Finding small regular graphs of girth 6, 8 and 12 as subgraphs of cages

G. Araujo-Pardo*

Instituto de Matemáticas

Universidad Nacional Autonóma de México

Ciudad Universitaria, México D.F. 04510, MÉXICO.

C. Balbuena†

Departament de Matemàtica Aplicada III

Universitat Politècnica de Catalunya

Campus Nord, Edifici C2, C/ Jordi Girona 1 i 3

E-08034 Barcelona, SPAIN.

T. Héger‡

Departament of Computer Science

Eötvös Loránd University

H-1117 Budapest, Pázmány Péter sétány 1/C, HUNGARY

Abstract

Let q be a prime power and $g \in \{6, 8, 12\}$. In this paper we obtain (q, g)-graphs on $2q^{g/2-3}(q^2-1)$ vertices for $g = 6, 8, 12$ as subgraphs of known $(q+1, g)$-cages. We also obtain $(k, 6)$-graphs on $2(kq-1)$ vertices, and $(k, 8)$-graphs on $2k(q^2-1)$ vertices and $(k, 12)$-graphs on $2kq^2(q^2-1)$, where k is a positive integer

*garaujo@math.unam.mx

† m.camino.balbuena@upc.edu

‡hetamas@cs.elte.hu

such that $q \geq k \geq 3$. Some of these graphs have the smallest number of vertices known so far among the regular graphs with girth $g = 6, 8, 12$.

Key words. Cage, girth.

1 Introduction

Throughout this paper, only undirected simple graphs without loops or multiple edges are considered. Unless otherwise stated, we follow the book by Godsil and Royle [14] for terminology and definitions.

Let $G = (V(G), E(G))$ be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *girth* of a graph G is the number $g = g(G)$ of edges in a smallest cycle. The *degree* of a vertex $v \in V$ is the number of vertices adjacent to v. A graph is called *regular* if all the vertices have the same degree. A *cage* is a k-regular graph with girth g having the smallest possible number of vertices. Simply counting the numbers of vertices in the distance partition with respect to a vertex yields a lower bound $n_0(k, g)$ on the number of vertices $n(k, g)$ in a cage, with the precise form of the bound depending on whether g is even or odd.

$$n_0(k,g) = \begin{cases} 1 + k + k(k-1) + \ldots + k(k-1)^{(g-3)/2} & \text{if } g \text{ is odd;} \\ 2(1 + (k-1) + \ldots + (k-1)^{g/2-1}) & \text{if } g \text{ is even.} \end{cases} \tag{1}$$

A (k, g)-cage with even girth g and $n_0(k, g)$ vertices is said to be a *generalized polygon graph.* Generalized polygon graphs exists if and only if $g \in \{4, 6, 8, 12\}$ [6]. When $g = 6$, the existence of a graph with $n_0(k, 6) = 2(k^2 - k + 1)$ vertices called *generalized triangle*, is equivalent to the existence of a projective plane of order $k - 1$, that is, a symmetric $(n_0/2, k, 1)$-design. It is known that these designs exist whenever $k - 1$ is a prime power, but the existence question for many other values remains unsettled. *Generalized quadrangles* when $g = 8$, and *generalized hexagons* when $g = 12$ are also known to exist for all prime power values of $k - 1$, see [4, 6, 14].

Biggs [6] call *excess* of a k-regular graph G the difference $|V(G)| - n_0(k, g)$. The question of the construction of graphs with small excess is a difficult one. Cages have been studied intensely since they were introduced by Tutte [22] in 1947. Erdős and Sachs [10] proved the existence of a graph for any value of the regularity k and the girth g, thus most of work carried out has been focused on constructing a smallest one [1, 2, 5, 8, 9, 11, 12, 16, 18, 19, 20, 23, 24]. Biggs is the author of an impressive report on distinct methods for constructing cubic cages [7]. More details about constructions on cages can be found in the survey by Wong [24] or in the survey by Holton and Sheehan [15].

