally apply Theorem 6.5 to the resulting graft. ∎

Let us consider two interesting special cases of Seymour's theorem.

A graph $G = (V,E)$ is called *series-parallel* if it cannot be contracted to K_4. This is equivalent to saying (easy) that G does not include a subdivided K_4 as a subgraph. G. Dirac proved (and this justifies the name) that connected series-parallel graphs are precisely those graphs which can be obtained from a node by applying four operations in arbitrary order: adding a loop, adding an edge connecting an existing node and a new node, adding an edge parallel to an existing one, subdividing an edge.

Clearly, a series-parallel graph cannot be T-contracted to $\mathbf{K_4}$ hence (6.3) holds for every even $T \subseteq V$.

Another interesting example is when G is planar and the elements of T are in the boundary of one face, say, the infinite face O of G. We then say that (G,T) is a *planar graft.* It is easy to see that a planar graft cannot be T-contracted to $\mathbf{K_4}$ and hence (6.3) holds.

Note that, by a clever elementary construction, the undirected edge-Menger theorem implies directly (6.3) for planar grafts.

(Hint. T subdivides the boundary of O into an even number of paths $P_1, \ldots, P_{2h}$. In the planar dual of G split the node corresponding to O

into two nodes s and t so that the edges corresponding to the edges of P_{2i} (respectively, P_{2i+1}) are incident to s (resp., t) and let G' denote the resulting graph. Show that a path in G' connecting s and t corresponds to a T-cut of G and a cut of G' separating s and t corresponds to a T-join of G.)

Theorem 6.5' provides a characterization for grafts (G, T) having the property $\tau(G, T, w) = \nu(G, T, w)$ for every non-negative integer-valued w. That is, in this case G and T are fixed. If each of G, T, w is fixed, then the problem to decide if $\tau(G, T, w) = \nu(G, T, w)$ holds is NP-complete (Theorem 3.1). There is one more natural question here to ask: what if only G and w are fixed. We consider the case when $w \equiv 1$. A (connected) graph G is called a *Seymour-graph* if for every even cardinality subset T of nodes $\nu(G, T)) = \tau(G, T)$. For example, bipartite graphs and series-parallel graphs are Seymour-graphs. B. Gerards [10] described a class of Seymour-graphs which includes both bipartite and series-parallel graphs. Generalizing this result and answering a conjecture of A. Sebő in the affirmative, A. Agaev, A. Kostochka and Z. Szigeti proved recently the following characterization of Seymour-graphs [1].

Theorem 6.6. *A graph G is Seymour if and only if for every conservative ± 1 weighting of the edges the subgraph formed by the negative edges and by the union of 0-circuits is bipartite.*

7. Planar edge-disjoint paths problem

We introduced the edge-disjoint paths problem in Section 3 and noted that it is equivalent to that of finding a complete packing of J-good circuits. For planar graphs, in turn, this latter problem is equivalent to the one of J-good cuts and hence we may apply the results developed in earlier sections to the planar edge-disjoint paths problem.

We use the notions and notation introduced in Section 3. In particular, we are given a supply graph G' a demand graph H so that $G := G' + H$ is planar. The cut condition (3.1b) is a necessary condition and may be re-formulated in the following way.

$$d_{G'}(X) \geq d_H(X) \text{ for every } X \subseteq V \tag{7.1}$$

Since the dual of a planar graph G is bipartite if and only if G is Eulerian, Theorem 4.2A of P. Seymour immediately yields:

Theorem 7.1. *If $G := G' + H$ is planar and Eulerian, the edge-disjoint paths problem has a solution if and only if the cut condition holds.* ∎

From Theorem 5.2 of Korach and Penn we obtain:

Theorem 7.2. *Suppose that $G := G' + H$ is planar and the cut condition holds. Then it is possible to leave out at most one demand edge from every finite face of G' so that the resulting planar edge-disjoint paths problem has a solution.* ∎

Note that the edges in the dual of G corresponding to the edges in H (:the demand edges) form a forest. Each component of this forest corresponds to a subset of demand edges belonging to one face of G'. The exceptional component K_0 in Theorem 5.2 corresponds to the subset of demand edges belonging to the infinite face of G'.

In other words Theorem 7.2 tells that the cut condition in the planar edge-disjoint path problem, though not sufficient in general, ensures a nearly complete solution. In the unlucky case, however, when each face of G' contains at most one demand edge, then the theorem is basically meaningless. But even in this situation the following weighted version may be extremely useful.

Let $w : E' \cup J \to \mathbb{Z}_+$ be a non-negative integer-valued weight-function. For a demand edge $f = s_i t_i$, $w(f)$ indicates that we want to have $w(f)$ paths in G' connecting s_i and t_i. For a supply edge $e \in E'$, $w(e)$ means that at most that many paths may use edge e.

Theorem 7.2B (Korach and Penn, [13]) *Suppose that $G := G' + H$ is planar and the weighted cut condition (3.1a) holds. Then it is possible to find at most one demand edge f from every finite face of G' and decrease its demand $w(f)$ by one so that the resulting (weighted) planar edge-disjoint paths problem has a solution.* ∎

If we are not allowed to loose any demand, then we are forced to assume something stronger than the cut condition. For a cut $B := \delta(X)$ we define the *surplus* $s(B) := d_{G'}(X) - d_H(X)$. The cut condition is equivalent to saying that the surplus is non-negative.

Theorem 5.4 gives rise to the following:

Theorem 7.3. (Frank and Szigeti, [9]) *Let $G' + H$ be planar. Suppose that the surplus $s(B)$ of every cut B is at least the number of those finite faces of G which contain a demand edge from B. Then the edge-disjoint paths problem has a solution.* ■

If in the planar edge-disjoint paths problem each demand edge is in one face of G', then the cut condition is sufficient. This may be seen directly quite easily, or it is a special case of Theorem 7.2. Let's assume now that the demand edges lie in two faces of G'. Beside the cut condition one can formulate the following necessary condition. Call a subset X of nodes and the cut $\delta(X)$ *tight* if $d_{G'}(X) = d_H(X)$.

Intersection condition. For tight sets X and Y, $d_G(X \cap Y)$ is even.

To see the necessity, assume that the edge-disjoint paths problem has a solution. Then every supply edge in a tight cut is used by a path. Therefore every supply edge in the cut $\delta(X \cap Y)$ is used. On the other hand, if $d_G(X \cap Y)$ were odd, then an odd number of supply edges in the cut $\delta(X \cap Y)$ were not used by any path.

Theorem 7.4. (Frank, [5]) *If $G' + H$ is planar and the demand edges lie in two faces of G', then the cut condition and the intersection condition is necessary and sufficient for the solvability of the edge-disjoint paths problem.*

By planar dualization this theorem may be derived from Theorem 6.3b of Korach. One has only to observe that two tight cuts violating the intersection condition corresponds to a bad-K_4 in the dual.

Corollary 7.5. *Assume that $G' + H$ is planar and the demand edges lie in two faces of G'. If the surplus of every cut is positive, then there is a solution to the edge-disjoint paths problem.* ■

Actually, this result follows not only from Theorem 7.4 but from Theorem 7.3, as well. The next example (Fig. 6.), due to E. Korach, shows that the analogous statement is no more true if the demand edges lie in three faces of the supply graph G'.

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