It is conjectured that cages with even girth are bipartite [21, 24]. In [3], $(k, 6)$-bipartite graphs of order $2(kq - 1)$ are obtained giving the incidence matrices where $k \geq 3$ is an integer and q is a prime power such that $q \geq k$. When q is a square prime power it has been proved [13] that $n(k, 6) \leq 2(kq - (q-k)(\sqrt{q}+1) - \sqrt{q})$ for all $k \leq q$ using geometrical techniques. This last result improves the above one when the smallest prime power $q \geq k$ happens to be a square. Otherwise even if a square prime power is very close to q, say $q + 2$, we obtain larger graphs. For example, for $k = 21$ a $(21, 6)$-regular graph on $2(23 \cdot 21 - 1) = 964$ vertices has been constructed in [3] giving explicitly its incidence matrix, while the result in [13] gives $n(21, 6) \leq 2(525 - 4(5+1) - 5) = 992$.

In this paper we obtain first (q, g)-graphs on $2q^{g/2-3}(q^2 - 1)$ vertices as subgraphs of known $(q+1, g)$-cages for $g = 6, 8, 12$. Second, using similar ideas, we exhibit $(k, 6)$-bipartite graphs on $2(kq - 1)$ vertices and $(k, 8)$-bipartite graphs on $2(q^2k - k)$ vertices. Finally, we also obtain $(q-1, 12)$-bipartite graphs on $2(q-1)^2(q^3 + q^2)$ vertices.

2 Results

To state our results we introduce some notation based on a standard decomposition for a graph G of even girth g. Choose an edge xy of G and define for $0 \leq i \leq g/2 - 1$ the following sets,

$$\begin{array}{rcl} X_i & = & \{u \in V(G) : \partial(u, x) = i,\ \partial(u, y) = i + 1\}, \\ Y_i & = & \{v \in V(G) : \partial(v, y) = i,\ \partial(v, x) = i + 1\}. \end{array} \tag{2}$$

The fact that the girth of G is g implies that the sets X_i, Y_i $(0 \leq i \leq g/2 - 1)$ are pairwise disjoint.

In the next theorem we find (q, g)-graphs for q a prime power and $g = 6, 8, 12$ as a subgraphs of some generalized polygons graphs. This construction extends to $g = 8, 12$ the results contained in [3] for $(q, 6)$-cages. Also this construction allows us to improve the upper bound $n(q, g) \leq 2(q^{\frac{g-2}{2}})$ shown in [2] for q a prime power and $g = 6, 8, 12$.

Theorem 2.1 *Let q be a prime power and $g = 6, 8, 12$. Then any $(q+1, g)$-cage contains as a subgraph a (q, g)-graph on $2q^{g/2-3}(q^2 - 1)$ vertices. Hence*

$$n(q, g) \leq 2q^{g/2-3}(q^2 - 1).$$

Proof. Let H be a generalized polygon graph of degree $q+1$ and girth $g = 6, 8, 12$. Choose an edge xy of H and consider the sets introduced in (2), which clearly partition $V(H)$. Let denote $X_1 = \{x_1, x_2, \ldots, x_q\}$ and $Y_1 = \{y_1, y_2, \ldots, y_q\}$. Let us partition X_i and Y_i, $i \in \{1, \ldots, g/2-1\}$ into the following sets:

$$D_{i-1}(x_j) = \{w \in X_i : \partial(w, x_j) = i-1\},\ j = 1, 2, \ldots, q.$$

$$D_{i-1}(y_j) = \{w \in Y_i : \partial(w, y_j) = i-1\},\ j = 1, 2, \ldots, q.$$

Let us show the following claim.

Claim 1 *Each one of the induced subgraphs $H[D_{g/2-2}(x_i) \cup D_{g/2-2}(y_j)]$, $i, j \in \{1, 2, \ldots, q\}$ of H define a perfect matching.*

Suppose that there is $u \in D_{g/2-2}(x_i)$ such that $|N(u) \cap D_{g/2-2}(y_j)| \geq 2$ for some $j \in \{1, 2, \ldots, q\}$. Then $uw_1, uw_2 \in E(H)$ for $w_1, w_2 \in D_{g/2-2}(y_j)$ yielding that the shortest (y_j, w_1)-path of length $g/2-2$, the shortest (y_j, w_2)-path of length $g/2-2$, and the edges uw_1, uw_2 create a cycle of length less than g which is a contradiction. Therefore $|N(u) \cap D_{g/2-2}(y_j)| \leq 1$ for all $u \in D_{g/2-2}(x_i)$ and $j = 1, 2, \ldots, q$. As every vertex $u \in D_{g/2-2}(x_i)$ has exactly one neighbor in $X_{g/2-2}$ and the other q neighbors of u must be in $Y_{g/2-1}$ because $g \geq 6$, then $|N(u) \cap D_{g/2-2}(y_j)| = 1$. Analogously every vertex $v \in D_{g/2-2}(y_j)$ has $|N(v) \cap D_{g/2-2}(x_j)| = 1$, hence the claim is valid. ■

Let G be the graph obtained from H by deleting X_i and Y_i for $0 \leq i \leq g/2-3$ and the set of vertices

$$\bigcup_{i=g/2-3}^{g/2-2} (D_i(x_q) \cup D_i(y_q)).$$

To illustrate the construction of this graph G, Figure 1 depicts on the left side the spanning tree of the $(4, 6)$-cage and the eliminated vertices from it are indicated inside a box. And on the right side the resulting $(3, 6)$-graph after the deletion of the indicated vertices is shown. Figure 2 depicts the spanning tree of the $(4, 8)$-cage and the deleted vertices are also indicated inside a box.

Clearly, all the remaining vertices of $X_{g/2-2} \cup Y_{g/2-2}$ have degree q in G since all these vertices have the same neighbors they had in H except the removed vertex belonging to $D_{g/2-3}(x_q) \cup D_{g/2-3}(y_q)$. Furthermore, all the remaining vertices of $X_{g/2-1} \cup Y_{g/2-1}$ have degree q in G, because they have the same neighbors they had in H except one neighbor which was in $D_{g/2-2}(x_q) \cup D_{g/2-2}(y_q)$.

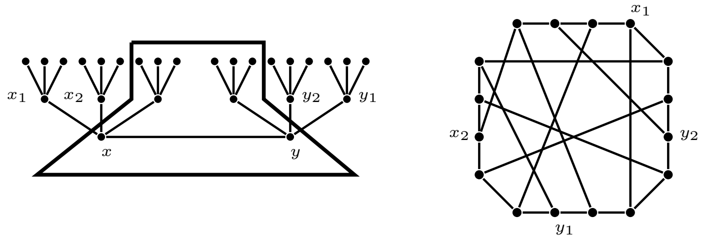

Figure 1: Deleted vertices in the $(4,6)$-cage and the resulting $(3,6)$-graph on 16 vertices.

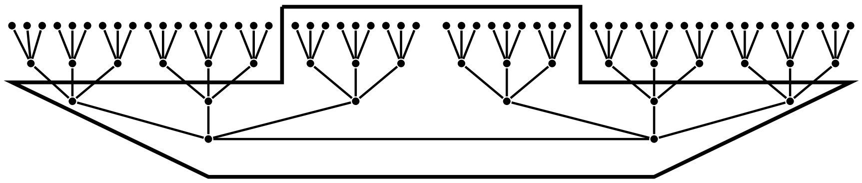

Figure 2: Deleted vertices in the $(4,8)$-cage.

Therefore, H is a (q,g)-graph with $g = 6, 8, 12$ on

$$|V(H)| = 2(q-1)(q^{\frac{g-6}{2}} + q^{\frac{g-4}{2}}) = 2q^{\frac{g-6}{2}}(q^2-1)$$

vertices. Thus, the theorem is true. ■

Let $k \geq 3$ be an integer, $(k,6)$-bipartite graphs for $q \geq k$ a prime power and on $2(kq-1)$ vertices are constructed in [3], via the incidence matrix. Next we give other graphical construction following the same ideas as in Theorem 2.1. We use the notation $N[x]$ to mean the set of vertices $N(x) \cup \{x\}$.

Theorem 2.2 *Let $k \geq 3$ be an integer and $q > k$ be a prime power. Then any $(q+1,6)$-cage contains as a subgraph a $(k,6)$-bipartite graph on $2(kq-1)$ vertices. Hence*

$$n(k,6) \leq 2(kq-1).$$

Proof. Let H be a generalized triangle graph of degree $q+1$ and girth $g = 6$. Let us choose an edge xy of H and consider again the sets $X_1 = \{x_1, x_2, \ldots, x_q\}$ and $Y_1 = \{y_1, y_2, \ldots, y_q\}$. Observe that X_2 and Y_2 are partitioned into the neighborhoods $N(x_i) - x$ and $N(y_j) - y$, $i, j \in \{1, 2, \ldots, q\}$, respectively. By Claim 1, each one of the induced subgraphs $H[(N(x_i)-x) \cup (N(y_j)-y)]$, $i, j \in \{1, 2, \ldots, q\}$ of H define a perfect matching. Let denote by $\ell_1, \ldots, \ell_{q-k} \in N(x_k) - x$ and by $r_1, \ldots, r_{q-k} \in N(y_k) - y$ such that $\ell_i r_i \in E(H)$, $i = 1, \ldots, q-k$. Then, the structure of H induce the following two injective mappings for all $t = 2, \ldots, q$.

$$f_t : \{\ell_1, \ldots, \ell_{q-k}\} \to N(y_t) - y$$

such that $\ell_i f_t(\ell_i) \in E(H)$, and

$$\varphi_t : \{r_1, \ldots, r_{q-k}\} \to N(x_t) - x$$

such that $r_i \varphi_t(r_i) \in E(H)$. Let G be the graph obtained from H by deleting the following set of vertices and edges.

$$\begin{array}{lcl} \text{vertices} & : & \displaystyle\bigcup_{t=k+1}^{q} (N[x_t] \cup N[y_t]) \cup (N[x_k] \setminus \{\ell_1, \ldots, \ell_{q-k}\}) \cup (N[y_k] \setminus \{r_1, \ldots, r_{q-k}\}); \\ \text{edges} & : & x_t \varphi_t(r_i),\ y_t f_t(\ell_i),\ t = 1, \ldots, k-1,\ i = 1, \ldots, q-k. \end{array}$$

Figure 3 depicts on the left side the spanning tree of a $(6,6)$-cage. The new graph G is obtained by eliminating the vertices indicated inside a box, and the deleted edges are indicated in dashed lines. On the right side of this figure, the resulting $(3,6)$-graph G on 28 vertices is shown.

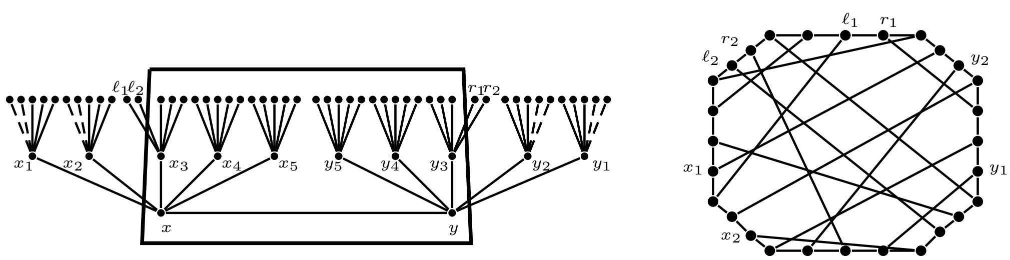

Figure 3: Eliminated vertices and edges in a $(6,6)$-cage and the resulting $(3,6)$-graph on 28 vertices.

Let us see that G is a k-regular graph.

The vertices $x_t \in X_1 \setminus \{x_k, x_{k+1}, \ldots, x_q\}$, $y_t \in Y_1 \setminus \{y_k, y_{k+1}, \ldots, y_q\}$ have degree k in G because they have the same neighbors as in H except the $q-k$ corresponding to the removed edges $x_t \varphi_t(r_i)$, $y_t f_t(\ell_i)$, $i = 1, \ldots, q-k$, and the edges incident with x, y.

Vertices $w \in X_2 \setminus \bigcup_{t=2}^{k} \varphi_t(\{r_1, \ldots, r_{q-k}\})$ have degree k in G because they have lost $q-k+1$ which are: one neighbor in $N(y_t) - y$ for each $t = k+1, \ldots, q$, and other more in $N(y_k) \setminus \{r_1, \dots, r_{q-k}\}$. Similarly, vertices $w \in Y_2 \setminus \bigcup_{t=2}^{k} f_t(\{\ell_1, \dots, \ell_{q-k}\})$ have degree k in G.

Vertices $f_t(\ell_i)$ have degree k because they are adjacent to r_i and have one neighbor in each $N(y_t) - y$, $t = 1, \dots, k-1$. Similarly $\varphi_t(r_i)$ is proved to have degree k. Finally, every ℓ_i, $i = 1, 2, \dots, q-k$, has degree k because it is adjacent to r_i and has other neighbor in $N_H(y_t) - y$ for each $t = 1, \dots, k-1$. Similarly, r_i has degree k.

The order of G is

$$|V(G)| = 2((k-1) + (k-1)q + (q-k)) = 2(kq-1),$$

and clearly G has girth at least 6. To state that the girth is exactly 6 it is enough to notice that $q - 1 \geq 3$ and the number of vertices of the constructed graph is strictly less than the lower bound given in (1) for $g = 8$. ■

Theorem 2.3 *Let $k \geq 3$ be an integer and $q > k$ be a prime power. Then any $(q+1, 8)$-cage contains as a subgraph a $(k, 8)$-bipartite graph on $2(kq^2 - k)$ vertices. Hence*

$$n(k, 8) \leq 2k(q^2 - 1).$$

Proof. Let H be a generalized quadrangle graph of degree $q+1$ and girth $g = 8$. Choose an edge xy of H and consider the sets introduced in (2), which clearly partition $V(H)$. Let denote by $X_1 = \{x_1, x_2, \dots, x_q\}$ and $Y_1 = \{y_1, y_2, \dots, y_q\}$. Let us partition X_i and Y_i, $i = 2, 3$ into the following sets:

$$D_{i-1}(x_j) = \{w \in X_i : \partial(w, x_j) = i-1\}, \ j = 1, 2, \dots, q.$$

$$D_{i-1}(y_j) = \{w \in Y_i : \partial(w, y_j) = i-1\}, \ j = 1, 2, \dots, q.$$

By Claim 1 each one of the induced subgraphs $H[D_2(x_i) \cup D_2(y_j)]$, $i, j \in \{1, 2, \dots, q\}$ of H define a perfect matching which induces the following one-to-one mapping:

$$f_{ij} : D_2(x_i) \to D_2(y_j),$$

such that $wf_{ij}(w) \in E(H)$ for all $w \in D_2(x_i)$. Let us denote $D_1(x_k) = \{x_{k1}, \dots, x_{kq}\}$, $D_1(y_t) = \{y_{t1}, \dots, y_{tq}\}$, $k, t = 1, \dots, q$, hence $D_2(x_k) = \bigcup_{j=1}^{q}(N_H(x_{kj}) - x_k)$ and $D_2(y_t) = \bigcup_{j=1}^{q}(N_H(y_{tj}) - y_t)$. Let us see that there is exactly one edge joining the set $N_H(x_{kj}) - x_k$ and the set $N_H(y_{tj}) - y_t$. Otherwise suppose that $|(N_H(y_{tj}) -$

$y_t) \cap f_{kt}(N_H(x_{ki}) - x_k)| \geq 2$ for some $y_{tj} \in D_1(y_t)$. Then there are two distinct vertices $u, v \in N_H(x_{ki}) - x_k$ such that $f_{kt}(u), f_{kt}(v) \in N_H(y_{tj}) - y_t$ yielding that $x_{ki}, u, f_{kt}(u), y_{tj}, f_{kt}(v), v, x_{ki}$ is a cycle of length 6 which is a contradiction. Therefore $|(N_H(y_{tj}) - y_t) \cap f_{kt}(N_H(x_{ki}) - x_k)| = 1$ for all $y_{tj} \in D_1(y_t)$.

Let denote $L = \bigcup_{i=1}^{q-k}(N_H(x_{ki}) - x_k) \subset D_2(x_k)$ and $R = \bigcup_{j=1}^{q-k}(N_H(y_{kj}) - y_k) \subset D_2(y_k)$. Now, let G be the induced subgraph $G = H[S] - M$ of H where $S \subset V(H)$ and $M \subset E(H)$ are the following:

$$\begin{aligned} S &= \bigcup_{t=1}^{k-1}\bigcup_{i=1}^{2}(D_i(x_t) \cup D_i(y_t)) \bigcup_{i=1}^{q-k}\{x_{ki}, y_{ki}\} \cup (L \cup R) \\ M &= \{uv \in E(H) : u \in X_2 \cup Y_2, v \in X_3 \cup Y_3, N(v) \cap (L \cup R) \neq \emptyset\}. \end{aligned}$$

By way of example, Figure 4 shows the spanning tree of a $(5, 8)$-cage. The new graph is the induced subgraph for the vertices outside of the box for the case $k = 3$.

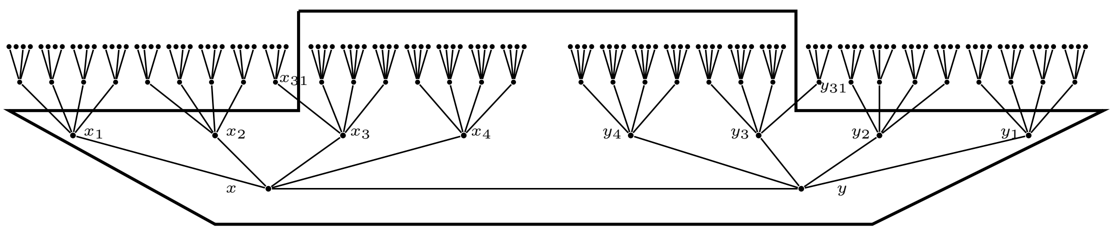

Figure 4: Eliminated vertices and edges in a $(5, 8)$-cage for $k = 3$.

Let us continue proving that the degree of G is k.

Every vertex $y_{tj} \in D_1(y_t)$ $(t = 1, \ldots, k-1,\ j = 1, \ldots, q)$ has degree k in G because $N_G(y_{tj}) = N_H(y_{tj}) \setminus (\{y_t\} \cup (N_H(y_{tj}) \cap f_{kt}(L))$. The same argument is valid for proving that the vertices $y_{k1}, \ldots, y_{k,q-k} \in D_1(y_k)$ have degree k. Similarly, every vertex $x_{tj} \in D_1(x_t)$ has degree k in G because $N_G(x_{tj}) = N_H(x_{tj}) \setminus (\{x_t\} \cup (N_H(x_{tj}) \cap f_{tk}^{-1}(R))$.

If $w \in f_{tk}(L)$ then w has degree k in the new graph G, because it has lost $q+1-k$ neighbors, one in each $D_2(x_j)$, $j = k+1, \ldots, q$ and one $y_{ij} \in \bigcup_{t=1}^{q-k} D_1(y_t)$ because of the eliminated edges. Analogously, $w \in f_{kt}^{-1}(R)$ is proved to have degree k.

If $w \in D_2(y_t) \setminus f_{kt}(L)$, then w has degree k in the new graph G, because it has lost $q+1-k$ neighbors, one in each $D_2(x_j)$, $j = k+1, \ldots, q$ and one other more in the removed part of $D_2(x_k)$. Analogously, $w \in D_2(x_t) \setminus f_{tk}^{-1}(R)$ is proved to have degree k.

Therefore we conclude that the degree is k as claimed. The order of G is

$$\begin{aligned} |V(G)| &= 2((k-1)(q^2+q)+(q-k)(q+1)) \\ &= 2(kq^2-k), \end{aligned}$$

and clearly G has girth at least $g=8$. If $q=4$ and $k=3$ the girth is 8 as it is can easily be checked working in the indicated way with a $(5,8)$-cage. To state that the girth is exactly 8 for other values of q and k, it is enough to notice that the order of the new graph G is strictly less than the lower bound given in (1) when the girth is 10. ■

Theorem 2.4 *Let $k \geq 3$ be an integer and $q \geq k$. Then any $(q+1,12)$-cage contains a $(k,12)$-graph as a subgraph on $2kq^2(q^2-1)$ vertices. Hence*

$$n(k,12) \leq 2kq^2(q^2-1).$$

Proof. Let H be a generalized hexagon graph of degree $q+1$ and girth $g=12$. Choose an edge xy of H and consider the sets introduced in (2), which clearly partition $V(H)$. For any vertex $u \in X_i$ and $j \leq 5-i$ let

$$D_j(u) = \{v \in X_{i+j} : \partial(u,v) = j\}.$$

For $u \in Y_i$ define $D_j(u) \subset Y_{i+j}$ similarly. Note that $|D_j(u)| = q^j$ and that $X_{i+j} = \bigcup_{z \in X_i} D_j(z)$, where the sets $D_j(z), z \in X_i$ are disjoint.

Claim 2 *Let $i+j \geq 6$ $(1 \leq i,j \leq 5)$. Then for any $u \in X_i$ and $v \in Y_j$ there is at most one edge between the sets $D_{5-i}(u) \subset X_5$ and $D_{5-j}(v) \subset Y_5$. If $i+j=6$, then there is exactly one edge.*

Proof. Suppose on the contrary that there are two edges between the sets $D_{5-i}(u)$ and $D_{5-j}(v)$, say u_1v_1 and u_2v_2. Then the natural walk given by the paths $u \to u_1$, u_1v_1, $v_1 \to v$, $v \to v_1$, v_2u_2, $u_2 \to u$ would contain a cycle of length at most $2(5-i)+2(5-j)+2 = 20-2(i+j)+2 \leq 10$, contradicting that the girth of H is 12.

On the second hand, suppose $i+j=6$. There are exactly $q \cdot q^{5-i} = q^{6-i}$ edges going from $D_{5-i}(u)$ to Y_5, where Y_5 is partitioned by the $q^j = q^{6-i}$ sets $D_{5-j}(z), z \in X_j$, and as seen before, only one edge can go to each set of the partition. The equal number of edges and sets proves the statement. ■

Let $X_1 = \{x_1, \ldots, x_q\}$ and $Y_1 = \{y_1, \ldots, y_q\}$, furthermore let $D_1(x_k) = \{x_{k1}, \ldots, x_{kq}\}$ and $D_1(y_k) = \{y_{k1}, \ldots, y_{kq}\}$. Let $X^* \subset D_4(x_k) \subset X_5$ and $Y^* \subset D_4(y_k) \subset Y_5$ be the sets $\bigcup_{i=1}^{q-k} D_3(x_{ki})$ and $\bigcup_{i=1}^{q-k} D_3(y_{ki})$, respectively.

Now let G be the subgraph $H[S] - M$ of H where $S \subset V(H)$ and $M \subset E(H)$ are the following:

$$S = \bigcup_{i=1}^{k-1} \bigcup_{j=3}^{4} (D_j(x_i) \cup D_j(y_i)) \cup \bigcup_{i=1}^{q-k} \bigcup_{j=2}^{3} (D_j(x_{ki}) \cup D_j(y_{ki})) \,,$$

$$M = \{uv \in E(H) : u \in X_4 \cup Y_4,\, v \in X_5 \cup Y_5,\, N(v) \cap (Y^* \cup X^*) \neq \emptyset\} \,.$$

Let us check if G is k-regular. Let $u \in V(G) \cap X_5$ and let the unique vertex of $N(u) \cap X_4$ be denoted by $r \in V(G)$. Then by Claim 2, u originally had exactly one neighbor in each set $D_4(y_i)$, $i = 1, \ldots, q$, say $u_1, \ldots, u_q$, among which $u_1, \ldots u_{k-1}$ are in $V(G)$, $u_{k+1}, \ldots, u_q$ are not in $V(G)$ and u_k is in $V(G)$ depending on whether $u_k \in Y^* \subset V(G)$ or not. In both cases u has exactly k neighbors in G, since either u_k is deleted and the kth neighbor of u is r or $u_k \in Y^*$, thus the edge $ur \in E(H)$ is not included in $E(G)$.

If $u \in V(G) \cap X_4$, then all the neighbors of u in G are found in $D_1(u) \subset X_5$. By Claim 2, there is exactly one edge between the sets $D_1(u)$ and $D_3(y_{ki})$, $i = 1, \ldots, q$, say $u_1v_1, \ldots, u_qv_q$. Since $v_i \in Y^*$ if and only $1 \leq i \leq q - k$, $uu_i \in E(G)$ if and only if $q - k + 1 \leq i \leq q$, hence the valency of u in G is k.

The order of G is

$$|V(G)| = 2\left((k-1)(q^4 + q^3) + (q-k)(q^3 + q^2)\right) = 2q^2(kq^2 - k),$$

which finishes the proof. ■

Acknowledgment

Research supported by the Ministry of Education and Science, Spain, and the European Regional Development Fund (ERDF) under project MTM2008-066200-C03-02/MTM .Also this research is supported by CONACyT-México under project 57371. The third author was supported by OTKA grant T 67867.

References

[1] M. Abreu, M. Funk, D. Labbate, V. Napolitano, On (minimal) regular graphs of girth 6, Australasian Journal of Combinatorics 35 (2006), 119-132.

[2] G. Araujo, D. González Moreno, J.J. Montellano, O. Serra, On upper bounds and conectivity of cages, Australasian Journal of Combinatorics. 38 (2007), 221–228.

[3] C. Balbuena, Incidence matrices of projective planes and other bipartite graphs of few vertices, Siam Journal of Discrete Maths. 22(4) (2008) 1351-1363.

[4] L.M. Batten, Combinatorics of finite geometries, Cambridge University Press, Cambridge, UK, 1997.

[5] C.T. Benson, Minimal regular graphs of girth eight and twelve, Canad. J. Math 18 (1966), 1091-1094.

[6] N. Biggs, Algebraic Graph Theory, Cambridge University Press, New York, 1996.

[7] N. Biggs, Construction for cubic graphs with large girth, The electronic Journal of combinatorics 5 (1998), #A1.

[8] E. Bannai and T. Ito, On finite Moore graphs, J. Fac. Sci. Univ. Tokio, Sect. I A Math 20 (1973), 191-208.

[9] G. Brinkmann, B. D. McKay and C. Saager, The smallest cubic graphs of girth nine, Combinatorics Probability and Computing 5, (1995), 1-13.

[10] P. Erdös and H. Sachs, Reguläre Graphen gegebener Taillenweite mit minimaler Knotenzahl, Wiss. Z. Uni. Halle (Math. Nat.), 12 (1963), 251–257.

[11] G. Exoo, A Simple Method for Constructing Small Cubic Graphs of Girths 14, 15 and 16, Electronic Journal of Combinatorics, 3, 1996.

[12] W. Feit and G. Higman, The non-existence of certain generalized polygons, J. Algebra 1 (1964), 114-131.

[13] A. Gács and T. Héger, On geometric constructions of (k, g)-graphs, Contributions to Discrete Mathematics, 3(1) (2008) 63-80.

[14] C. Godsil and G. Royle, Algebraic Graph Theory, Springer, NY 2000.

[15] D.A. Holton and J. Sheehan, The Petersen Graph, Chapter 6: Cages, Cambridge University (1993).

[16] F. Lazebnik, V.A. Ustimenko, A.J. Woldar New upper bounds on the order of cages. Electronic Journal of Combinatorics 14 R13, (1997), 1–11.

[17] H. Van Maldeghem, Generalized Polygons, Birkhäuser, Basel, 1998.

[18] M. Meringer, Fast generation of regular graphs and construction of cages, Journal of Graph Theory 30 (1999), 137–146.

[19] M. O'Keefe and P.K. Wong, A smallest graph of girth 10 and valency 3, Journal of Combinatorial Theory (B) 29 (1980), 91–105.

[20] M. O'Keefe and P.K. Wong, The smallest graph of girth 6 and valency 7, J. Graph Theory 5 (1981), no. 1, 79–85.

[21] T. Pisanski, M. Boben, D. Marusic, A. Orbanic, A. Graovac, The 10-cages and derived configurations, Discrete Mathematics 275 (2004), 265-276.

[22] W. T. Tutte, A family of cubical graphs, Proc. Cambridge Philos. Soc. (1947), 459–474.

[23] P.K. Wong, A regular graph of girth 6 and valency 11, Internat. J. Math. Math. Sci 9, (1986), 561–565.

[24] P. K. Wong, Cages-a survey, Journal of Graph Theory 6 (1982), 1–22.